Kissing Number Cells and Integral Conjecture

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Abstract: Kissing Numbers (*1*) appear to be the product of dimension number and the dimension's simplex vertex number for 0-3 Euclidean spatial dimensions, but depart from the linear product of dimension and dimension+1 relationship at 4-dimensions and above increasing away from this exponentially. For 0-8 dimensions there is a Coxeter Number root system type relationship. The author proposes a very simple relationship which satisfies both aforementioned patterns, but extends from dimension 0 infinitely upwards. The conjecture is seen to satisfy the non-root system 24-dimensions and leads to prediction. The simplex nature of this work may be utilised in Quantum Gravity theories similar to Causal Dynamical Triangulation.

Introduction

We compare the (Newton) Kissing Number Problem (*1*) to Simplexes, from 0 to 3 dimensions (n), resulting in a simple relationship when applied between n=0-8 produces Coxeter Numbers. Up to and beyond dimension 8 the author conjectures Kissing Numbers (K) being composed of integral multiples of dimension number. We call these multiples Kissing Cells (c). Empirically we find that this is true for all known Kissing Numbers, and remarkably even when applied to the non root system n=24, and it is also frequently the case for best known bounds. We predict a Kissing Number in 5-dimensons. This relationship is extremely simple with very little mathematics, hence may prove useful or be of interest outside its own arena.

Classic problem, Reduced

Aside from highly symmetrical Euclidean spatial dimensions, such as n=8 and n=24, the classic Kissing Number Problem at 5-dimensions and above remains unknown. For these dimensions, once the apparent maximum number of outer spheres all touch the central sphere, a lot of space remains, hence a precise Kissing Number is difficult to prove.

Kissing Cells for n=0-3 are self evident and trivially proven as 1, 2, 3 and 4 respectively, i.e. they are derived from the n-simplex polytope's vertices, being the number of spheres which all touch each other or are equidistant from each other, without involving a central sphere (Figs. 1-3) (calculation 1).

Calculation 1

K = cn

K = Kissing Number for dimension n
c = Kissing Cell for dimension n
n = Dimension Number

At 4-dimensional space and above the simplex multiple relationship stops, where the Kissing Number is 24 (*3*). Our calculation shows the Kissing Cell is 6 spheres – matching the Coxeter number.
We have dimensions 0-4 where the Kissing Number divided by the dimension number, n gives the Coxeter Number / Kissing Cell.

Although it is known that Coxeter Numbers to dimension 8 give the best known classic Kissing Numbers from root systems, table 1 shows that at dimension 24 the multiple relationship holds for the known Kissing Number 196560 with an integral Kissing Cell of 8190, while it is not from a root system itself. In fact many of the calculated current best bounds also result in integer Kissing Cells (table 1), whether from root systems or not.

The Integral Kissing Cell conjecture is based upon simplex vertices being constrained at 3-dimensional space.
At 4-dimensions the simplex is the 5-cell which has 1 too few vertices to give the Kissing Number when multiplied by n.
In the case of 4-dimension's 24-cell (the known Kissing Number configuration), which has 24 octahedral cells, there are 6 of these meeting at each vertex, which are 3-dimensional shapes not matching the native dimension-4. Hence the theory that 3-dimensional space constrains Kissing Cell geometry and hence Kissing Numbers.

In the cases of n=0-4 native polytopes give Kissing Cells.

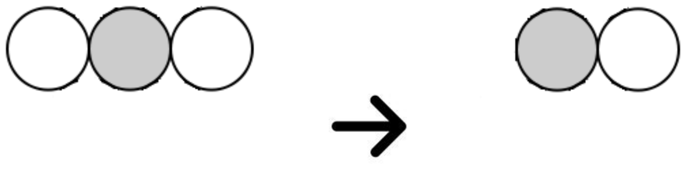

Fig. 1. 1-dimensional Kissing Number of 2 and Kissing Cell from the 2 vertices of the 1-simplex line segment.

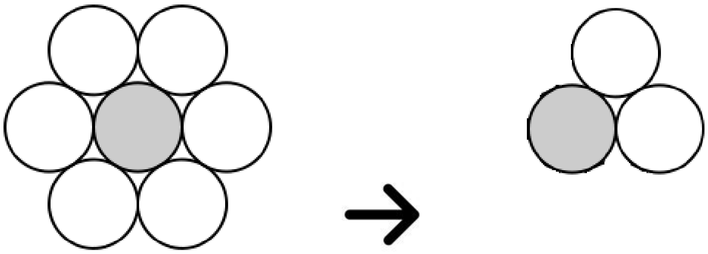

Fig. 2. 2-dimensional Kissing Number of 6 and Kissing Cell from the 3 vertices of the 2-simplex triangle.

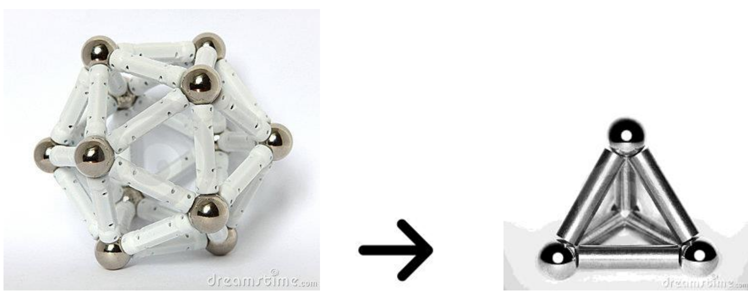

Fig. 3. 3-dimensional Kissing Number of 12 and Kissing Cell from the 4 vertices of the 3-simplex tetrahedron (*7*).

