On a New and Novel Solution to Einstein's Famous Twin Paradox Without Invoking Accelerations of the Travelling Twin

(Paper I)

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This is the first instalment in a four part series, the aim of the work being to introduce absolute motion into Einstein's Special Theory of Relativity (STR). In the traditional treatment of Einstein's famous twin paradox, it is argued that the stay at home twin will age more than the "travelling" twin and the asymmetry is attributed to the fact that the travelling twin's reference system is not an inertial reference system during the periods of acceleration and deceleration thus making it "illegal" for the "travelling" twin to use the STR in their reference system, hence "resolving" the paradox altogether. From within the domains, confines and provinces of Einstein's STR, we argue without considering the accelerations and decelerations, where we show that, indeed, it is the "travelling" twin that is younger at the point of reunion. This brings us to a point of admission that there is indeed a twin who really does the travelling and another that does the staying at home. Hidden within the labyrinth of its seemingly coherent and consistent structure and fabric, does Einstein's STR imply absolute motion – we ask? This is the question that we leave hanging in the mind of the reader. In the next reading, we propose a new version of the twin paradox, where the scenario is truly symmetric from either of the twin's reference systems – we have coined this, the "Symmetric Twin Paradox (STP)". This version (STP) unearths an irretrievable contradiction hidden at the deepest and subtle level of Einstein's STR. It is shown that Einstein's STR is unable to resolve this irretrievable contradiction, even if the accelerations and decelerations are taken into. Not even Einstein's General Theory of Relativity can be brought to the rescue in the case of the STP. In our third instalment, we shall setforth a new version of the STR where absolute motion is permitted. This version solves the symmetric twin paradox and any known paradox of relativity. Lastly, we apply this new STR where absolute motion is permitted to experimental efforts that have been made to measure absolute motion. Most well trained physicists tend to ignore completely, readings purporting to go against Einstein's STR. We would like to persuade our reader to make a brief stop and consider for a minute, what we have to say in our four part series of readings.

"At the heart of science is an essential balance between two seemingly contradictory attitudes – an openness to new ideas no matter how bizarre or counterintuitive they may be, and the most ruthless skeptical scrutiny of all ideas, old and new. This is how deep truths are winnowed from deep nonsense."

– Carl Sagan (1934 – 1996)

Contents

1 Introduction

On April 20, 1905, the soon to be Dr. Albert Einstein (1879 – 1955) – then a mere twenty six year old working far from the provinces and shores of professional academia as a 3^{rd} Class Patent Clerk in Bern, Switzerland; he [Einstein] is said to have shaken the very foundations of physics with his theory of relativity, which on Max Karl Ernst Ludwig Planck (1858 – 1947)'s suggestion[*], come to be known as the Special Theory

[*]When Einstein first submitted his revolutionary paper, its initial title was "*The Theory of Invariants*". Planck, a founder and the first editor of the

of Relativity (STR) [1]. Some simple call it Special Relativity. This theory come as nothing short of a revolution in science and human thought since man set his eyes on the stars – first because it demolished at the outset, the long held notion that before then appeared immutable, namely that, time was an absolute and universal physical quality. This assumption syphoned directly from common sense is one of the central tenets of the great Sir Isaac Newton (1642 – 1727)'s mechanics which at the time, had stood the test of theoretical and experimental philosophy for nearly two and half centuries. In Newton's own words:

> *"Absolute, true, and mathematical time, of itself, and from its own nature, flows equable without relation to anything external ..."*

Completely at odds with natural intuition, from Einstein's STR, one learns of the unexpected and surprising fact that time is not an absolute and universal physical quantity flowing off and by itself with no relation to anything external as Newton had wanted us to believe, but, that its flow varies from one inertial system of reference to the other – simple, it flows inequitably for different observers.

The second reason is that Einstein's STR – with the simple remark "*superfluous*" – in an all-sweeping manner, it [the STR] rendered as unnecessary the idea that there exists a preferred or privileged system of reference in the Universe – *i.e.*, the *aether* is not necessary for the physical description of the Universe. Hypothesised (postulated) by Maxwell James Clerk (1831 – 1879) in 1869, the aether was thought to be essential and necessary for the propagation of light through empty space [9]. According to Maxwell's ideas, in this hypothetical aether medium, for all observers – irrespective of their state of motion; light travels at the constant and seemingly sacrosanct speed $c = 2.99792458 \times 10^8\,\mathrm{ms}^{-1}$.

