Two numerical counterexamples contrary to the phase matching condition for quantum search

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September 24, 2013

Abstract

In this short comment, by giving two numerical counterexamples directly contrary to the phase matching condition presented by Long *et al.*, I show that Grover's conclusion that his algorithm can be extended to the case when the two phase inversions are replaced by arbitrary phases (Phys. Rev. Lett. **80**, 4329–4332, 1998) in a sense is correct.

Keywords: *Grover's quantum search algorithm; Phase matching condition; Two-dimensional complex subspace; Multiphase matching equation; Numerical counterexample.*

1 Two numerical counterexamples

In this short comment, I want to put forward the following two problems related to the generalized Grover's quantum search algorithm. I shall, unless otherwise indicated, adopt the same notations and symbols as in Ref. [1] throughout this comment.

(I) When the quantum search space is strictly confined to the two-dimensional complex subspace spanned by $|\alpha\rangle$ and $|\beta\rangle$, i.e. $\omega = \pi/2$ or $\Phi = 0$, Long *et al.*

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concluded that to guarantee the effectiveness of their generalized Grover's algorithm, the phase matching condition shall be obeyed [2], and explicitly pointed out that ***a useful quantum search algorithm can be constructed if and only if the two phase rotations are equal***, i.e., $\phi' = \theta'$ [3] (see the last line on p. 143 of Section.1 in [3]), where ϕ' and θ' are the rotation angles for the desired state and the $|0\rangle$ state respectively. In [4], Grover foresaw that his algorithm can be extended to the case when the two phase inversions are replaced by arbitrary phases; however, the number of operations required to reach a unique desired state (or the the superposition of the desired states) $|\beta\rangle$ will be greater. From the result obtained in [2], Long *et al.* [5] asserted that the Grover's conclusion [4] is incorrect. As a matter of fact, the Grover's conclusion in a sense is completely correct. In order to substantiate Grover's prediction, I shall give a numerical counterexample directly contrary to the conclusions drawn in Refs. [2, 3, 5].

Suppose that the two rotation angles ϕ and θ appearing in Eqs. (3) and (4) of Ref. [1] are defined in the real domain (denoted by R) only for the sake of computational convenience. In the case of $\omega = \pi/2$, Eq. (10) of Ref. [1] would then become

$$Q_{\omega=\pi/2} = \begin{pmatrix} -1+\left(1-e^{i\theta}\right)\cos^2 t & e^{i\phi}\left(1-e^{i\theta}\right)\sin t\cos t e^{-i\Lambda} & 0 \\ \left(1-e^{i\theta}\right)\sin t\cos t e^{i\Lambda} & e^{i\phi}\left(1-e^{i\theta}\right)\sin^2 t - e^{i\phi} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$\equiv \begin{pmatrix} -1+\left(1-e^{i\theta}\right)\cos^2 t & e^{i\phi}\left(1-e^{i\theta}\right)\sin t\cos t e^{-i\Lambda} \\ \left(1-e^{i\theta}\right)\sin t\cos t e^{i\Lambda} & e^{i\phi}\left(1-e^{i\theta}\right)\sin^2 t - e^{i\phi} \end{pmatrix}, \tag{1}$$

where $\Lambda = t_2 - t_1 \in R$. Without loss of generality, we assume that the initial superposition of the quantum system is given by

$$|\gamma_0\rangle = \cos\beta_0 |\alpha\rangle + e^{i\varsigma}\sin\beta_0 |\beta\rangle, \tag{2}$$

where $\varsigma \in R$. According to Eqs. (1) and (2), writing

$$\begin{pmatrix} u_j \\ d_j \end{pmatrix} = Q_{\omega=\pi/2}\left(\phi_j, \theta_j\right) Q_{\omega=\pi/2}\left(\phi_{j-1}, \theta_{j-1}\right) \cdots Q_{\omega=\pi/2}\left(\phi_1, \theta_1\right) \begin{pmatrix} \cos\beta_0 \\ e^{i\varsigma}\sin\beta_0 \end{pmatrix} \tag{3}$$

allows one to compute the success probability of finding a desired state $P_j = \left|d_j\right|^2$ for any positive integer j. Because the exhaustive algorithm for obtaining the numerical counterexample listed in Table 1 is somewhat complicated, I have to ignore the details and directly give the numerical results. Here the values for the necessary parameters are taken as follows: $N = 500$, $M = 1$, $t = \beta_0 = \arcsin\left(\sqrt{1/500}\right)$, $\varsigma = \pi/4$, and $\Lambda = 2\pi/3$.

Table 1: A numerical evidence of finding a unique desired state with the maximum success probability almost close to 100% for the case when any two phase rotation angles corresponding to each iteration of Grover operator are completely independent and all of the pairs of phases are different from one another

