The sum and number of primes between any two positive integers

Zhen Liu

Abstract Using the method for equation reconstruction of prime sequence, this paper gives the proof that there is at least one prime between positive integers n^2 and $(n+1)^2$.The sum and number formulae of primes between any two positive integers are also given out.

1 Introduction

Because it can not be divisible except 1 and itself, primes are difficult to be described by appropriate expressions. This property makes prime sequence be difficult to be described such as arithmetic progression, geometric progression with the determined term formula. However, this property can make prime number establish some diophantine equations. And prime numbers can be decided by whether there is positive whole number solutions of these diophantine equations. Therefore, the expressions for solutions of these diophantine equations and its transform are used to describe the divisible property of prime number, and forming an equivalent sequence for the property. Thus, this will be easy to find the key node and the law implied to solve the problem. To this end, the theorem for equation reconstruction of prime sequence is presented and proved. Using the method, this paper gives the proof that there is at least one prime between positive integers n^2 and $(n+1)^2$.The sum and number formulae of primes between two positive integers are also given out. It could be hope to provide an idea and methods to solve similar problems.

Zhen Liu

Doctor, Associate Professor of School of Marine Sciences, Sun Yat-sen University, China

Adds: Sun Yat-sen University, No.135, Xingang Xi Rd, Haizhu District, Guangzhou, Guangdong, 510275, China

E-mail: liuzh8@mail.sysu.edu.cn

In this paper, all parameters are positive whole number except where stated.

2 Proof of the theorem for equation reconstruction of prime sequence

Lemma: The prime sequence could be equivalent to the sequence with the determined general term formula through equation reconstruction of prime number for the divisible property.

Proof.

Any prime number c could be expressed as $3a \pm 1$ (a is an positive even), $4a \pm 1$ or $6a \pm 1$.

Proof is carried out the following in the case of $3a \pm 1$ first.

If $3a \pm 1$ is a prime number, it certainly can not be written $3a \pm 1 = (3x_1 \pm 1)(3x_2 \pm 1)$,

otherwise, and vice versa.

Case 1: $3a + 1$

$$3a + 1 = (3x_1 + 1)(3x_2 + 1) = 9x_1x_2 + 3(x_1 + x_2) + 1$$

or

$$3a + 1 = (3x_1' - 1)(3x_2' - 1) = 9x_1'x_2' - 3(x_1' + x_2') + 1$$

Where, let $-x_1' = x_1$, $-x_2' = x_2$.

Then there is $a = 3x_1x_2 + (x_1 + x_2)$.

It is easy to see that whether $3a + 1$ is a prime number depends entirely on the a.

Namely $3a + 1$ is a prime number that is equivalent x_1 and x_2 are both positive whole number in $a = 3x_1x_2 + (x_1 + x_2)$.

Let $x_1 + x_2 = -q$, $x_1x_2 = p$

According to Vieta's formulas, equation (1) is established.

$$x^2 + qx + p = 0 \tag{1}$$

Then there is $x_{1,2} = \dfrac{-q \pm \sqrt{q^2 - 4p}}{2}$

Therefore, if x_1 and x_2 of equation (1) roots are not both positive whole number, $3a + 1$

must be a prime number. Otherwise, it will be a composite number.

There is

$$a = 3p - q$$

Obviously, if $3a+1$ is a prime number, q and $\sqrt{q^2 - 4p}$ are not both even numbers.

Therefore, in the divisible property of prime number, c_i in prime sequence $\{c_n\}$ is equivalent to $a_i = 3p_i - q_i$ in sequence $\{a_n\}$, namely prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n\}$.

Here q_i and $\sqrt{q_i^2 - 4p_i}$ are not both even numbers.

In order to facilitate the expression, let $q = 2s$, $p = 2r$.

Here s and r are real numbers.

$\therefore x_{1,2} = s \pm \sqrt{s^2 - 2r}$

Let $\sqrt{s^2 - 2r} = t$

There is

$$a = 12st - 12t^2 - 2s$$

Therefore, $a_i = 3p_i - q_i$ in sequence $\{a_n\}$ (q_i and $\sqrt{q_i^2 - 4p_i}$ are not both even numbers) is equivalent to $a_i' = 12s_i t_i - 12t_i^2 - 2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'\}$.