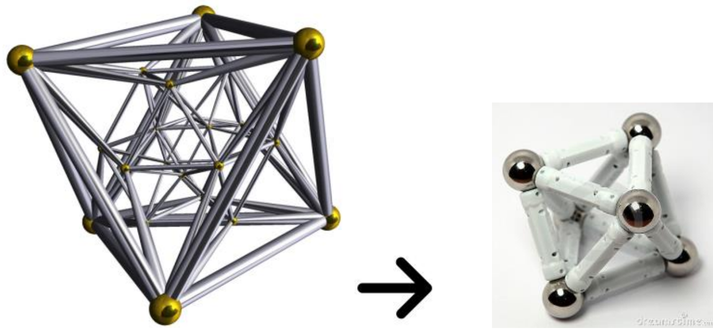

Fig. 4. 4-dimensional Kissing Number of 24 (24-cell) (*8*) and Kissing Cell from the 6 vertices of the 3-dimensional Octahedron (*7*), not the 5-cell simplex (figure 5). The simplest unit cell in the 24-cell to achieve Kissing Number is the 3-dimensional octahedron.

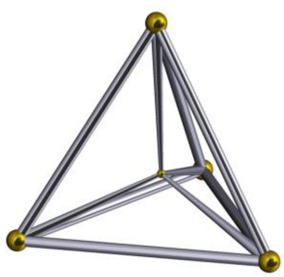

Fig. 5. 4-dimensional simplex 5-cell (*9*) has just 5 vertices, which cannot give the 4-dimensional Kissing Number of 24 as a product of n and c.

Discussion

Table 1 shows the calculated Kissing Cells follow a Coxeter Number type pattern up to 8-dimensions, yielding integers, which is to be expected as these limits are calculated from root systems. However, integrals are yielded in a number of higher dimensional best bound cases that are not root systems and particularly remarkable is the presence of an integer for the Kissing Cell from the precisely known dimension-24 (table 1).

The Kissing Number would indeed seem to be a multiple of both the dimension number and Kissing Cell, or more precisely a product of the two.

Notable in table 1 is that we change from c>K at n=0, to c=K at n=1 to c<K at n=2, which is expected for n=0 since we can have an infinitesimally small point, but no room for any Kissing spheres.

It has recently been realised that the perhaps seemingly abstract Kissing Cell number of 8190 for dimension 24, when divided by the odd lattice number of 273 (*4*), results in 30, i.e. the Kissing Cell for dimension 8. This may have potential usage in lattice study, particularly between the symmetries of E_8 and the Leech Lattice.

Further, the author has researched best known classic Kissing Numbers past dimension 24 (*5-7*), and many of these result in integral Kissing Cell bounds (table 1).

The author predicts that all future proven Kissing Numbers will be multiples of dimension number n. If our conjecture is correct then dimension-5 is already solved as a Kissing Number of 40 from an integral Kissing Cell of 8, because the upper bound currently calculates as 8.8 and the lower is already 8, the nearest tighter integer is 8.

This work may be utilised in Quantum Gravity theories and is the subject of an ongoing theory of everything by the author.

References and Notes:

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Table 1. Kissing Numbers and best bounds with calculated Kissing Cells

Dimension (n)	**Kissing Number Lower Bound (*2*)**	**Kissing Number Upper Bound (*2*)**	**Calculated Kissing Cell Lower Bound**	**Calculated Kissing Cell Upper Bound**
0	0	0	1	1
1	**2**	**2**	2	2
2	**6**	**6**	3	3
3	**12**	**12**	4	4
4	**24**	**24**	6	6
5	40	44	8	8.8
6	72	78	12	13
7	126	134	18	19.14285714
8	**240**	**240**	30	30
9	306	364	34	40.44444444
10	500	554	50	55.4
11	582	870	52.90909091	79.09090909
12	840	1357	70	113.0833333
13	1154	2069	88.76923077	159.1538462
14	1606	3183	114.7142857	227.3571429
15	2564	4866	170.9333333	324.4
16	4320	7355	270	459.6875
17	5346	11072	314.4705882	651.2941176
18	7398	16572	411	920.6666667
19	10688	24812	562.5263158	1305.894737
20	17400	36764	870	1838.2
21	27720	54584	1320	2599.238095
22	49896	82340	2268	3742.727273
23	93150	124416	4050	5409.391304
24	**196560**	**196560**	8190	8190

32	146880, 261120 (*5*)		4590, 8160	
44	2708112 (*5*)		61548	
48	52416000 (*6*)		1092000	
64	9694080 (*5*)		151470	
72	6218175600 (*6*)		86363550	
128	1260230400 (*5*)		9845550	

Table 1 shows the calculated Kissing Cells, the majority of which are integers. Bold type denotes the currently known Kissing Numbers.