In 1881 and later in 1879, Michelson and Michelson & Mosley set themselves the task to equal this delicate and surmountable task which would make them the first men to measure the aether [6, 7]. Much to their "chagrin" and as-well to the surprise of the scientific community of the day, they reported to the world their famous result that the experiment(s) yielded no proof of the existence of the aether.

Without (perhaps) the full knowledge of the Michelson-Mosley Experiment (MME), it strongly appears that Einstein had already independently reasoned that it was not necessary to invoke this hypothetical aether medium. He reasoned that naturally, the Laws *of* Physics must be the same for all inertial observers – this seemingly simple idea, he called the *Principle of Relativity*. The Principle of Relativity formed the powerful basis of all his reasoning. If this were true, and concurrently the speed of light were an absolute constant and independent of an observer's state of motion as predicted by Maxwell's celebrated electromagnetic theory – then the speed of light ought to be a universal and absolute constant. This must be true for every observer every-when and anywhere in the Universe – Einstein contended.

From this kind of simple but deep reasoning, Einstein – with a rare mastery stroke of brilliance; overturned *Newtonian Physics* forever thus replacing it with his newly discovered STR which was derived from the universal constancy of the speed of light and the Principle of Relativity. Because of the experimental success of Einstein's theory, from a phenomenological standpoint, it [the STR] hardly can be wrong. However, its philosophy can be replaced while upholding its phenomenology. This is the attempt that we make in our four part series of readings. Whether this attempt is successful or not, we leave this to the reader and the experimenters of prosperity.

Our strategy in our presentation of facts is as follows. In the present reading, we demonstrate (argue) that there is no need for one to invoke the accelerations of the "travelling" twin in-order to arrive at a solution to the twin paradox as is the case in the textbook solution of the problem. By making use of the "clandestine" symmetry hidden in the scenario of the set of twins, one can argue from within the logical confines of the STR that, indeed, one twin will emerge younger while the other emerges older. This naturally brings us to the paradoxical scenario where truly one of the twins is travelling and the other is truly not travelling.

In the second follow-up reading, we present a truly paradoxical scenario of the symmetric twins who both travel and on return. In this symmetric scenario, if we are to follow Einstein's philosophy of relative motion, then, they [twins] can not and can never ascertained who between the two of them is younger than the other on reunion. This case of the symmetric twin paradox lets the "*cat out of the bag*". The internal logical inconsistency of Einstein's STR is out for full public viewing, in the light of the day for all to see. This however does not spell doom for Einstein's theory as the two sacrosanct postulates of relativity can still be upheld – *albeit*, at the expense of introducing absolute motion into Einstein's STR. Thus, in the third reading, we develop the STR in which absolute motion is possible. In the forth and final reading, we apply this new STR in which absolute motion is possible to the MME where upon it is seen that, this experiment may very well have been the first experiment to detect the existence of absolute motion.

2 Twin Paradox as Commonly Understood

Judging from the response of first year students encountering this for the first time and even well acquaintanced and seasoned physicists, it is (perhaps) safe and fair to say the twin paradox can be confusing*. Before going into its intimate details, we would like to give a brief background of its origins.

newly established Journal, *Annalen der Physik*, suggested to Einstein that he changed the title to "*Special Theory of Relativity*".

*See *e.g.* `http://twinparadox.net/`, visited on this day 14 Aug. 2013@15*h*55 GMT+2. A survey and perusal of this site will – amongst others, reveal why the *Twin Paradox* can not be considered a solved problem. Debates on this problem are still very much alive.

