Non-Solvable Equation Systems with Graphs Embedded in $\mathbb{R}^n$

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June 28, 2013
§1. Introduction

Consider two systems of linear equations following:

\[
(LES_4^N) \quad \begin{cases}
  x + y &= 1 \\
  x + y &= -1 \\
  x - y &= -1 \\
  x - y &= 1
\end{cases} \quad (LES_4^S) \quad \begin{cases}
  x = y \\
  x + y = 2 \\
  x = 1 \\
  y = 1
\end{cases}
\]

\( (LES_4^N) \) is non-solvable \hspace{1cm} \( (LES_4^S) \) is solvable

What is the geometrical essence of a non-solvable or solvable system of linear equations?
Fig. 1

\[(\text{LES}_4^N)\]

\[
x - y = -1
\]
\[
x - y = 1
\]
\[
x + y = 1
\]
\[
x + y = -1
\]

\[(\text{LES}_4^S)\]

\[
x = 1
\]
\[
x = y
\]
\[
y = 1
\]
\[
x + y = 2
\]
(LES^n_4) is non-solvable but (LES^S_4) solvable in sense because of

\[ L_{x+y=1} \bigcap L_{x+y=-1} \bigcap L_{x-y=1} \bigcap L_{x-y=-1} = \emptyset \]

and

\[ L_{x=y} \bigcap L_{x=1} \bigcap L_{y=1} \bigcap L_{x+y=2} = \{(1, 1)\} \]
Generally,

\[
(ES_m) \begin{cases} 
  f_1(x_1, x_2, \cdots, x_n) = 0 \\
  f_2(x_1, x_2, \cdots, x_n) = 0 \\
  \cdots \cdots \cdots \\
  f_m(x_1, x_2, \cdots, x_n) = 0 
\end{cases}
\]

\((ES_m)\) is solvable or not dependent on \(\bigcap_{i} S_{f_i} = \emptyset\) or \(\neq \emptyset\).

**Proposition 1.1** Any system \((ES_m)\) of algebraic equations with each equation solvable posses a geometrical figure in \(\mathbb{R}^n\), no matter it is solvable or not.
Conversely, for a geometrical figure $\mathcal{G}$ in $\mathbb{R}^n$, $n \geq 2$,

*how can we get an algebraic representation for geometrical figure $\mathcal{G}$?*

As a special case, let $G$ be a graph embedded in Euclidean space $\mathbb{R}^n$ and

$$(E S_e) \quad \begin{cases} f_1^e(x_1, x_2, \cdots, x_n) = 0 \\ f_2^e(x_1, x_2, \cdots, x_n) = 0 \\ \cdots \cdots \cdots \\ f_{n-1}^e(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

be a system of equations for determining an edge $e \in E(G)$ in $\mathbb{R}^n$. 
Then the system

\[
\begin{align*}
  f_1^e(x_1, x_2, \cdots, x_n) &= 0 \\
  f_2^e(x_1, x_2, \cdots, x_n) &= 0 \\
  \quad \vdots \\
  f_{n-1}^e(x_1, x_2, \cdots, x_n) &= 0 \\
\end{align*}
\]  \quad \forall e \in E(G)

is a non-solvable system of equations.
For example, let $G$ be a planar graph, shown in Fig.2.

Fig.2

$$\begin{align*} &v_1 \quad y = 8 \\
v_4 \quad y = 2 &v_3 \quad x = 12 \\
x = 2 &x = 12 \\
y = 2 &y = 2 \\
3x + 5y = 46. &3x + 5y = 46. \end{align*}$$

**Proposition 1.2** Any geometrical figure $G$ consisting of $m$ parts, each of which is determined by a system of algebraic equations in $\mathbb{R}^n$, $n \geq 2$ possesses an algebraic representation by system of equations, solvable or not in $\mathbb{R}^n$. 
§2. Smarandache Systems with Labeled Topological Graphs

**Definition 2.1([5-7])** A rule $\mathcal{R}$ in a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set $\Sigma$, i.e., validated and invalidated, or only invalidated but in multiple distinct ways.

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule $\mathcal{R}$.

