A Preliminary Study on Rational Structure and Barred Galaxies

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Abstract Newton's understanding of solar system opened a new era of human civilization. Accordingly, the understanding of galaxies will stimulate a new period of human social harmony. Galaxies are the most independent components of the universe. A massive Hubble Space Telescope photos survey reveals that the diversity of galaxies in the early universe was as varied as the many galaxy types seen today. In 2001, I proposed the rational model of galaxy structure. Now in this paper I present a preliminary rational structure theory. The mathematical background needed is no more than the preliminary results of traditional college courses of complex functions and partial differential equations. Its application to barred galaxies is discussed. If the application is successful then three simple and important examples of its testification are, the structural simulation of barred galaxy light distribution with rational structure, the simulation of spiral arms with the Darwin curves of rational structure, and the simulation of observational galaxy kinematics (e.g., rotation curves) with the New Universal Gravity dictated by the rational structure and the principle of force line conservation.

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Contents

1 Rational Structure Equation

A structure is a two or three dimensional distribution of similar matters. This paper deals with two dimensional structure only. A curve in the plane of the distribution is called a Darwin curve if the matter density on one side of the curve is in constant ratio to that on the other side of the curve. If there exists an orthogonal net of Darwin curves in the plane, the distribution of matters is called a rational structure. We found many evidences that galaxies are rational stellar distribution. We list a few examples. Firstly, galaxy components (disks and bars) can be fitted with rational structure. Secondly, spiral arms can be fitted with Darwin curves. Thirdly, rational structure dictates New Universal Gravity which explains constant rotation curves simply and elegantly.

Rational structure in two dimension

$$\rho(x, y) \tag{1}$$

means that not only there exists an orthogonal net of curves in the plane

$$x = x(\lambda, \mu),\ y = y(\lambda, \mu) \tag{2}$$

but also, along each curve, the matter density on one side of the curve is in constant ratio to the one on the other side of the curve. Such a curve is called a proportion curve or a Darwin curve. Such a distribution of matter is called a rational structure. Because the density ratio is equivalent to the derivative to the logarithm of the density

$$f(x, y) = \ln \rho(x, y) \tag{3}$$

we from now on, are only concerned with the logarithmic density $f(x, y)$. We know that, given the two partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \tag{4}$$

the structure $f(x, y)$ is determined provided that the Green's theorem is satisfied

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{5}$$

The above is the well–known identity of mixed derivatives. Now we are interested in rational structure. Instead of calculating the partial derivatives (4), we calculate the directional derivatives along the tangent direction to the above curves

$$\frac{\partial f}{\partial l_\lambda}, \quad \frac{\partial f}{\partial l_\mu} \tag{6}$$

where l_λ is the linear length on the *physical* plane (x, y) and is along the row curves whose parameter is λ while l_μ is the linear length along the column curves whose parameter is μ. Given the two partial derivatives (6), however, the structure $f(x, y)$ may not be determined. A similar Green's theorem, which is called rational structure equation, must be satisfied

$$\frac{\partial}{\partial \mu}\left(P\,\frac{\partial f}{\partial l_\lambda}\right) - \frac{\partial}{\partial \lambda}\left(Q\,\frac{\partial f}{\partial l_\mu}\right) = 0 \tag{7}$$

where

$$\begin{aligned} P(\lambda, \mu) &= \sqrt{x_\lambda'^2 + y_\lambda'^2}, \\ Q(\lambda, \mu) &= \sqrt{x_\mu'^2 + y_\mu'^2} \end{aligned} \tag{8}$$

are the lengths or the magnitudes of the vectors (x'_λ, y'_λ) and (x'_μ, y'_μ), respectively. Note that we have used the simple notation $x'_\lambda = \frac{\partial x}{\partial \lambda}$. From now on, we always use the similar simple notations.

It is straightforward to show that the partial derivatives of rational structure in the parameter space (λ, μ) are

$$f'_\lambda = P\,\frac{\partial f}{\partial l_\lambda}, \quad f'_\mu = Q\,\frac{\partial f}{\partial l_\mu} \tag{9}$$

Therefore, the rational structure equation (7) is nothing but the following simple equation

$$\frac{\partial^2 f}{\partial\lambda\partial\mu} = \frac{\partial^2 f}{\partial\mu\partial\lambda} \tag{10}$$

To simplify the expression of our equations, we introduce the important new notations for our directional derivatives along the tangent direction of row and column curves in the physical space (x, y)

$$u(\lambda,\mu) = \frac{\partial f}{\partial l_\lambda}, \quad v(\lambda,\mu) = \frac{\partial f}{\partial l_\mu} \tag{11}$$