This paradox was first pointed out by Einstein himself, not as a paradox but as a straight forward logical deduction from his STR. In its original form, Einstein stated [2]:

> *"If we placed a living organism in a box . . . one could arrange that the organism, after any arbitrary lengthy flight, could be returned to its original spot in a scarcely altered condition, while corresponding organisms which had remained in their original positions had already long since given way to new generations. For the moving organism the lengthy time of the journey was a mere instant, provided the motion took place with approximately the speed of light."*

It was the prominent and venerated French physicist – Paul Langevin (1872 – 1946), in 1911 that rephrased this into what we now know as the twin paradox by replacing the organisms with the twins[†]. Since then, the twin paradox has been the subject of analysis in philosophy, physics, biology, chemistry and other esoteric fields of human endeavour. A natural source of this confusion for those encountering the STR for the first in their endeavour to comprehend the time-dilation effect and this is where the fascination and confusion comes from when one is dealing with the twin paradox. The real confusion lays in fathoming who is moving and who is not.

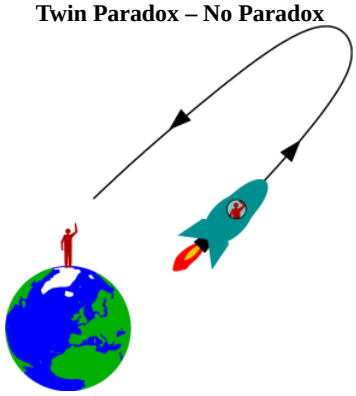

Fig. 1: Adapted from Markus Pössel [8]. Tandiwa rockets into space on a round trip with his equally agile twin bother staying put on planet Earth.

Suppose we have a set of twins – instead of Alice and Bob, let us for whatever reason – veer off from tradition and call them Takunda (T_1) and Tadiwa (T_2). Tadiwa decides to celebrate his 21$^{\text{th}}$ birthday in style by rocketing at a constant relativistic speed (*i.e.* speeds comparable to the speed of light, for which the effects predicted by the STR become important and significant) to the nearest star to planet Earth – which is α-Centauri. Takunda and Tadiwa are recent *kum laud* physics graduates who understand very well Professor Albert Einstein's 1905 STR. Tadiwa makes a round-trip, *i.e.*, he travels to α-Centauri at a constant relativistic speed and upon arrival, he immediately makes an about-turn and returns back to planet Earth. The other twin Takunda decides to stay at home and not join his adventurous twin brother.

According to the *bona fide* and prevalent Philosophy of Relativity due Einstein, Takunda sees Tadiwa moving away from the Earth and at the sametime, Tadiwa has equal claim in his own system of reference that he is not moving but Takunda is moving away from him at the same speed as that Takunda sees him move albeit in the opposite direction. The paradox arises because according to the STR, the one that is "moving" will experience time dilation, so the question is; since each sees the other as "moving", who then amongst the two of them is the one that has experienced this time dilation? and thus seems younger to the other upon reunion?

3 Textbook Solution to the Twin Paradox

Virtually all standard textbooks that we have had the opportunity to peruse through, rightly state that the twin paradox is not a paradox and the solution they offer is as follows. They correctly state that the apparent paradox arises from an incorrect application of the Principle of Relativity to the description of the story from the travelling twin's point of view. From his point of view [*i.e.*, the travelling twin Tadiwa], the argument goes; his non-adventurous stay-at-home brother is the one who travels backward on a receding Earth, and then returns as the Earth approaches the spaceship again; while in the system of reference fixed to the spaceship, the astronaut twin is not moving at all. Thus, it would then seem that the twin on Earth is the one whose biological clock should tick more slowly and not the one on the spaceship. Also, from Takunda's point of view, it is Tadiwa that is moving and thus must be younger on his return thus raising the apparent paradoxical situation – who really is younger on reunion?

The textbooks state that the flaw in the reasoning is that the Principle of Relativity only applies to systems that are in motion at constant velocity relative to one another. This is correct, the question is, does this really solve the problem in manner that renders absolute motion superfluous? The astronaut twin's system of reference, is a non-inertial system because his spaceship must accelerate when it leaves until it reaches its desired speed, decelerate when it reaches its destination before turning back for the return journey, and then repeat the whole process (acceleration-deceleration) again on the way back home. Their experiences are not equivalent, because the astronaut twin feels accelerations and decelerations thus leading to the conclusion that the travelling twin will be younger when they are reunited. That is very true – the question is, "Does this render absolute motion superfluous?"