It is obvious that

$$t = \frac{s \pm \sqrt{s^2 + \frac{2s-a}{3}}}{2}$$

Let $s^2 + \frac{2s-a}{3} = e^2$

Then there is

$$a = 3s^2 + 2s - 3e^2$$

Therefore, $a'_i = 12s_i t_i - 12t_i^2 - 2s_i$ in sequence $\{a'_n\}$ (s_i and t_i are not both positive whole number solutions) is equivalent to $a''_i = 3s_i^2 + 2s_i - 3e_i^2$ in sequence $\{a''_n\}$ (s_i and e_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a''_n\}$.

It is obvious that

$$s = \frac{-1 \pm \sqrt{9e^2 + 3a + 1}}{3}$$

Let $9e^2 + 3a + 1 = (3h+1)^2$

Then there is

$$3a + 1 = (3h+1)^2 - (3e)^2$$

$$a = 3h^2 + 2h - 3e^2$$

Therefore, $a''_i = 3s_i^2 + 2s_i - 3e_i^2$ in sequence $\{a''_n\}$ (s_i and e_i are not both positive whole number solutions) is equivalent to $3a'''_n + 1 = (3h_i + 1)^2 - (3e_i)^2$ in sequence $\{a''_n\}$ (e_i and h_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a'''_n\}$.

Case 2: $3a - 1$

$$3a - 1 = (3x'_1 + 1)(3x'_2 - 1) = 9x'_1 x'_2 + 3(x'_2 - x'_1) - 1$$

Where, let $-x'_1 = x_1$, $x'_2 = x_2$.

Then there is $a = -3x_1 x_2 + (x_1 + x_2)$.

Namely $3a - 1$ is a prime number that is equivalent x_1 and x_2 are both positive whole number in $a = -3x_1 x_2 + (x_1 + x_2)$.

Let $x_1 + x_2 = -q$, $x_1 x_2 = p$. Here p is negative whole number.

According to Vieta's formulas, equation (2) is established.

$$x^2 + qx + p = 0 \tag{2}$$

Then there is $x_{1,2} = \dfrac{-q \pm \sqrt{q^2 - 4p}}{2}$.

Therefore, if x_1 and x_2 of equation (2) roots are not both positive whole number, $3a-1$ must be a prime number. Otherwise, it will be a composite number.

There is

$$a = -3p - q$$

Obviously, if $3a-1$ is a prime number, q and $\sqrt{q^2 - 4p}$ are not both even numbers.

Therefore, in the divisible property of prime number, c_i in prime sequence $\{c_n\}$ is equivalent to $a_i = -3p_i - q_i$ in sequence $\{a_n\}$, namely prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n\}$.

Using the same argument as in the case 1, we can easily get

$$a = 12t^2 - 12st - 2s$$

Therefore, $a_i = -3p_i - q_i$ in sequence $\{a_n\}$ (q_i and $\sqrt{q_i^2 - 4p_i}$ are not both even numbers) is equivalent to $a_i' = 12t_i^2 - 12s_i t_i - 2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'\}$.

It is obvious that

$$t = \frac{s \pm \sqrt{s^2 + \dfrac{2s+a}{3}}}{2}$$

Let $s^2 + \dfrac{2s+a}{3} = e^2$

Then there is

$$a = 3e^2 - 3s^2 - 2s$$

Therefore, $a_i' = 12t_i^2 - 12s_i t_i - 2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) is equivalent to $a_i'' = 3e_i^2 - 3s_i^2 - 2s$ in sequence $\{a_n''\}$ (s_i and e_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n''\}$.

It is obvious that

$$s = \frac{-1 \pm \sqrt{9e^2 - 3a + 1}}{3}$$

Let $9e^2 - 3a + 1 = (3h + 1)^2$

Then there is

$$3a - 1 = (3e)^2 - (3h + 1)^2$$

$$a = 3e^2 - 3h^2 - 2h$$

Therefore, $a_i'' = 3e_i^2 - 3s_i^2 - 2s$ in sequence $\{a_n''\}$ (s_i and e_i are not both positive whole number solutions) is equivalent to $3a_n''' - 1 = (3e_i)^2 - (3h_i + 1)^2$ in sequence $\{a_n''\}$ (e_i and h_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'''\}$.

