Unification of Mass and Gravitation in a Common 4D Higgs Compatible Theory

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Since the 1920s, the formulas of mass and gravitation have worked perfectly with high accuracy. However, the basic principle of these two phenomena remains unknown. What is the origin of mass? How can spacetime be curved by mass? What is the mechanism by which spacetime creates gravitation? ... The solution to these enigmas lies in general relativity. A thorough examination of the Einstein Field Equations highlights some minor inconsistencies concerning the sign and the meaning of tensors. Solving these inconsistencies fully explains the curvature of spacetime, mass, and gravitation, without modifying the mathematics of general relativity. Moreover, this explanation shows that mass and gravitation are two similar phenomena that can be unified in a common 4D Higgs compatible theory. Applications to astrophysics are also very important: black holes, dark matter, dark energy...

1. Introduction

When Einstein devised General Relativity (GR), knowledge in physics was very poor. For example, in the 1910s, the atom was regarded as a pudding in which raisins represented electrons. The proton and neutron were found later, in 1919 and 1932.

Thus, in the 1910s, GR was grounded on a partially erroneous view of reality. The result is that the mathematics of GR works perfectly, but the enigmas remain. For example, assuming that "a mass curves spacetime" does not explain the mechanism of the curvature of spacetime by mass. This problem is common in physics. For example, Newton found the formulation of gravitation without explaining the mechanism.

Since the 1910s, physics has evolved considerably. Today, we must reconsider GR in the current context, excluding the "pudding model" but including quarks and the Higgs mechanism. This leads to a new and modern view of GR [1] that explains the curvature of spacetime, mass, and gravitation, without modifying the mathematics of GR.

This article begins with a short summary of GR, followed by a list of inconsistencies [3]–[10] of the Einstein Field Equations (EFE). Based on a new definition of "Volume", this paper shows that the solution to these inconsistencies is right before our eyes, in the EFE. This modern view of GR also solves several enigmas of physics, such as:

– *What is the mechanism of the increase of the mass of relativistic particles?*
– *What is the origin and principle of the attractive force of gravitation?*
– *What is the mechanism by which mass can be converted into energy in* $E = mc^2$?

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2. Origins of General Relativity

2.1 Preliminary note

EFE can be studied in different ways. This article is grounded on the original 1910-1915 Einstein-Grossmann's version of GR [3][4][8][9][17][18].

2.2 Introduction

In the 1910s, Einstein studied gravity. Following the reasoning of Faraday and Maxwell, he thought that if two objects are attracted to each other, there would be some medium. The only medium he knew in 1910 was spacetime. He then deduced that the gravitational force is an indirect effect carried by spacetime. He concluded that any mass perturbs spacetime, and that the spacetime, in turn, has an effect on mass, which is gravitation. In other words, Einstein assumed that the carrier of gravitation is the *curvature of spacetime*. Thus, he tried to find an equation [11][12] connecting:

1. The curvature of spacetime. This mathematical object, called the "Einstein tensor", is the left hand side of the EFE (Eq. 1).

2. The properties of the object that curves spacetime. This quantity, called the "Energy–Momentum tensor", is $T_{\mu\nu}$ in the right hand side of the EFE.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad (1)$$

2.3 The Einstein Tensor

The left hand side of the EFE expresses the curvature of spacetime. Einstein started with a curved Gaussian space. Without knowing the mechanism for the curvature of spacetime, he posed the question of how to have a four-dimensional Gaussian space. This leads to a more general concept, Riemannian space. For a 4D space, using the Christoffel symbols, the curvature of spacetime is given by the Riemann–Christoffel tensor, which becomes the Ricci tensor after reduction. From this, Einstein created a new tensor, the "Einstein tensor", which combines the Ricci tensor $R_{\mu\nu}$, the metric of spacetime $g_{\mu\nu}$, and the scalar curvature R [13][14].

2.4 Fluid Mechanics

In fluid mechanics, the medium has effects on objects. For example, air, i.e., the medium, imposes a pressure on airplanes, producing perturbations around them. So, Einstein thought that spacetime could be assimilated to a Newtonian fluid [15].