These textbooks go on to say that the GTR must be used during the accelerations and decelerations of the "travelling" twin. These accelerations and decelerations - they say; bring about asymmetric and it is this asymmetric that solves the apparent paradox. While these textbooks say the GTR solves

[†]See *e.g.* http://arxiv.org/ftp/arxiv/papers/1205/1205.0922.pdf

the problem none that we have had the opportunity to peruse through, do make the GTR calculation to verify their claim. We have even surveyed GTR books, and again, none make this calculation. One can find a calculation on *Wikipedia*[*] which only mathematically proves that the travelling twin is really the one that is younger on reunion. To what extent do the accelerations and decelerations affect the ageing process? no answers can be found on this website. Off cause, because of accelerations and decelerations, the adventurous twin is the one that really is moving. During the period when the adventurous twin is not experiencing any accelerations and decelerations, whose clock is tricking slower? Once again, one finds no answers to these questions in the textbooks, *Wikipedia* or related websites.

It is clear from the above that the *"real trick and relativity's sleight of hand"* lays in the accelerations and decelerations experienced by the travelling twin; these bring about the much needed asymmetry which leads to Tadiwa being the one that experiences the time dilation. Despite the fact these accelerations and decelerations experienced by the travelling twin are accepted as a resolution of the paradox, we hold a view to the contrary namely that these accelerations are not key to the resolution of the problem. As will be demonstrated soon, we believe there is a deep underlying asymmetry that solves this problem within the confines of the STR and this asymmetry, as shall be argued, invariably and intimately connects the STR to the existence of a fixed, immovable, all-pervading and permuting cosmic background or medium. This solution we provide in the next section.

4 New and Novel Solution to the Twin Paradox

From a purely idealized standpoint, we can neglect these accelerations and decelerations. If we do this, we will be lead to a scenario that appears at face value symmetric and this would certainly lead to irretrievable contradictions? With the accelerations and decelerations neglected, the scenario is actually asymmetric and this conclusion we draw from the fact the twin's succinct description of their experience reveal a deep underlying asymmetry.

If two persons where to give a succinct description of their experiences and these experiences where truly symmetric, one would not be able to differentiate the difference in their statements, because their experiences would appear exactly the same (equivalent) if we swapped or interchanged some keywords in their statements. This is not the case with the present scenario as will be clarified soon. A succinct description of the twins experiences is as follows:

***According to Takunda* (Earth bound twin):**
He is stationery and Tadiwa is moving toward the constellation α-Centauri and this constellation α-Centauri is not moving.

***According to Tadiwa* (travelling twin):**
He is stationery while both Takunda and the constellation α-Centauri are moving as a whole unit like a rigid body.

NB: According to Tadiwa, Takunda and α-Centauri move as a rigid body because they are stationery relative to each other – this is where the asymmetric lays and this asymmetric solves the twin paradox but rises a question about absolute motion. Without any doubt, α-Centauri is a third fixed reference point and it is this point that resolves the paradox from within the confines of the STR without need to invoke the asymmetries that come in with the accelerations and decelerations. According to Einstein's philosophy of relative motion, two reference points are sufficient for the complete description of motion *i.e.* the "stationery observer" and the "moving observer". The third point α-Centauri is a stationery fixed point relative to the "stationery observer". The "stationery observer" together with α-Centauri as a point, these form "rod" which moves back and forth according to the "moving observer". As we all know, according to Einsten's STR, a moving rod will appear shorter by a factor $\sqrt{1-v^2/c^2}$ according to the observer observing this motion. This is where our approach to the twin paradox lies!

If the reader agrees with us so far, then, we can proceed. Clearly, the description of events by the Takunda and Tadiwa are not equivalent hence not symmetric. For example, Tadiwa sees Takunda and α-Centauri moving as a whole unit like a rigid body, while Takunda sees himself and α-Centauri at relative rest. In order to better understand what we mean by "the description of events by each of the observers must be the same (equivalent) or symmetric" and as-well what we mean by:

> *'If their experiences where symmetric, then, the description of their experiences would appear exactly the same if we swapped (or interchanged) some keywords in their succinct statements'*

the reader may have to wait until the penultimate of the subsequent section. The asymmetry seen in the description of events here is all one needs in order to come to the conclusion that the Tadiwa is older at the moment of reunion. We show and re-enforce our point in the subsequent paragraphs. We hope our reader will pay particular attention to the fact that "according to the observer in motion, the earth and α-Centauri form or comprise a 'fixed' length redolent a rod."