The prime sequence that prime number c could be expressed as $4a \pm 1$ or $6a \pm 1$, have equivalent methods that are similar to the case of $3a \pm 1$. It can be proved in the same way as shown in the case of $3a \pm 1$ before. Of course, some new equivalent sequences are reconstructed through establishing other forms equations.

This completes the proof.

According to above proof, in the divisible property of prime number, the prime sequence $\{c_n\}$ without term formula is analyzed by using the sequence $\{a_n\}$、$\{a_n'\}$、$\{a_n''\}$、$\{a_n'''\}$ with term formula. This will be easy to find the key node and the law implied to solve the problem.

3 Proof of existing at least one prime between positive integers n^2 and $(n+1)^2$

Theorem1: There is at least one prime between positive integers n^2 and $(n+1)^2$, where n is any positive integer.

Proof.

It proves the Theorem with the reduction to absurdity follows.

If the Theorem is not true, it becomes: there is a positive integer n_0 that makes all integers between n_0^2 and $(n_0+1)^2$ be composite numbers.

According to the Lemma for equation reconstruction of prime sequence, there are

$$3a-1=(3x_1+1)(3x_2-1) \tag{3}$$

$$3a+1=(3x_1'+1)(3x_2'+1) \text{ or } 3a+1=(3x_1'-1)(3x_2'-1) \tag{4}$$

Where, $a=2l$, l is positive whole number.

Therefore, when a is large enough, at least one of equation (9) and equation (10) has integer solutions.

Using the same argument as in the proof of equation reconstruction of prime sequence, we can easily get this statement fellows.

For equation (3), there are

$$\frac{a}{2}=6t_1^2-6s_1t_1-s_1$$

$$t_1=\frac{s_1\pm\sqrt{s_1^2+\dfrac{2s_1+a}{3}}}{2}$$

Let $s_1^2+\dfrac{2s_1+a}{3}=e_1^2$

Then, there is

$$s_1=\frac{-1\pm\sqrt{9e_1^2-6l+1}}{3}$$

Let $9e_1^2-6l+1=(3h_1+1)^2$, Namely it makes s_1 be a positive whole number.

Then, there is

$$6l-1=9e_1^2-(3h_1+1)^2$$

$$a=2l=3e_1^2-3h_1^2-2h_1$$

$$3a-1=9e_1^2-(3h_1+1)^2 \qquad (5)$$

For equation (4), there are

$$\frac{a}{2}=6s_2t_2-6t_2^2-s_2$$

$$t_2=\frac{s_2\pm\sqrt{s_2^2+\frac{2s_2-a}{3}}}{2}$$

Let $s_2^2+\frac{2s_2-a}{3}=e_2^2$

Then, there is

$$s_2=\frac{-1\pm\sqrt{9e_2^2+6l+1}}{3}$$

Let $9e_2^2+6l+1=(3h_2+1)^2$, Namely it makes s_2 be a positive whole number.

Then, there is

$$6l+1=(3h_2+1)^2-9e_2^2$$

$$a=2l=3h_2^2+2h_2-3e_2^2$$

$$3a+1=(3h_2+1)^2-9e_2^2 \qquad (6)$$

According to the equation (5) and equation (6), there is

$$3a\pm1=\left|(3h+1)^2-9e^2\right|$$

Let $3a\pm1=n_0^2+m$, $0<m<2n_0+1$.

Then, if the Theorem is not true, it becomes: there is a positive integer n_0 that makes n_0^2+m be composite numbers. Namely there is a positive integer n_0 that makes $n_0^2+m=\left|(3h+1)^2-3e^2\right|$.

For n_0, there are 3 cases: $n_0=3k_0-1$, $n_0=3k_0$ and $n_0=3k_0+1$.

Let when k_0 is a even, $k_0 = 2d_0$. Let when k is odd, $k_0 = 2d_0 - 1$.

Then,

Case $n_0 = 3k_0 - 1$:

when $k_0 = 2d_0$, $n_0^2 = 36d_0^2 - 12d_0 + 1$,

when $k_0 = 2d_0 - 1$, $n_0^2 = 36d_0^2 - 48d_0 + 16$.

Csae $n_0 = 3k_0$:

when $k_0 = 2d_0$, $n_0^2 = 36d_0^2$,

when $k_0 = 2d_0 - 1$, $n_0^2 = 36d_0^2 - 36d_0 + 9$.