2.5 The Energy–Momentum Tensor

In fluid mechanics, the motion of an elementary volume $dxdydz$ is described by the Navier–Stokes equations. This leads to a more general formulation, the Cauchy–stress tensor. In order to get a relativistic tensor, Einstein included the 4D "four-vectors" of special relativity in fluid mechanics. Thus, the original 3D Cauchy–stress tensor became the 4D energy–momentum tensor (Fig. 1, next page).

2.6 The Einstein Field Equations (EFE)

Finally, Einstein added the coefficient $8\pi G/c^4$ to homogenize the two sides of the EFE. This coefficient is calculated to get back Newton's Law from the EFE in the case of a static sphere in a weak field [16].

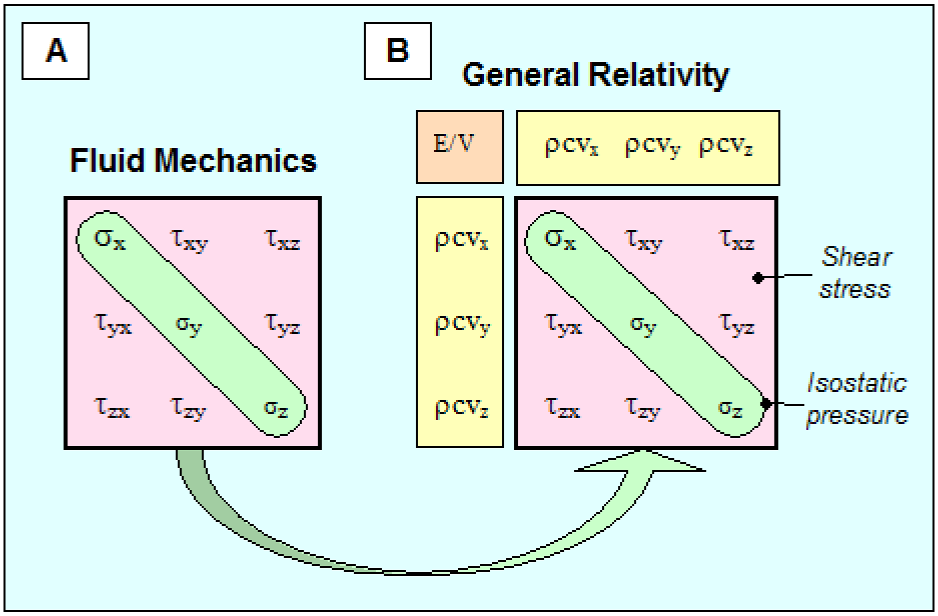

Figure 1. The origin of the energy–momentum tensor of GR, in the original 1910s fluid mechanics version, is the Cauchy–stress tensor. Since the stress tensor is included in the energy–momentum tensor, the two tensors must have the same meaning, at least on the spatial part (in pink in the figure).

3. Inconsistencies of the EFE

To date, the situation is clear:

– The mathematics of GR, i.e., the EFE, work perfectly . . .
– . . . but enigmas remain, such as "What is the origin of gravity?"

In such a situation, the only thing to do is to rebuild GR from scratch, taking into account the evolution of physics since the 1910s. Here is a list of inconsistencies found in the original construction of the EFE (fluid mechanics version) by Einstein and Grossmann. Appendix E covers a full mathematical description of the EFE.

Inconsistency #1

All elements of the Cauchy stress tensor (Fig. 1A) have a pressure-like dimension. Indeed, the fluid exerts a *pressure* on objects (Fig. 2A, next page). Since Einstein built the energy–momentum tensor from the Cauchy stress tensor, each term must keep its original meaning. This is not the case. Question: *Why has the original pressure force been replaced by an attractive force (gravitation)?*

This inconsistency is also highlighted by the split method used in fluid mechanics to calculate the Hooke tensor. This thought experiment splits the object into two parts to determine the pressure forces on each point (Fig. 2A). If we replace the two halves by two spheres (Fig. 2B), the pressure forces still exist.