j	ϕ_j	θ_j	P_j
1	4.47691852225397	3.86564750136926	0.01597636683063
2	2.39985397616049	1.67532285905716	0.03491761450790
3	1.62682858315984	1.56193055107000	0.06052028476760
4	5.47514521429465	0.40562308764187	0.05700913029382
5	5.07660860396279	4.75035279035777	0.08516026183164
6	1.88766024824765	5.41391236605466	0.06681909845621
7	3.98249699397130	1.20405220860577	0.05754029739753
8	4.18963049355497	3.75176971789826	0.10300670684107
9	1.01204403679963	0.13870274962554	0.10488680795531
10	0.48292846003932	3.43414230660624	0.07326004645406
11	5.49055951096594	3.26987901995537	0.12632809841112
12	1.50100709231654	2.64530126314932	0.13814472386638
13	3.73241787761449	0.37169823391148	0.14498164019590
14	0.69451916346778	2.45769180516456	0.20786499683523
15	3.05583268809717	2.91754658939600	0.27381722454255
16	4.30800333723039	4.14788490412821	0.29747009903199
17	2.59304302189212	2.47304055388295	0.36863772596513
18	3.04314293709985	3.15879006245116	0.43753602376110
19	4.34421860580483	1.82544724328038	0.38163672190798
20	7.72984351058501	3.86425735312075	0.42012462934596
21	-2.15475234993836	-0.95896714849203	0.45396376865470
22	5.23670329166646	4.19393652082960	0.49148989042532
23	4.97797578323003	-0.10794225661714	0.49466434837179
24	3.08277659579260	0.09312885169068	0.49756117934816
25	0.75467543553487	1.63964319213892	0.54782092456201
26	1.62618946041987	3.15542689128986	0.63539510329060
27	3.14761090698791	2.77407116409396	0.71707078394616
28	2.74509008509969	3.36621248912693	0.79154641878018
29	3.33509937355549	2.55077089638525	0.85577608615215
30	2.52011004402567	3.91844808433375	0.90028376571870
31	3.87146339082840	1.91057277845084	0.93770179601169
32	1.85496962258905	1.77254965984372	0.96438414877729
33	1.78706221588899	0.72432675890097	0.96918204081119
34	0.74923849230603	2.955747313196098	0.99215942299348
35	-110.15332871550700	-90.83731191554520	1.00000000000000
Σ	-9.51641349000195	-9.51639373488229	

Actually, we can prove that given an initial superposition as defined in Eq. (2), there exist two sets of the rotation angles $\left\{\phi_1, \phi_2, \cdots, \phi_j\right\}$ and $\left\{\theta_1, \theta_2, \cdots, \theta_j\right\}$ for some j such that a unique desired state can be found with certainty in the two-dimensional complex subspace. This result fully shows that in order for the outcome to be a desired state with success probability exactly equal to unity when we measure the quantum system, there is no need to appeal to the previous phase-matching conditions derived from literature. It can be shown theoretically that finding a desired state with certainty in the two-dimensional complex subspace depends on the exact multiphase matching equation

$$\sum\nolimits_{j=1}^{k''} \phi_j - \sum\nolimits_{j=1}^{k''} \theta_j = |\delta| , \tag{4}$$

where $|\delta| \in R$ is very small in general, and k'' is some positive integer. If we let $\phi_1 = \phi_2 = \cdots = \phi_j = \phi$ and $\theta_1 = \theta_2 = \cdots = \theta_j = \theta$, then we recover the different accurate phase formulae in the literature, which are the special cases of Eq. (4). Hence, Grover's conclusion that his algorithm can be extended to the case when the two phase inversions are replaced by arbitrary phases [4] in a sense is correct. We see that Table 1 also supports his opinion.

(II) Similarly, the special case for the Grover's problem was regarded as a general conclusion, leading to another incorrect conclusion that arbitrary phase rotation of the marked state can not be used for Grover's quantum search algorithm [6]. However, Table 2 is shown to be relevant contrary to what has been claimed in Ref. [6]. As to Table 2, the necessary parameters are taken as follows: $N = 100$, $M = 1$, $t = \beta_0 = \arcsin\left(\sqrt{1/100}\right)$, $\varsigma = \pi/4$, and $\Lambda = 2\pi/3$.

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Table 2: Another numerical evidence of finding a unique desired state with the maximum success probability almost close to 100% for the case when any two phase rotation angles corresponding to each iteration of Grover operator are completely independent and all of the pairs of phases are different from one another

j	ϕ_j	$\theta_j = \pi$	P_j
1	2.74693546171360	3.14159265358979	0.04366720905957
2	0.39828214016856	3.14159265358979	0.00054554900806
3	2.48833695270913	3.14159265358979	0.04236514463243
4	1.73964289056824	3.14159265358979	0.10042469718810
5	0.58043353704706	3.14159265358979	0.01566909697381
6	4.45838129862463	3.14159265358979	0.02889814813813
7	3.94062881006929	3.14159265358979	0.13113440664562
8	2.26321781589114	3.14159265358979	0.24991790499712
9	2.24415685987322	3.14159265358979	0.28072174486909
10	1.00275699685193	3.14159265358979	0.13012427131784
11	2.75632915768029	3.14159265358979	0.06581602684862
12	6.05617274918125	3.14159265358979	0.10865748067297
13	4.36972813166616	3.14159265358979	0.25520219912522
14	4.05749371815842	3.14159265358979	0.34617385077964
15	3.89135508767492	3.14159265358979	0.32873877913832
16	3.64438547831505	3.14159265358979	0.25064081311826
17	0.05998018813862	3.14159265358979	0.33814365765471
18	3.86135920128299	3.14159265358979	0.52848663455115
19	3.18486364339076	3.14159265358979	0.71631782902581
20	3.29347389448605	3.14159265358979	0.87429387628613
21	3.49908183896486	3.14159265358979	0.97040478005902
22	1.43286253370092	3.14159265358979	0.95485558950199
23	0.15138778378410	3.14159265358979	0.96039946932160
24	13.71855915018830	3.14159265358979	0.99999999999600
$\sum$	75.83980532012950	75.39822368615504	

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