Case $n_0 = 3k_0 + 1$:

when $k_0 = 2d_0$, $n_0^2 = 36d_0^2 + 12d_0 + 1$,

when $k_0 = 2d_0 - 1$, $n_0^2 = 36d_0^2 - 24d_0 + 4$.

∵ Any prime could be expressed as $6b \pm 1$.

∴ Just consider only a situation that $n_0^2 + m$ could be expressed as $6b \pm 1$, while the rest are composite numbers.

Because of $n_0^2 + m$ corresponding to be expressed as $6b + 1$ and $6b - 1$, there are 6 cases as follows:

when $n_0^2 = 36d_0^2 - 12d_0 + 1$, $m = 6c$ or $m = 6c + 4$,

when $n_0^2 = 36d_0^2 - 48d_0 + 16$, $m = 6c + 1$ or $m = 6c - 1$,

when $n_0^2 = 36d_0^2$, $m = 6c + 1$ or $m = 6c - 1$,

when $n_0^2 = 36d_0^2 - 36d_0 + 9$, $m = 6c + 4$ or $m = 6c + 2$,

when $n_0^2 = 36d_0^2 + 12d_0 + 1$, $m = 6c + 4$ or $m = 6c$,

when $n_0^2 = 36d_0^2 - 24d_0 + 4$, $m = 6c + 1$ or $m = 6c + 3$.

When $n_0^2 + m = (3h + 1)^2 - (3e)^2$, let

$$n_0^2 + m_0 = 3a_0 + 1 = (3h_0 + 1)^2 - (3e_0)^2 = A$$

Where, m_0 is the minimum of m.

Then the equation $n_0^2 + m_0 + m = [3(h_0 + \Delta_h) + 1]^2 - [3(e_0 + \Delta_e)]^2$ has integer solutions for any m.

There is

$$\Delta_h = \frac{1}{3}\left[-3h_0 - 1 \pm \sqrt{(3h_0 + 1)^2 + (9\Delta_e^2 + 18e_0\Delta_e + m)}\right]$$

$\therefore (3h_0 + 1)^2 + (9\Delta_e^2 + 18e_0\Delta_e + m)$ is a square number.

Then let

$$(3h_0 + 1)^2 + (9\Delta_e^2 + 18e_0\Delta_e + m) = f^2$$

$$(3h_0 + 1)^2 = B$$

There is

$$\Delta_e = \frac{1}{3}\left[-3e_0 \pm \sqrt{9e_0^2 - m - B + f^2}\right]$$

$\therefore 9e_0^2 - m - B + f^2$ is also a square number.

Then, there is $9e_0^2 - m - B + f^2 = \delta_{c_0+c}^2$ for arbitrary continuous m.

And there is

$$\left|\delta_{c_0+c}^2 - \delta_{c_0}^2\right| = \Delta m = 6c$$

It is easy to see that $6c$ are not all the difference between square numbers. This is also in contradiction with the difference between square numbers $\delta_{m_0+m}^2 - \delta_{m_0}^2$.

When $n_0^2 + m = (3e)^2 - (3h + 1)^2$, let

$$n_0^2 + m_0 = 3a_0 - 1 = (3e_0)^2 - (3h_0 + 1)^2 = A$$

Where, m_0 is the minimum of m.

Then the equation $n_0^2 + m_0 + m = [3(e_0 + \Delta_e)]^2 - [3(h_0 + \Delta_h) + 1]^2$ has integer solutions for any m.

There is

$$\Delta_h = \frac{1}{3}\left[-3h_0 - 1 \pm \sqrt{(3h_0+1)^2 + \left(9\Delta_e^2 + 18e_0\Delta_e - m\right)}\right]$$

$\therefore (3h_0+1)^2 + \left(9\Delta_e^2 + 18e_0\Delta_e - m\right)$ is a square number.

Then let

$$(3h_0+1)^2 + \left(9\Delta_e^2 + 18e_0\Delta_e - m\right) = f^2$$

$$(3h_0+1)^2 = B$$

There is

$$\Delta_e = \frac{1}{3}\left[-3e_0 \pm \sqrt{9e_0^2 + m - B + f^2}\right]$$

$\therefore 9e_0^2 + m - B + f^2$ is also a square number.