**Definition 2.2([5-7],[11])** Let $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \ldots, (\Sigma_m; \mathcal{R}_m)$ be $m \geq 2$ mathematical spaces, different two by two. A Smarandache multi-space $\tilde{\Sigma}$ is a union $\bigcup_{i=1}^{m} \Sigma_i$ with rules $\tilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i$ on $\tilde{\Sigma}$, denoted by $(\tilde{\Sigma}; \tilde{\mathcal{R}})$. 
Such a typical example is the proverb of blind men with an elephant.
Definition 2.3(([5-7])) Let \((\tilde{\Sigma}; \tilde{R})\) be a Smarandache multi-space with \(\tilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i\) and \(\tilde{R} = \bigcup_{i=1}^{m} R_i\). Then a inherited graph \(G[\tilde{\Sigma}, \tilde{R}]\) of \((\tilde{\Sigma}; \tilde{R})\) is a labeled topological graph defined by

\[
V\left(G[\tilde{\Sigma}, \tilde{R}]\right) = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_m\},
\]

\[
E\left(G[\tilde{\Sigma}, \tilde{R}]\right) = \{ (\Sigma_i, \Sigma_j) \mid \Sigma_i \cap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m \}
\]

with an edge labeling

\[
l^E : (\Sigma_i, \Sigma_j) \in E\left(G[\tilde{\Sigma}, \tilde{R}]\right) \rightarrow l^E(\Sigma_i, \Sigma_j) = \overline{\omega}\left(\Sigma_i \cap \Sigma_j\right),
\]

where \(\overline{\omega}\) is a characteristic on \(\Sigma_i \cap \Sigma_j\) such that \(\Sigma_i \cap \Sigma_j\) is isomorphic to \(\Sigma_k \cap \Sigma_l\) if and only if \(\overline{\omega}(\Sigma_i \cap \Sigma_j) = \overline{\omega}(\Sigma_k \cap \Sigma_l)\) for integers \(1 \leq i, j, k, l \leq m\).
For example, let $S_1 = \{a, b, c\}$, $S_2 = \{c, d, e\}$, $S_3 = \{a, c, e\}$ and $S_4 = \{d, e, f\}$. Then the multi-space $\tilde{S} = \bigcup_{i=1}^{4} S_i = \{a, b, c, d, e, f\}$ with its labeled topological graph $G[\tilde{S}]$ is shown in Fig.4.
The labeled topological graph $G \left[ \tilde{\Sigma}, \tilde{R} \right]$ reflects the notion that there exists linkage between things in philosophy. In fact, each edge $(\Sigma_i, \Sigma_j) \in E \left( G \left[ \tilde{\Sigma}, \tilde{R} \right] \right)$ is such a linkage with coupling $\varpi(\Sigma_i \cap \Sigma_j)$. For example, let $a = \{ \text{tusk} \}$, $b = \{ \text{nose} \}$, $c_1, c_2 = \{ \text{ear} \}$, $d = \{ \text{head} \}$, $e = \{ \text{neck} \}$, $f = \{ \text{belly} \}$, $g_1, g_2, g_3, g_4 = \{ \text{leg} \}$, $h = \{ \text{tail} \}$ for an elephant $\mathcal{E}$. Then its labeled topological graph is shown in Fig.5,

Fig.5

which implies that one can characterizes the geometrical behavior of an elephant combinatorially.
§3. Non-Solvable Systems of Ordinary Differential Equations

3.1 Linear Ordinary Differential Equations

For integers $m$, $n \geq 1$, let

$$\dot{X} = F_1(X), \quad \dot{X} = F_2(X), \ldots, \dot{X} = F_m(X) \quad (DES_m^1)$$

be a differential equation system with continuous $F_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $F_i(\mathbf{0}) = \mathbf{0}$, particularly, let

$$\dot{X} = A_1 X, \ldots, \dot{X} = A_k X, \ldots, \dot{X} = A_m X \quad (LDES_m^1)$$

be a linear ordinary differential equation system of first order with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \leq k \leq m$, $1 \leq i, j \leq n$. 
Definition 3.1 An ordinary differential equation system \((DES_m^1)\) or \((LDES_m^1)\) are called non-solvable if there are no function \(X(t)\) hold with \((DES_m^1)\) or \((LDES_m^1)\) unless the constants.