The condition of rational structure is that u depends only on λ and v depends only on μ

$$u = u(\lambda),\ v = v(\mu) \tag{12}$$

Now we prove the condition. Assume you walk along a row curve. The logarithmic ratio of the density on your left side to the immediate density on your right side is approximately the directional derivative of $f(x, y)$ along the column direction. That is, the logarithmic ratio is approximately the directional derivative $v(\lambda, \mu)$. Because $v(\lambda, \mu)$ is constant along the row curve (rational), $v(\lambda, \mu)$ is independent of λ, $v = v(\mu)$. Similarly, we can prove that $u(\lambda, \mu) = u(\lambda)$.

In the case of rational structure, the directional derivatives, $u = \frac{\partial f}{\partial l_\lambda}$ and $v = \frac{\partial f}{\partial l_\mu}$, are the functions of the single variables λ and μ, respectively (see the formula (12)). Therefore, the Green's theorem (7) turns out to be much simpler

$$u(\lambda)P'_\mu = v(\mu)Q'_\lambda \tag{13}$$

which is our rational structure equation [1,2]. The following equation is the necessary and sufficient condition for the curves to be orthogonal

$$x'_\lambda x'_\mu + y'_\lambda y'_\mu = 0 \tag{14}$$

In fact, rational structures depend only on the geometric curves, not the choice of coordinate parameters. Therefore, the general coordinate parameters $(\sigma,\ \tau)$ for the orthogonal net of curves are

$$\begin{cases} x = x(g(\sigma),\ h(\tau)), \\ y = y(g(\sigma),\ h(\tau)) \end{cases} \tag{15}$$

where $\lambda = g(\sigma), \mu = h(\tau)$ are arbitrary functions. All these expressions give the same orthogonal net of Darwin curves and generate the same rational structure. In the following Section, we choose some special coordinate parameters.

2 Harmonic Coordinate System and Rational Structure Equation

A harmonic coordinate system is the choice of coordinate parameters such that the following equations hold

$$\begin{aligned} x'_\lambda &= y'_\mu, \\ x'_\mu &= -y'_\lambda \end{aligned} \tag{16}$$

They are the well known Cauchy-Riemann equations and the following complex function in the space (λ, μ)

$$\Psi(\lambda, \mu) = x(\lambda, \mu) + \mathrm{i}\, y(\lambda, \mu) \tag{17}$$

must be analytic. The real part $x(\lambda, \mu)$ and the imaginary part $y(\lambda, \mu)$ are determined by each other and both are harmonic functions,

$$\begin{aligned} x''_{\lambda\lambda} + x''_{\mu\mu} &= 0 \\ y''_{\lambda\lambda} + y''_{\mu\mu} &= 0 \end{aligned} \tag{18}$$

The imaginary part is usually called the conjugate of the real part (see some textbook on analytic complex functions).

Does there exist a harmonic coordinate parameter for any net of curves? We prove that the necessary and sufficient condition for the existence of harmonic coordinates is that the net of curves is orthogonal. Application of the derivative chain–rule to the composite functions (15)

$$\begin{aligned} x'_\sigma &= x'_\lambda g'_\sigma, \quad x'_\tau = x'_\mu h'_\tau \\ y'_\sigma &= y'_\lambda g'_\sigma, \quad y'_\tau = y'_\mu h'_\tau \end{aligned} \tag{19}$$

The Cauchy-Riemann equations for the harmonic coordinate parameter (σ, τ) are

$$\begin{aligned} x'_\sigma &= y'_\tau, \\ x'_\tau &= -y'_\sigma \end{aligned} \tag{20}$$

That is,

$$\begin{cases} x'_\lambda g'_\sigma - y'_\mu h'_\tau = 0, \\ y'_\lambda g'_\sigma + x'_\mu h'_\tau = 0 \end{cases} \tag{21}$$

The above linear equation system has non-zero solution $\lambda = g(\sigma), \mu = h(\tau)$ if and only if the determinant of the equation system is zero. The condition turns out to be the orthogonal condition (14). Because rational structure is always defined on orthogonal net of curves, its harmonic coordinate parameters always exist. Therefore, we from now on, assume that the curve parameters (λ, μ) are themselves harmonic. A immediate result is the following

$$P(\lambda, \mu) = \sqrt{x'^2_\lambda + y'^2_\lambda} \equiv \sqrt{x'^2_\mu + y'^2_\mu} = Q(\lambda, \mu) \tag{22}$$