***According to Takunda* (Earth bound twin T_1):**
He is stationery and Tadiwa is moving toward α-Centauri and the constellation α-Centauri is not moving. Takunda, know-

[*]See `http://en.wikipedia.org/wiki/Twin_paradox`, visited on this day 14 Aug. 2013@15*h*46 GMT+2.

ing that the proper distance from the Earth to α-Centauri in his rest system is L_0; and that Tadiwa is moving at a speed v relative to him and given that Takunda is an astute physics graduate; it follows that he knows that the time lapse for a round trip for Takunda will be $\Delta t_1 = 2L_0 = v$ (the accelerations and decelerations have been neglected here). This time lapse is the time he has aged which Tandiwa has been rocketing. Hence, Takunda will boldly conclude that he has aged $\Delta t_1 = 2L_0/v$ (years) at reunion since Tandiwa's departure.

NB: It is important at this point of the reading to note that the rest length L_0 which is the proper distance between the Earth and the constellation α-Centauri and as-well the relative speed v between the set of twins are accessible to both twins. Equipped with this information, these two observers can calculate the age or the other – these ages, they will compare at reunion. Already, from the above argument, the Earth bound twin T_1 knows that he has aged $\Delta t_1 = 2L_0/v$ (years) since Tandiwa's departure. The question is, 'if we go into the rocketing twin's frame of reference, will this twin also find that their age is $2L_0/v$?' The answer is no, they will calculate a different age altogether. The problem with Einstein's twin paradox as commonly presented in standard textbooks is that – without the accelerations and decelerations, it is assumed that the situation of the twins is symmetric in such a manner that each of the twins will compute the same duration for their separation, leading to the paradoxical scenario of who really is younger at reunion? This symmetry is broken by the accelerations and decelerations, leading to an acceptable solution to the paradox.

***According to Tadiwa* (Travelling twin T_2):**

He is stationery while both Takunda and the constellation α-Centauri are moving as a whole unit like a rigid body. That is, Takunda and α-Centauri are stationery relative to each other hence they behave like a rigid body just like a rod. In Tadiwa's rest system, this rigid body rod will be measured to have a length L which is different from the proper length L_0 since it is "moving". Since this rigid body is not stationery in Tadiwa's reference system but is moving at a speed v past his "stationery" system of reference. Since Tadiwa is also an astute physics graduate, he knows very well that the rod will "appear" to him to have a length $L = L_0\sqrt{1 - v^2/c^2}$ and since this rod is moving at a speed v, it means that the time it takes this rod to move back and forth will be $t_2 = 2L_0\sqrt{1 - v^2/c^2}/v$. It follows that:

$$\Delta t_2 = \frac{t_1}{\sqrt{1 - v^2/c^2}}, \tag{1}$$

hence, Tadiwa will boldly conclude that his time since he journeyed to the stars is Δt_2. From the (1) above, it follows that $\Delta t_2 \neq \Delta t_1$ *i.e.* $\Delta t_2 > \Delta t_1$. The time lapses t_2 & t_1 are the times that the twins will present to each other at reunion as being the duration of their separation as measured in their respective system of reference. Clearly, there is no contradiction or debate let alone a paradox about who is younger or older at reunion. The facts are as clear as the Earth's atmosphere occurring when there is a blue sky. Figure (2) below captures very well what we have said above.