As we known, the general solution of the \(i\)th differential equation in \((LDES_m^1)\) is a linear space spanned by the elements in the solution basis

\[
\mathcal{B}_i = \left\{ \beta_k(t)e^{\alpha_k t} \mid 1 \leq k \leq n \right\}
\]

for integers \(1 \leq i \leq m\), where

\[
\alpha_i = \begin{cases} 
\lambda_1, & \text{if } 1 \leq i \leq k_1; \\
\lambda_2, & \text{if } k_1 + 1 \leq i \leq k_2; \\
\cdots & \\
\cdots & \\
\lambda_s, & \text{if } k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq n,
\end{cases}
\]

\(\lambda_i\) is the \(k_i\)-fold zero of the characteristic equation

\[
\det(A - \lambda I_{n \times n}) = |A - \lambda I_{n \times n}| = 0
\]

with \(k_1 + k_2 + \cdots + k_s = n\) and \(\beta_i(t)\) is an \(n\)-dimensional vector consisting of polynomials in \(t\) with degree\(\leq k_i - 1\).
In this case, we can simplify the labeled topological graph \( G \left[ \sum_i, \tilde{R} \right] \) replaced each \( \sum_i \) by the solution basis \( \mathcal{B}_i \) and \( \sum_i \cap \sum_j \) by \( \mathcal{B}_i \cap \mathcal{B}_j \) if \( \mathcal{B}_i \cap \mathcal{B}_j \neq \emptyset \) for integers \( 1 \leq i, j \leq m \), denoted by \( G[LD_{m}ES_{m}^{1}] \).

For example, let \( m = 4 \) and

\[
\mathcal{B}_1^0 = \{e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\}, \quad \mathcal{B}_2^0 = \{e^{\lambda_3 t}, e^{\lambda_4 t}, e^{\lambda_5 t}\}, \quad \mathcal{B}_3^0 = \{e^{\lambda_1 t}, e^{\lambda_3 t}, e^{\lambda_5 t}\}
\]

\[
\mathcal{B}_4^0 = \{e^{\lambda_4 t}, e^{\lambda_5 t}, e^{\lambda_6 t}\}, \text{ where } \lambda_i, \ 1 \leq i \leq 6 \text{ are real numbers different two by two.}
\]

Then \( G[LD_{m}ES_{m}^{1}] \) is shown in Fig.6.

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**Fig.6**

[Diagram showing the labeled topological graph with nodes \( \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 \) and edges connecting them with labels \( \{e^{\lambda_3 t}\}, \{e^{\lambda_1 t}, e^{\lambda_3 t}\}, \{e^{\lambda_3 t}, e^{\lambda_5 t}\}, \{e^{\lambda_4 t}, e^{\lambda_5 t}\}, \{e^{\lambda_5 t}\} \).]
Theorem 3.2([10]) Every linear homogeneous differential equation system \((LDES^1_m)\) uniquely determines a basis graph \(G[LDES^1_m]\) inherited in \((LDES^1_m)\). Conversely, every basis graph \(G\) uniquely determines a homogeneous differential equation system \((LDES^1_m)\) such that \(G[LDES^1_m] \simeq G\).

Such a basis graph \(G[LDES^1_m]\) is called the \(G\)-solution of \((LDES^1_m)\).

Theorem 3.3([10]) Every linear homogeneous differential equation system \((LDES^1_m)\) has a unique \(G\)-solution, and for every basis graph \(H\), there is a unique linear homogeneous differential equation system \((LDES^1_m)\) with \(G\)-solution \(H\).
Example 3.4

Let \((LDE^m_n)\) be the following linear homogeneous differential equation system

\[
\begin{cases}
\ddot{x} - 3\dot{x} + 2x = 0 & (1) \\
\ddot{x} - 5\dot{x} + 6x = 0 & (2) \\
\ddot{x} - 7\dot{x} + 12x = 0 & (3) \\
\ddot{x} - 9\dot{x} + 20x = 0 & (4) \\
\ddot{x} - 11\dot{x} + 30x = 0 & (5) \\
\ddot{x} - 7\dot{x} + 6x = 0 & (6)
\end{cases}
\]