Then, there is $9e_0^2 + m - B + f^2 = \delta_{c_0+c}^2$ for arbitrary continuous m.

And there is

$$\left|\delta_{c_0+c}^2 - \delta_{c_0}^2\right| = m = 6c$$

It is easy to see that $6c$ are not all the difference between square numbers. This is also in contradiction with the difference between square numbers $\delta_{m_0+m}^2 - \delta_{m_0}^2$.

It is now obvious that the theorem holds.

There is at least one prime between positive integers n^2 and $(n+1)^2$.

This completes the proof.

4. Proof of the sum and number formulae of primes between two positive integers

Theorem2: The sum formula $\sigma(\alpha,\beta)$ and number formula $\pi(\alpha,\beta)$ of primes between positive integers α and β, where $\beta > \alpha$.

Proof.

According to the Lemma for equation reconstruction of prime sequence, any prime could be expressed as $3a \pm 1$, where $\alpha < p < \beta$. And

$$3a \pm 1 = \left|(3h+1)^2 - (3e)^2\right|$$

Then

$$\frac{\alpha \mp 1}{3} < a < \frac{\beta \mp 1}{3}$$

$\therefore$ The equation $3a \pm 1 = \left|(3h+1)^2 - (3e)^2\right|$ does not have integer solutions for all primes $3a \pm 1$ between α and β, while the equation $3a \pm 1 = \left|(3h+1)^2 - (3e)^2\right|$ has integer solutions for composite numbers.

$\therefore$ The sum formula $\sigma(\alpha, \beta)$ of primes between positive integers α and β is

$$\sigma(\alpha,\beta) = \sum_{a_1}^{a_n}(3a_i \pm 1) - \sum_{h_1}^{h_n}\sum_{e_1}^{e_n}\left|(3h_i+1)^2 - (3e_i)^2\right|$$

Where, $a_1 = \left[\frac{\alpha \mp 1}{3}\right] + 1$, $a_n = \left[\frac{\beta \mp 1}{3}\right]$, $3a_1 \pm 1 = \left|(3h_1+1)^2 - (3e_1)^2\right|$,

$3a_n \pm 1 = \left|(3h_n+1)^2 - (3e_n)^2\right|$.

And $\left|(3h_i+1)^2 - (3e_i)^2\right| \neq \left|(3h_j+1)^2 - (3e_j)^2\right|$. if $\left|(3h_i+1)^2 - (3e_i)^2\right| = \left|(3h_j+1)^2 - (3e_j)^2\right|$, it could only take one of (h_i, e_i) and (h_j, e_j) in the calculations.

And there are $3a+1 = (3h_2+1)^2 - 9e_2^2$, $3a-1 = 9e_1^2 - (3h_1+1)^2$.

$\therefore$ The number formula $\pi(\alpha, \beta)$ of primes between positive integers α and β is

$$\pi(\alpha,\beta) = \sum_{a_1}^{a_n}(3a_i+1) - \sum_{h_1}^{h_n}\sum_{e_1}^{e_n}\left(9h_i^2 + 6h_i - 9e_i^2\right) + \left|\sum_{a_1}^{a_n}(3a_i-1) - \sum_{h_1}^{h_n}\sum_{e_1}^{e_n}\left(9e_i^2 - 9h_i^2 - 6h_i\right)\right|$$

It could also be written as

$$\pi(\alpha,\beta) = \sum_{a_1}^{a_n}(3a_i \pm 1) - \sum_{h_1}^{h_n}\sum_{e_1}^{e_n}\left|9h_i^2 + 6h_i - 9e_i^2\right| + 2n$$

5. Proof of the upper and lower limit formula of number of primes between two positive integers

Theorem3: The upper and lower limit formula of number $\pi(\alpha, \beta)$ for primes between positive integers α and β, where $\beta > \alpha$.

Proof.

Using the same method as in the proof of Theorem1, for α, there are 3 cases: $\alpha = 3\varphi - 1$,

$\alpha = 3\varphi$ and $\alpha = 3\varphi + 1$.

Let when φ is a even, $\varphi = 2\gamma$. Let when φ is odd, $\varphi = 2\gamma - 1$.