View According to Traveling Twin

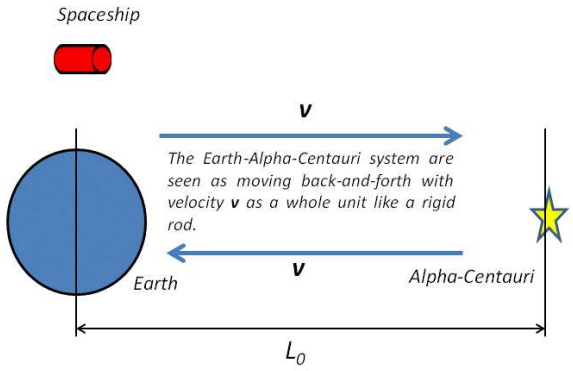

Fig. 2: The traveling twin sees the fixed distance between the Earth and α-Centauri as truly Lorentz-contracted by a factor $\sqrt{1 - v^2/c^2}$ as envisoned by Lorentz and Fitzgerald [3–5]. From his thorough knowledge of Einstein's STR, he will conclude that the time for the Earth and α-Centauri to move back and forth is $2L_0\sqrt{1 - v^2/c^2}/v$, hence he can compute the age of the Earth bound twin!

From the foregoing, the rocking twin is the one that ages less, and his ageing is real and not apparent and accepting this leads us to a "problem", namely that the twin that ages less than the other is really the one that experiences motion in the true sense. The solution has come from the very fact that in Tadiwa's system of reference, Takunda and α-Centauri move as a rigid body because they are stationery relative to each other and Takunda can never say that about Tadiwa and α-Centauri. *This asymmetric, as just demonstrated, is – in our modest view, the durable and correct solution that solves the twin paradox once and for altime.* While it solves the twin paradox from within the provinces of the STR, it rises a question about absolute motion.

That is, while the travelling twin will see, the stay at home twin as being in motion and he being stationery, this motion is not real but apparent and only the motion seen by the Earth bound observer is what is real and the rest is nothing but an illusion since in the true sense, it is the travelling twin that really ages and we need not the accelerations and decelerations to justify this.

We should say that we have never encountered this kind of solution to the problem of the twin paradox in the literature that we have had the good fortune to lay our hands. Hence, we believe this may be the first time such a solution is ap-

pearing. Because we have no better way to express ourself, we strongly believe the reader should go through this again to really convince themselves that the solution lays in the asymmetry as stated above.

5 Discussion and Conclusion

5.1 Discussion

We do not know what the reader has to say about the present presentation. To ourself, we are of the strong feeling that this reading spells it out clearly without ambiguity, that the twin paradox – without having to invoke the GTR; it does have a durable solution from within the internal logic, coherency and consistency of Einstein's STR. However, this solution directly points to the unpleasant fact that there is one twin that is *truly moving* and one that is *truly stationery*. The one that is truly moving is the one that is younger at reunion.

Sacrilegiously, this new solution pits us *face-against-the-wall* with a central tenant of Einstein's STR, for one somehow is strictly forbidden from talking of special relativity and absolute motion and or space in the same theory. In his landmark and seminal paper on the STR, with the simple remark "superfluous", Einstein – at a stroke – dismissed the notion of absolute motion and or space, at with occasion, he sweepingly, relegated this once great idea to the peripheries of physics of great but failed ideas. For Einstein, absolute motion and or space are not necessary for the description of physical phenomenon. So, there is no need to invoke what is not necessary, hence the remark "superfluous".

It appears we have to consider the real possibility that Einstein's STR points to the existence of absolute motion. This is what this work will do. In the next instalment, we will demonstrate (convincingly) that Einstein's STR contains a non-repairable logical flaw that can only be resolved by invoking absolute motion. This forces us to develop a new theory of relativity in which absolute motion is possible. This theory will have to uphold the two postulates of Einstein's STR while at the sametime it in-cooperates absolute motion. This is what we will do in the third instalment. In the forth and final instalment, we apply this new theory to experimental efforts that have been made so far to determine the absolute motion of the Earth whereby we demonstrate that there exists a significant non-zero absolute motion of the Earth.

5.2 Conclusion

In conclusion, we would like to say that it is our strong view that this reading has urged consistently, coherently and efficiently, well within the accepted bounds of physics and common logic that Einstein's twin paradox has a solution well within the framework of the STR. Indeed, the stay at home twin is really older than the stay at home twin at reunion, thus making him the truly stationery twin and the other the travelling twin. This solution points to the real possibility of the existence of absolute motion. The contraction of the Earth-α-Centauri distance as seen by the rocketing twin is real and not apparent because if it was apparent, the ageing would too be apparent. This truly points to the existence of some absolute space to which mechanics facts can be refereed to.

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