Fig. 7 A basis graph
3.2 Combinatorial Characteristics of Linear Differential Equations

Definition 3.5 Let \((LDES_m^1), (LDES_m^1)\)' be two linear homogeneous differential equation systems with \(G\)-solutions \(H, H'\). They are called combinatorially equivalent if there is an isomorphism \(\varphi : H \rightarrow H'\), thus there is an isomorphism \(\varphi : H \rightarrow H'\) of graph and labelings \(\theta, \tau\) on \(H\) and \(H'\) respectively such that \(\varphi \theta(x) = \tau \varphi(x)\) for \(\forall x \in V(H) \cup E(H)\), denoted by \((LDES_m^1) \cong (LDES_m^1)'\).

Definition 3.6 Let \(G\) be a simple graph. A vertex-edge labeled graph \(\theta : G \rightarrow \mathbb{Z}^+\) is called integral if \(\theta(uv) \leq \min\{\theta(u), \theta(v)\}\) for \(\forall uv \in E(G)\), denoted by \(G^{I\theta}\).

Let \(G_1^{I\theta}\) and \(G_2^{I\tau}\) be two integral labeled graphs. They are called identical if \(G_1 \cong G_2\) and \(\theta(x) = \tau(\varphi(x))\) for any graph isomorphism \(\varphi\) and \(\forall x \in V(G_1) \cup E(G_1)\), denoted by \(G_1^{I\theta} = G_2^{I\tau}\).
For example, these labeled graphs shown in Fig.8 are all integral on $K_4 - e$, but $G_{1}^{I_{\theta}} = G_{2}^{I_{\tau}}$, $G_{1}^{I_{\theta}} \neq G_{3}^{I_{\sigma}}$.

![Graphs](image)

Fig.8

**Theorem 3.5([10])** Let $(LDES_{m}^{1})$, $(LDES_{m}^{1})'$ be two linear homogeneous differential equation systems with integral labeled graphs $H$, $H'$. Then $(LDES_{m}^{1}) \cong (LDES_{m}^{1})'$ if and only if $H = H'$. 
3.3 Non-Linear Ordinary Differential Equations

If some functions $F_i(X)$, $1 \leq i \leq m$ are non-linear in $(DES_m^1)$, we can linearize these non-linear equations $\dot{X} = F_i(X)$ at the point $\bar{0}$, i.e., if

$$F_i(X) = F'_i(\bar{0})X + R_i(X),$$

where $F'_i(\bar{0})$ is an $n \times n$ matrix, we replace the $i$th equation $\dot{X} = F_i(X)$ by a linear differential equation

$$\dot{X} = F'_i(\bar{0})X$$

in $(DES_m^1)$. 
§4. Cauchy Problem on Non-Solvable Partial Differential Equations

Let \((PDES_m)\) be a system of partial differential equations with

\[
\begin{aligned}
F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) &= 0 \\
F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) &= 0 \\
&\quad \vdots \\
F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) &= 0
\end{aligned}
\]

on a function \(u(x_1, \cdots, x_n, t)\). Then its symbol is determined by

\[
\begin{aligned}
F_1(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) &= 0 \\
F_2(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) &= 0 \\
&\quad \vdots \\
F_m(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) &= 0,
\end{aligned}
\]

i.e., substitute \(p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_n^{\alpha_n}\) into \((PDES_m)\) for the term \(u_{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}\), where \(\alpha_i \geq 0\) for integers \(1 \leq i \leq n\).

**Definition 4.1** A non-solvable \((PDES_m)\) is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.
Theorem 4.2([11]) A Cauchy problem on systems

\[
\begin{align*}
F_1(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) &= 0 \\
F_2(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) &= 0 \\
\vdots \\
F_m(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) &= 0
\end{align*}
\]

of partial differential equations of first order is non-solvable with initial values

\[
\begin{align*}
x_i \big|_{x_n = x_n^0} &= x_i^0(s_1, s_2, \ldots, s_{n-1}) \\
u \big|_{x_n = x_n^0} &= u_0(s_1, s_2, \ldots, s_{n-1}) \\
p_i \big|_{x_n = x_n^0} &= p_i^0(s_1, s_2, \ldots, s_{n-1}), \quad i = 1, 2, \ldots, n
\end{align*}
\]