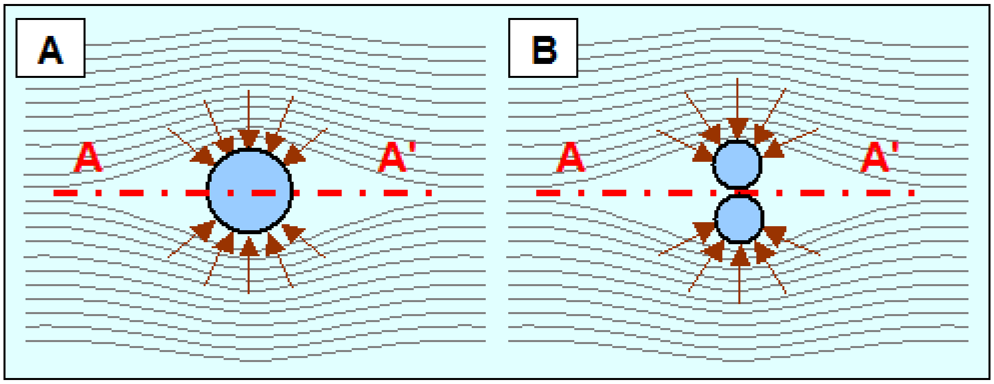

Figure 2. Replacing the two halves (A) by two spheres (B) in the split method explains gravitation, which is not an attractive force, but a pressure force exerted by the curvature of spacetime on the surface of objects.

Therefore, the split method can also be used to understand gravitation, which is not an attractive force as we think but a pressure force exerted by the curvature of spacetime on objects that tends to bring them closer to each other (Fig. 2B).

Inconsistency #2

In fluid mechanics, the curvature of the fluid made by an object is *convex* (Fig. 3A). The scientific literature shows that the curvature of spacetime produced by a mass such as a black hole is always *concave* (Fig. 3B). There is a debate about this subject, but since the stress tensor is at the origin of the energy–momentum tensor, it is obvious that the curvature of spacetime must be *convex*, not *concave*.

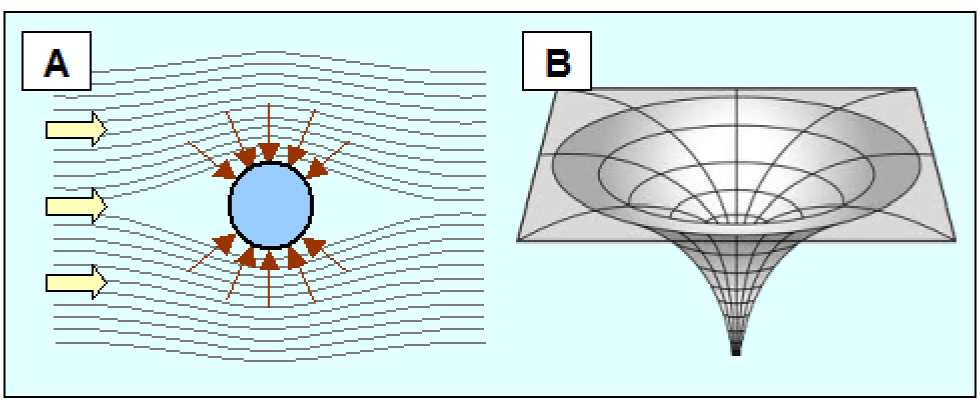

Figure 3. Since the stress tensor and the energy–momentum tensor are identical, there is no reason to replace the convex curvature of the stress tensor by the concave curvature of GR.

Inconsistency #3

In fluid mechanics, the curvature of the fluid is made by the *volume* (Fig. 3A). In GR, it is made by the *mass*. Here too, we have no reason to modify the meaning of the stress tensor, replacing *volume* by *mass*. However, experiments show that the curvature of spacetime is really produced by mass. Sections 4–6 solve this inconsistency.

Inconsistency #4

To date, no one has proposed a formulation of mass in 4D $m = f_{(x,y,z,t)}$. The variable m, which is embedded in energy and momentum, is therefore a new dimension. There is no other alternative since m is not a dimensionless constant. Thus, we are faced with a lack of homogeneity since the left hand side of the EFE is devised in 4D $f_{(x,y,z,t)}$, whereas the right hand side is in 5D $f_{(x,y,z,t,m)}$. The only way to solve this problem is to find a 4D expression for m: $m = f_{(x,y,z,t)}$. Section 6 covers this inconsistency.