Then,

Case $\alpha = 3\varphi - 1$:

when $\varphi = 2\gamma$, $\alpha = 6\gamma - 1$,

when $\varphi = 2\gamma - 1$, $\alpha = 6\gamma - 4$.

Case $\alpha = 3\varphi$:

when $\varphi = 2\gamma$, $\alpha = 6\gamma$,

when $\varphi = 2\gamma - 1$, $\alpha = 6\gamma - 3$.

Case $\alpha = 3\varphi + 1$:

when $\varphi = 2\gamma$, $\alpha = 6\gamma + 1$,

when $\varphi = 2\gamma - 1$, $\alpha = 6\gamma - 2$.

∵ Any prime could be expressed as $6b \pm 1$.

∴ Just consider only a situation that $\alpha + \Delta$ could be expressed as $6b \pm 1$, while the rest are composite numbers. Where $\alpha + \Delta < \beta$.

Because of $\alpha + \Delta$ corresponding to be expressed as $6b + 1$ and $6b - 1$, there are 6 cases as follows:

when $\alpha = 6\gamma - 1$, $\Delta = 6\delta + 2$ or $\Delta = 6\delta$,

when $\alpha = 6\gamma - 4$, $\Delta = 6\delta + 5$ or $\Delta = 6\delta + 3$,

when $\alpha = 6\gamma$, $\Delta = 6\delta + 1$ or $\Delta = 6\delta + 5$,

when $\alpha = 6\gamma - 3$, $\Delta = 6\delta + 4$ or $\Delta = 6\delta + 2$,

when $\alpha = 6\gamma + 1$, $\Delta = 6\delta$ or $\Delta = 6\delta + 4$,

when $\alpha = 6\gamma - 2$, $\Delta = 6\delta + 3$ or $\Delta = 6\delta + 1$.

Let $\Delta = 6\delta + \chi$, then it could be marked

when $\alpha = 6\gamma - 1$, $(\chi_{1,} \chi_2) = (2, 0)$,

when $\alpha = 6\gamma - 4$, $(\chi_{1,} \chi_2) = (5, 3)$,

when $\alpha = 6\gamma$, $(\chi_{1,} \chi_2) = (1, 5)$,

when $\alpha = 6\gamma - 3$, $(\chi_{1,} \chi_2) = (4, 2)$,

when $\alpha = 6\gamma + 1$, $(\chi_{1,} \chi_2) = (0, 4)$,

when $\alpha = 6\gamma - 2$, $(\chi_{1,} \chi_2) = (3, 1)$.

Using the same method as in the proof of Theorem1, there are

When $\alpha + \Delta = (3h+1)^2 - (3e)^2$, let

$$\alpha + \Delta_0 = 3a_0 + 1 = (3h_0 + 1)^2 - (3e_0)^2 = A$$

Where, Δ_0 is the minimum of Δ.

If $\alpha + \Delta$ is a composite number, the equation $\alpha + \Delta_0 + \Delta = [3(h_0 + \Delta_h) + 1]^2 - [3(e_0 + \Delta_e)]^2$ has integer solutions.

There is

$$\Delta_h = \frac{1}{3}\left[-3h_0 - 1 \pm \sqrt{(3h_0 + 1)^2 + (9\Delta_e^2 + 18e_0\Delta_e + \Delta)}\right]$$

$\therefore (3h_0 + 1)^2 + (9\Delta_e^2 + 18e_0\Delta_e + \Delta)$ is a square number.

Then let

$$(3h_0 + 1)^2 + (9\Delta_e^2 + 18e_0\Delta_e + \Delta) = f^2$$

$$(3h_0 + 1)^2 = B$$

There is

$$\Delta_e = \frac{1}{3}\left[-3e_0 \pm \sqrt{9e_0^2 - \Delta - B + f^2}\right]$$

$\therefore 9e_0^2 - \Delta - B + f^2$ is also a square number.

And $\Delta_{\max} \le \beta - \alpha$

$\therefore$ The number of Δ that make $\alpha+\Delta$ be expressed as $6b+1$ is $\left[\frac{\beta-\alpha+\chi_1-1}{6}\right]$.

The number of Δ that make $9e_0^2-\Delta-B+f^2$ be square number and be divided evenly by 6 is $\left[\sqrt{\frac{\beta-\alpha+\chi_1-1}{6}}\right]$.