if and only if the system

\[F_k(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) = 0, \quad 1 \leq k \leq m\]

is algebraically contradictory, in this case, there must be an integer \(k_0, \ 1 \leq k_0 \leq m\) such that

\[F_{k_0}(x_1^0, x_2^0, \ldots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \ldots, p_n^0) \neq 0\]

or it is differentially contradictory itself, i.e., there is an integer \(j_0, \ 1 \leq j_0 \leq n - 1\) such that

\[
\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.
\]
Corollary 4.3  Let

\[ \begin{align*}
F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) &= 0 \\
F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) &= 0
\end{align*} \]

be an algebraically contradictory system of partial differential equations of first order. Then there are no values \( x^0_i, u_0, p^0_i, \ 1 \leq i \leq n \) such that

\[ \begin{align*}
F_1(x^0_1, x^0_2, \cdots, x^0_{n-1}, x_n^0, u_0, p^0_1, p^0_2, \cdots, p^0_n) &= 0, \\
F_2(x^0_1, x^0_2, \cdots, x^0_{n-1}, x_n^0, u_0, p^0_1, p^0_2, \cdots, p^0_n) &= 0.
\end{align*} \]

Corollary 4.4  A Cauchy problem \((LPDES^C_m)\) of quasilinear partial differential equations with initial values \( u|_{x_n-x^0_n} = u_0 \) is non-solvable if and only if the system \((LPDES_m)\) of partial differential equations is algebraically contradictory.
Denoted by \( \hat{G}[PDES_m^C] \) such a graph \( G[PDES_m^C] \) eradicated all labels. Particularly, replacing each label \( S^{[i]} \) by \( S_0^{[i]} = \{ u_0^{[i]} \} \) and \( S^{[i]} \cap S^{[j]} \) by \( S_0^{[i]} \cap S_0^{[j]} \) for integers \( 1 \leq i, j \leq m \), we get a new labeled topological graph, denoted by \( G_0[PDES_m^C] \). Clearly, \( \hat{G}[PDES_m^C] \simeq \hat{G}_0[PDES_m^C] \).

**Theorem 4.5([11])** For any system \( (PDES_m^C) \) of partial differential equations of first order, \( \hat{G}[PDES_m^C] \) is simple. Conversely, for any simple graph \( G \), there is a system \( (PDES_m^C) \) of partial differential equations of first order such that \( \hat{G}[PDES_m^C] \simeq G \).

**Corollary 4.6** Let \( (LPDES_m) \) be a system of linear partial differential equations of first order with maximal contradictory classes \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s \) on equations in \( (LPDES) \). Then \( \hat{G}[LPDES_m^C] \simeq K(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s) \), i.e., an \( s \)-partite complete graph.
Definition 4.7 Let \((PDES^C_m)\) be the Cauchy problem of a partial differential equation system of first order. Then the labeled topological graph \(G[PDES^C_m]\) is called its topological graph solution, abbreviated to \(G\)-solution.

Combining this definition with that of Theorems 4.5, the following conclusion is hold.

Theorem 4.8([11]) A Cauchy problem on system \((PDES_m)\) of partial differential equations of first order with initial values \(x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}, 1 \leq i \leq n\) for the \(k\)th equation in \((PDES_m)\), \(1 \leq k \leq m\) such that

\[
\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0
\]

is uniquely \(G\)-solvable, i.e., \(G[PDES^C_m]\) is uniquely determined.
§5. Global Stability of Non-Solvable Differential Equations

Definition 5.1 Let $H$ be a spanning subgraph of $G[LD{E}S^1_m]$ of systems $(LD{E}S^1_m)$ with initial value $X_v(0)$. Then $G[LD{E}S^1_m]$ is called sum-stable or asymptotically sum-stable on $H$ if for all solutions $Y_v(t), v \in V(H)$ of the linear differential equations of $(LD{E}S^1_m)$ with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$,

$$\left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| < \varepsilon,$$

or furthermore,

$$\lim_{t \to 0} \left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| = 0.$$
Similarly, a system \((PDESC^C_m)\) is sum-stable if for any number \(\varepsilon > 0\) there exists \(\delta_v > 0, \ v \in V(\hat{G}[0])\) such that each \(G(t)\)-solution with \(u'_0^{[v]} - u_0^{[v]} < \delta_v, \forall v \in V(\hat{G}[0])\) exists for all \(t \geq 0\) and with the inequality