(see the formulas (8)). Accordingly the rational structure equation is

$$u(\lambda) P'_\mu = v(\mu) P'_\lambda \tag{23}$$

The symbol of quantity $Q(\lambda, \mu)$ is no longer needed. Given $u(\lambda)$ and $v(\mu)$, the above rational structure equation is a first order partial differential equation whose unknown is

$P(\lambda, \mu)$. Its general solution can be obtained with the standard characteristic method. The characteristic equation is

$$\frac{d\lambda}{v(\mu)} = \frac{d\mu}{-u(\lambda)} \tag{24}$$

which is an ordinary differential equation. The general solution to the ordinary equation is

$$U(\lambda) + V(\mu) = c \tag{25}$$

where c is an arbitrary constant, and $U(\lambda)$ and $V(\mu)$ are the indefinite integrals of the directional derivatives $u(\lambda)$ and $v(\mu)$, respectively

$$\begin{cases} u(\lambda) = U'_\lambda(\lambda), \\ v(\mu) = V'_\mu(\mu) \end{cases} \tag{26}$$

From now on, $U'_\lambda(\lambda)$ may simply be denoted by $U'(\lambda)$ or U'. The notation applies to other quantities when confusion may be avoided. Now the general solution for the original rational structure equation (23) is

$$W(P, C) = 0 \tag{27}$$

where $W(P, C)$ is an arbitrary function of the two variables P and C, and P is replaced with the unknown $P(\lambda, \mu)$ and C is replaced with the left–hand side of the equation (25) where the constant c must be in the right–hand side. If we do not pay much attention to possible multi-valued functions, the general solution (27) is essentially the fact that P is a function of the single variable $C = U(\lambda) + V(\mu)$

$$P = P(C) = P(U(\lambda) + V(\mu)) \tag{28}$$

The above is our second result derived with harmonic coordinate parameters. Note that $P(\lambda, \mu)$ and $P(C)$ are two different functions but we use the same symbol P to denote the two different functions. Which is which can be recognized with its accompanying symbols of derivatives. We follow this convention for simplicity also for other functions if applicable.

The formulas (9) now becomes

$$\begin{aligned} f'_\lambda &= u(\lambda)P(\lambda, \mu), \\ f'_\mu &= v(\mu)P(\lambda, \mu) \end{aligned} \tag{29}$$

Therefore,

$$u(\lambda)f'_\mu = v(\mu)f'_\lambda \tag{30}$$

Given $u(\lambda)$ and $v(\mu)$, the above first order partial differential equation is identical to the one (23). Following the same procedure for its general solution, we have

$$f = f(C) = f(U(\lambda) + V(\mu)) \tag{31}$$

Therefore,

$$\begin{aligned} f'_\lambda &= u(\lambda)f'_c(C), \\ f'_\mu &= v(\mu)f'_c(C) \end{aligned} \tag{32}$$

Comparison of the above equation with the equation (29) leads to

$$P(C) = f_c'(C) \tag{33}$$

Now we understand that the original rational structure equation (23) is solved The general solution is simply an arbitrary function of the single variable C: $P = P(C)$ or $f = f(C)$ (see the expression (28) or (31)) which is a composite function whose middle variable C is the sum of two other single–variable functions $U(\lambda)$ and $V(\mu)$

$$C = U(\lambda) + V(\mu) \tag{34}$$

However, the function expression (28) or (31) alone is not the sufficient condition for the solution of our rational structure. It is the necessary one. This is because the net of curves (2) solved from the following equation

$$P(\lambda, \mu) = \sqrt{x_\lambda'^2 + x_\mu'^2} \tag{35}$$

may not be orthogonal.

3 Necessary and Sufficient Condition for Rational Structure

The net of curves (2) is orthogonal and its parameters are harmonic if and only if the Cauchy-Riemann equations (16) hold. The equations hold if and only if $\mathrm{x}(\lambda, \mu)$ is a harmonic function. The function $\mathrm{x}(\lambda, \mu)$ is harmonic if and only if its derivatives

$$\Phi(\lambda, \mu) = x_\lambda' + \mathrm{i}\, x_\mu' \tag{36}$$

satisfy Cauchy-Riemann equations. The above complex function satisfies Cauchy-Riemann equations if and only if the logarithm of its modulus

$$\ln P(\lambda, \mu) = \ln \sqrt{x_\lambda'^2 + x_\mu'^2} \tag{37}$$

is a harmonic function. Finally, applying harmonic operation to the above function gives the necessary condition for orthogonal net of curves. We use the symbol L to denote the logarithm

$$L(\lambda, \mu) = \ln P(\lambda, \mu) = \ln P(C) = \ln P(U(\lambda) + V(\mu)) \tag{38}$$