The number of Δ that make Δ_e be positive integers is $\left[\frac{1}{3}\sqrt{\frac{\beta-\alpha+\chi_1-1}{6}}\right]$.

But this Δ could not make Δ_h be positive integers.

$\therefore$ The number of primes be expressed as $6b+1$ between α and β is

$$\left[\frac{\beta-\alpha+\chi_1-1}{6}\right]-\left[\frac{1}{3}\sqrt{\frac{\beta-\alpha+\chi_1-1}{6}}\right]-1\le\pi(\alpha,\beta)\le\left[\frac{\beta-\alpha+\chi_1-1}{6}\right]$$

When $\alpha+\Delta=(3e)^2-(3h+1)^2$, let

$$\alpha+\Delta_0+\Delta=3a_0-1=(3e_0)^2-(3h_0+1)^2=A$$

Where, Δ_0 is the minimum of Δ.

If $\alpha+\Delta$ is a composite number, the equation $\alpha+\Delta_0+\Delta=[3(e_0+\Delta_e)]^2-[3(h_0+\Delta_h)+1]^2$ has integer solutions.

There is

$$\Delta_h=\frac{1}{3}\left[-3h_0-1\pm\sqrt{(3h_0+1)^2+\left(9\Delta_e^2+18e_0\Delta_e-\Delta\right)}\right]$$

$\therefore (3h_0+1)^2+\left(9\Delta_e^2+18e_0\Delta_e-\Delta\right)$ is a square number.

Then let

$$(3h_0+1)^2+\left(9\Delta_e^2+18e_0\Delta_e-\Delta\right)=f^2$$

$$(3h_0+1)^2=B$$

There is

$$\Delta_e=\frac{1}{3}\left[-3e_0\pm\sqrt{9e_0^2+\Delta-B+f^2}\right]$$

$\therefore 9e_0^2+\Delta-B+f^2$ is also a square number.

And $\Delta_{\max} \le \beta - \alpha$

$\therefore$ The number of Δ that make $\alpha + \Delta$ be expressed as $6b - 1$ is $\left[\frac{\beta - \alpha + \chi_2 + 1}{6}\right]$.

The number of Δ that make $9e_0^2 + \Delta - B + f^2$ be square number and be divided evenly by 6 is $\left[\sqrt{\frac{\beta - \alpha + \chi_2 + 1}{6}}\right]$.

The number of Δ that make Δ_e be positive integers is $\left[\frac{1}{3}\sqrt{\frac{\beta - \alpha + \chi_2 + 1}{6}}\right]$.

But this Δ could not make Δ_h be positive integers.

$\therefore$ The number of primes be expressed as $6b + 1$ between α and β is

$$\left[\frac{\beta - \alpha + \chi_2 + 1}{6}\right] - \left[\frac{1}{3}\sqrt{\frac{\beta - \alpha + \chi_2 + 1}{6}}\right] - 1 \le \pi(\alpha, \beta) \le \left[\frac{\beta - \alpha + \chi_2 + 1}{6}\right]$$

$\therefore$The upper and lower limit of number $\pi(\alpha, \beta)$ for primes between positive integers α and β is

$$\pi(\alpha, \beta) \le \left[\frac{\beta - \alpha + \chi_1 - 1}{6}\right] + \left[\frac{\beta - \alpha + \chi_2 + 1}{6}\right]$$

$$\pi(\alpha, \beta) \ge \left[\frac{\beta - \alpha + \chi_1 - 1}{6}\right] + \left[\frac{\beta - \alpha + \chi_2 + 1}{6}\right] - \left[\frac{1}{3}\sqrt{\frac{\beta - \alpha + \chi_1 - 1}{6}}\right] - \left[\frac{1}{3}\sqrt{\frac{\beta - \alpha + \chi_2 + 1}{6}}\right] - 2$$

This completes the proof.

References

1. Vinogradov Ivan Matveevich: Elements of Number Theory. State izdatelstvo technical and theoretical literature (1952). (Qiu Guangming Translate, Harbin Institute of Technology Press (2011))
2. Pan Chengdong, Pan Chengbiao: Elementary proof of the prime number theorem. Shanghai Science and Technology Press (1988).
3. Wang Yuan. Dictionary of Mathematics. Science Press of China (2010).