\[
\left| \sum_{v \in V(\hat{G}[t])} u'_v^{[v]} - \sum_{v \in V(\hat{G}[t])} u_v^{[v]} \right| < \varepsilon
\]

holds, denoted by \(G[t] \overset{\Sigma}{\sim} G[0]\). Furthermore, if there exists a number \(\beta_v > 0, \ v \in V(\hat{G}[0])\) such that every \(G'[t]\)-solution with \(u'_0^{[v]} - u_0^{[v]} < \beta_v, \forall v \in V(\hat{G}[0])\) satisfies

\[
\lim_{t \to \infty} \left| \sum_{v \in V(\hat{G}[t])} u'_v^{[v]} - \sum_{v \in V(\hat{G}[t])} u_v^{[v]} \right| = 0,
\]

then the \(G[t]\)-solution is called asymptotically stable, denoted by \(G[t] \overset{\Sigma}{\rightarrow} G[0]\).
Theorem 5.2([10]) A zero $G$-solution of linear homogenous differential equation systems $(LDES_m^1)$ is asymptotically sum-stable on a spanning subgraph $H$ of $G[LDES_m^1]$ if and only if $\text{Re} \alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in B_v$ in $(LDES_1^1)$ hold for $\forall v \in V(H)$.

Example 5.3 Let a $G$-solution of $(LDES_m^1)$ or $(LDE_m^n)$ be the basis graph shown in Fig.4.1, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 = \{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the zero $G$-solution is sum-stable on the triangle $v_4v_5v_6$, but it is not on the triangle $v_1v_2v_3$. In fact, it is prod-stable on the triangle $v_1v_2v_3$.

![Diagram](image)
For partial differential equations, let the system \((PDES_m^C)\) be
\[
\begin{align*}
\frac{\partial u}{\partial t} &= H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\
u|_{t=t_0} &= u_0^i(x_1, x_2, \cdots, x_{n-1})
\end{align*}
\] \(1 \leq i \leq m\)  
\((APDES_m^C)\)

A point \(X_0^i = (t_0, x_{10}^i, \cdots, x_{(n-1)0}^i)\) with \(H_i(t_0, x_{10}^i, \cdots, x_{(n-1)0}^i) = 0\) for \(1 \leq i \leq m\) is called an equilibrium point of the \(i\)th equation in \((APDES_m)\). Then we know that

**Theorem 5.4([11])** Let \(X_0^i\) be an equilibrium point of the \(i\)th equation in \((APDES_m)\) for each integer \(1 \leq i \leq m\). If \(\sum_{i=1}^{m} H_i(X) > 0\) and \(\sum_{i=1}^{m} \frac{\partial H_i}{\partial t} \leq 0\) for \(X \neq \sum_{i=1}^{m} X_0^i\), then the system \((APDES_m)\) is sum-stability, i.e., \(G[t] \sim G[0]\). Furthermore, if \(\sum_{i=1}^{m} \frac{\partial H_i}{\partial t} < 0\) for \(X \neq \sum_{i=1}^{m} X_0^i\), then \(G[t] \rightarrow G[0]\).
§6. Applications

6.1 Application to Geometry

Theorem 6.1([11]) Let the Cauchy problem be \((PDES_m^C)\). Then every connected component of \(\Gamma[PDES_m^C]\) is a differentiable \(n\)-manifold with atlas \(\mathcal{A} = \{(U_v, \phi_v) | v \in V(\tilde{G}[0])\}\) underlying graph \(\tilde{G}[0]\), where \(U_v\) is the \(n\)-dimensional graph \(G[u^{[v]}] \simeq \mathbb{R}^n\) and \(\phi_v\) the projection \(\phi_v : ((x_1, x_2, \ldots, x_n), u(x_1, x_2, \ldots, x_n)) \rightarrow (x_1, x_2, \ldots, x_n)\) for \(\forall (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\).

