The Position-Momentum Commutator

September 26, 2013.

José Francisco García Juliá

jfgj1@hotmail.es

It is shown that the position-momentum commutator is a diagonal matrix.

Key words: position-momentum commutator, diagonal matrix.

In a recent paper [1], we have implied that the position-momentum commutator is a diagonal matrix

$$[x, p] = xp - px = i\hbar I \tag{1}$$

where x and p are the position and the momentum of the atomic electron, respectively, $i^2 = -1$, $\hbar = h/2\pi$, and h is the Planck's constant, and I the unit (or identity) matrix.

And where:

$x_{ij}(t) = x_{ij}(0)\ exp(i2\pi f_{ij}t)$; f_{ij} being the oscillation frequency of the atomic electron, and therefore also the frequency of the light emitted by it: $f_{ij} = (E_i - E_j)/h = E_{ij}/h$, when it goes from the level i to the level j ($i > j$), and where E_i and E_j are the energies of the electron corresponding to these levels, and t is the time

$p_{ij}(t) = m\ dx_{ij}(t)/dt = m\ x_{ij}(0)\ exp(i2\pi f_{ij}t)\ i2\pi f_{ij} = m\ x_{ij}(t)\ i2\pi f_{ij} = a\ x_{ij}(t)\ f_{ij}$; m being the electron moving mass and $a = m\ i2\pi$

$x = [x_{ij}]$, $p = [p_{ij}]$, $x_{ji} = x_{ij}^*$, $f_{ji} = (E_j - E_i)/h = -f_{ij}$

$$\begin{aligned}
[x, p] &= xp - px = [\Sigma_k\ x_{ik}\ p_{kj}] - [\Sigma_k\ p_{ik}\ x_{kj}] = [\Sigma_k\ x_{ik}\ a\ x_{kj}\ f_{kj}] - [\Sigma_k\ a\ x_{ik}\ f_{ik}\ x_{kj}] \\
&= a\ [\Sigma_k\ (f_{kj} - f_{ik})\ x_{ik}\ x_{kj}] = a\ [(\Sigma_k\ (f_{kj} - f_{ik})\ x_{ik}\ x_{kj})_{i\neq j} + (\Sigma_k\ (f_{kj} - f_{ik})\ x_{ik}\ x_{kj})_{i=j}] \\
&= a\ [0 + \Sigma_k\ (f_{kj} - f_{jk})\ x_{jk}\ x_{kj}] = a\ [\Sigma_k\ 2\ f_{kj}\ x_{kj}^*\ x_{kj}] = a\ [\Sigma_k\ 2\ f_{kj}\ |x_{kj}|^2] \\
&= i\ [\Sigma_k\ 2\ m\ 2\pi f_{kj}\ |x_{kj}|^2] = i\hbar\ I
\end{aligned}$$

That is

$$[x, p] = i\ [\Sigma_k\ 2\ m\ 2\pi f_{kj}\ |x_{kj}|^2] = i\hbar\ I \tag{2}$$

For the last relation, note that the stationary orbit condition of Bohr for the atomic electron was: $m\ v\ r = n\hbar$; then, $n\hbar = m\ v\ r = m\ \omega\ r\ r = m\ 2\pi f\ r^2$, where n is a positive integer, $\omega = 2\pi f$ the angular frequency and r the radius of the orbit.

Note also that it would be

$$(\Sigma_k\,(f_{kj} - f_{ik})\,x_{ik}\,x_{kj})_{i\neq j} = 0 \tag{3}$$

which implies that $[x, p] = xp - px$ is a diagonal matrix.

To demonstrate this, Jordan [2] used the Hamilton's equations: $\dot{q} = dq/dt = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$, where q, p and H are the (canonical) position, the momentum and the Hamiltonian, respectively. Then

$$d[q,p]/dt = d(qp - pq)/dt = \dot{q}p + q\dot{p} - \dot{p}q - p\dot{q} = (q\dot{p} - \dot{p}q) + (\dot{q}p - p\dot{q}) = (0) + (0) = 0$$

As

$$[x, p] = [\Sigma_k\, a\,(f_{kj} - f_{ik})\,x_{ik}\,x_{kj}] = [\Sigma_k\, a\,(f_{kj} - f_{ik})\,x_{ik}(0)\,exp(i2\pi f_{ik}t)\,x_{kj}(0)\,exp(i2\pi f_{kj}t)] \tag{4}$$

where $f_{ij} \neq 0$ for $i \neq j$ but $f_{ij} = 0$ for $i = j$, and as $f_{ik} + f_{kj} = f_{ij}$, then $[x, p]$ is a matrix of the type: $g = [g_{ij}\,exp(i2\pi f_{ij}t)]$. As $dg/dt = [g_{ij}\,exp(i2\pi f_{ij}t)\,i2\pi f_{ij}]$, then $dg/dt = 0$ only if g is diagonal ($i = j$): $g = [g_{ii}]$, which corresponds with (1), (2) and (3).

Now, let be the equation [3]:

$dy/dt = B\,y$, where B is a constant (independent of t) matrix with distinct characteristic roots (the equation, $det(B - \lambda I) = 0$, has distinct values of λ)

Doing $y = T\,z$, where T is a non singular ($det\,T \neq 0$) constant matrix whose columns are the eigenvectors of B, we have

$T\,dz/dt = B\,T\,z$

$dz/dt = T^{-1}\,B\,T\,z$

If $T^{-1}\,B\,T = \Lambda$, where Λ is a diagonal matrix, then we have k equations

$dz_k/dt = \lambda_k\,z_k$

whose solutions are

$z_k(t) = exp(\lambda_k t)\,z_k(0)$

But, using only matrices, it is also

$Z(t) = [exp(\lambda_k t)]\,Z(0)$

$Y(0) = T\,Z(0),\ Z(0) = T^{-1}\,Y(0) = T^{-1}\,I = T^{-1},\ doing\ Y(0) = I$

$Y(t) = T\,Z(t) = T\,[exp(\lambda_k t)]\,Z(0) = T\,[exp(\lambda_k t)]\,T^{-1}$

If for the matrix B, it can be obtained a diagonal matrix $T\,[exp(\lambda_k t)]\,T^{-1}$; then, from our matrix (4), it can also be obtained the diagonal matrix (2) with the condition (3).

In summary, the position-momentum commutator is a diagonal matrix.

[1] José Francisco García Juliá, A Criticism to the Quantum Mechanics, September 20, 2013. viXra:1308.0001 [Quantum Physics] http://vixra.org/abs/1308.0001

[2] M. Born and P. Jordan, Zur Quantenmechanik, Zeitschrift für Physik, **34**, 858-888, 1925 (received September 27, 1925).

[3] Richard Bellman, Introducción al Análisis Matricial, pp. 207-210, Reverté, Barcelona, 1965. Original edition, Introduction to Matrix Analysis, McGraw-Hill, New York, 1960.