Its derivatives are

$$\begin{aligned} L_{\lambda\lambda}'' &= \tfrac{P''P - P'^2}{P^2} u^2(\lambda) + \tfrac{P'}{P} u'(\lambda), \\ L_{\mu\mu}'' &= \tfrac{P''P - P'^2}{P^2} v^2(\mu) + \tfrac{P'}{P} v'(\mu) \end{aligned} \tag{39}$$

Finally, the harmonic equation $L_{\lambda\lambda}'' + L_{\mu\mu}'' = 0$ leads to the following necessary condition for the harmonic parameters (λ, μ)

$$\frac{L_{cc}''}{L_c'} = -\frac{u'(\lambda) + v'(\mu)}{u^2(\lambda) + v^2(\mu)} \tag{40}$$

That is,

$$\frac{P''_{cc}}{P'_c} - \frac{P'_c}{P} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} \tag{41}$$

Finally, the above equation is the necessary condition for the harmonic parameters (λ, μ). It is also the sufficient and necessary condition for the general rational structure if we always remember the additional condition, i. e., the existence of the single variable function (28) whose single–variable is the formula (34)

4 Solving the Sufficient and Necessary Condition for Rational Structure

We know that the equation (41) is the sufficient and necessary condition for rational structure. Now we try to solve it. Rational structure (3) can be classified into two types, axi-symmetric and non-axisymmetric. Any axi-symmetric structure is rational because it has a net of orthogonal curves (the polar coordinate curves, i. e., the radial lines and the circles with center at the origin). The directional derivatives along the circles, i. e., $v(\mu)$, is zero while the directional derivatives along the radial lines, i. e., $u(\lambda)$, depends solely on the single variable $r \sim \lambda$. Therefore, any axi-symmetric structure is rational. In this Section we study non-axisymmetric rational structure.

Now we deal with the partial differential equation (41). Firstly we try the simplest example of the function $P(C)$, a power function,

$$P = P(C) = C^k \tag{42}$$

where k is a real constant. The equation (41) becomes

$$-\frac{1}{U+V} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} \tag{43}$$

which is irrelevant to the constant l. The above equation is

$$U(\lambda)V''(\mu) + V(\mu)U''(\lambda) = U'^2(\lambda) + V'^2(\mu) - U(\lambda)U''(\lambda) - V(\mu)V''(\mu) \tag{44}$$

Therefore, the mixed partial derivatives to the left hand side of the above equation must be zero

$$U'(\lambda)V'''(\mu) + V'(\mu)U'''(\lambda) = 0 \tag{45}$$

Finally we have a general solution to the rational structure equation (41)

$$\begin{aligned} &U(\lambda) = m_0 + m_1 e^{l\lambda} + m_2 e^{-l\lambda}, \\ &V(\mu) = n_0 + n_1 \cos(l\mu) + n_2 \sin(l\mu), \\ &m_0 = -n_0, \quad 4m_1 m_2 = n_1^2 + n_2^2 \end{aligned} \tag{46}$$

where$l, m_0, m_1, m_2, n_0, n_1, n_2$ are real constants.

Now we want to solve the corresponding net of curves which is given by the equation (35). If we choose $k = 1/2$ then the equation is

$$\sqrt{U(\lambda) + V(\mu)} = \sqrt{x'^2_\lambda + x'^2_\mu} \tag{47}$$

Amazingly, the resulting harmonic coordinate $x(\lambda, \mu)$, its conjugate, and the rational structure are all the same Heaven Breasts pattern (see the graph in [3]). Therefore, we call the above complete solution (46) the Heaven Breasts structure. To have the breasts aligned in the horizontal direction, we need to choose $n_2 = 0$.

If we try an exponential function of C, $P(C) = \exp(C)$, then the net of orthogonal curves for the resulting rational structure are all straight lines, a trivial result. If we try other types of functions, e. g., a polynomial $P(C) = c_1 C^{k_1} + c_2 C^{k_2}$, and follow the similar procedure to the equations (42) through (45) then we find out that the resulting equations are highly nonlinear and it is hard for their solution. These have to be left for future exploration.

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