Theorem 6.2([11]) For any integer \(m \geq 1\), let \(U_i, 1 \leq i \leq m\) be open sets in \(\mathbb{R}^n\) underlying a connected graph defined by

\[
V(G) = \{U_i | 1 \leq i \leq m\}, \quad E(G) = \{(U_i, U_j) | U_i \cap U_j \neq \emptyset, 1 \leq i, j \leq m\}.
\]

If \(X_i\) is a vector field on \(U_i\) for integers \(1 \leq i \leq m\), then there always exists a differentiable manifold \(M \subset \mathbb{R}^n\) with atlas \(\mathcal{A} = \{(U_i, \phi_i) | 1 \leq i \leq m\}\) underlying graph \(G\) and a function \(u_G \in \Omega^0(M)\) such that

\[
X_i(u_G) = 0, \quad 1 \leq i \leq m.
\]
6.2 Global Control of Infectious Diseases

Consider two cases of virus for infectious diseases:

Case 1  There are $m$ known virus $\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_m$ with infected rate $k_i$, heal rate $h_i$ for integers $1 \leq i \leq m$ and an person infected a virus $\mathcal{V}_i$ will never infects other viruses $\mathcal{V}_j$ for $j \neq i$.

Case 2  There are $m$ varying $\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_m$ from a virus $\mathcal{V}$ with infected rate $k_i$, heal rate $h_i$ for integers $1 \leq i \leq m$.

We are easily to establish a non-solvable differential model for the spread of infectious viruses by applying the SIR model of one infectious disease following:

\[
\begin{align*}
\dot{S} &= -k_1 SI \\
\dot{I} &= k_1 SI - h_1 I \\
\dot{R} &= h_1 I \\
\end{align*}
\]

\[
\begin{align*}
\dot{S} &= -k_2 SI \\
\dot{I} &= k_2 SI - h_2 I \\
\dot{R} &= h_2 I \\
\end{align*}
\]

\[
\begin{align*}
\dot{S} &= -k_m SI \\
\dot{I} &= k_m SI - h_m I \\
\dot{R} &= h_m I
\end{align*}
\]  \quad \text{(DES}_{m}^1)
Conclusion 6.3([10]) For $m$ infectious viruses $\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_m$ in an area with infected rate $k_i$, heal rate $h_i$ for integers $1 \leq i \leq m$, then they decline to 0 finally if $0 < S < \frac{\sum_{i=1}^{m} h_i}{\sum_{i=1}^{m} k_i}$, i.e., these infectious viruses are globally controlled. Particularly, they are globally controlled if each of them is controlled in this area.
6.3 Flows in Network

*How can we characterize the behavior of flow $F$?*

![](image)
Denote the rate, density of flow $f_i$ by $\rho^{[i]}$ for integers $1 \leq i \leq m$ and that of $F$ by $\rho^{[F]}$

\[
\frac{\partial \rho^{[i]}}{\partial t} + \phi_i(\rho^{[i]}) \frac{\partial \rho^{[i]}}{\partial x} = 0, \quad 1 \leq i \leq m.
\]

Replacing each $\rho^{[i]}$ by $\rho$, $1 \leq i \leq m$ enables one getting a non-solvable system

\[
\left\{ \begin{array}{l}
\frac{\partial \rho}{\partial t} + \phi_i(\rho) \frac{\partial \rho}{\partial x} = 0 \\
\rho|_{t=t_0} = \rho^{[i]}(x, t_0)
\end{array} \right. \quad 1 \leq i \leq m.
\]

Applying Theorem 5.4, if

\[
\sum_{i=1}^{m} \phi_i(\rho) < 0 \quad \text{and} \quad \sum_{i=1}^{m} \phi(\rho) \left[ \frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho) \left( \frac{\partial \rho}{\partial x} \right)^2 \right] \geq 0
\]

for $X \neq \sum_{k=1}^{m} \rho_0^{[i]}$, then we know that the flow $F$ is stable and furthermore, if

\[
\sum_{i=1}^{m} \phi(\rho) \left[ \frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho) \left( \frac{\partial \rho}{\partial x} \right)^2 \right] < 0
\]

for $X \neq \sum_{i=1}^{m} \rho_0^{[i]}$, then it is also asymptotically stable.
Thanks for your Attention!