Recapitulation

Since the energy–momentum tensor is built from the stress tensor, it is obvious that these two tensors must have the same foundation and principle of construction. This is not the case. A thorough examination of the EFE shows that:

#1: Spacetime must exert a *pressure force* on objects, not an *attractive force*,
#2: The curvature of spacetime must be *convex*, not *concave*,
#3: Spacetime is curved by *volume*, not by *mass* (see sections 4–6).
#4: The EFE have a lack of homogeneity: $[L^3][T] = [L^3][T][M]$.

4. Modern Definition of Volume

One tends to assume that "a volume is a volume". This assumption must be dropped, because some volumes curve spacetime, but others do not. Since GR is grounded on the curvature of spacetime, the first thing to do is to separate the volumes that curve spacetime from the volumes that do not.

Contrary to what we could think, this study about volumes is very important because it leads to the expression of mass: $M = f_{(volume)}$. In reality, spacetime is not curved by mass (or vice-versa), but by a special class of volume. Replacing $mass$ by $volume$ changes all the meaning of GR and opens new horizons in astrophysics.

Therefore, we do not have only one type of volume, but five, as described below:

1/ "Closed Volumes": Volumes that curve spacetime

Example: leptons. Their internal spacetime "pushes" the surrounding spacetime to make room. Thus, closed volumes produce a convex curvature of spacetime (Fig. 3A). Since the latter has the properties of elasticity (Einstein), it exerts a pressure on the surface of these volumes that increases their resistance to movement. As a result, a *mass effect* appears, i.e., an effect having all the characteristics of mass.

For example, consider a particle of volume V crossing a cube of volume 1,000,000 V. The cube will contain 1,000,001 elements of volumes but its overall volume will remain 1,000,000 V because spacetime has the properties of elasticity. Due to this elasticity, the particle will be subject to a pressure from the 1,000,000 other elements of volume.

This pressure, determined by the constraints of the stress tensor (Fig. 1A), leads to a *mass effect*. See Section 6 and appendix C to get the mass expression $m = f_{(x,y,z,t)}$.

2/ "Open Volumes": Volumes that do not curve spacetime.

Open volumes are simply vacuum, but are often found in various forms, such as the volumes of orbitals. These volumes exist but are transparent regarding spacetime. Therefore, open volumes are massless since *no curvature* $\equiv$ *no pressure of spacetime*, and *no pressure of spacetime* means *no mass effect*.

3/ Apparent volumes: Combination of closed and open volumes.

Objects we use daily are defined as:

$$\textit{Apparent Volumes} = \sum \textit{Closed Volumes} + \sum \textit{Open Volumes}$$

These volumes, mainly atoms and molecules, are combinations of volumes with mass (closed volumes) and massless volumes (open volumes). In atoms, for example, the nucleus and electrons curve spacetime and have mass, whereas the orbitals do not.

We have the feeling that mass and volume are two different quantities. In reality, it is the proportion of closed to open volumes in "apparent volumes" that varies from one atom to another, from one molecule to another, and from one object to another, which gives us this feeling. It means that mass and (closed) volume are two different views of the same quantity, such as particles and waves in the wave-particle duality.

4/ Hermetic Volumes.

These volumes are also combinations of closed and open volumes but their global volume is hermetic regarding spacetime. For example, a nucleus is made of nucleons (closed volumes), separated by empty space (open volume). Whatever the content of this empty space (gluons...), this volume exists. The behaviour of this global volume is that of a closed volume regarding spacetime. Consequently, the whole volume of the nucleus deforms spacetime and, therefore, gets mass. This explains why the volume ($\equiv$ mass) of the nucleus is greater than those of its nucleons (mass excess). This also explains why the volume of protons, 938 MeV/c^2, is much greater than the volumes of the three quarks, 2.3 + 2.3 + 4.8 = 9.4 MeV/c^2, taking into account the volumes of gluons and the empty space (open volumes) between the components.

5/ Special Volumes.

These kinds of volumes are called "special" because we do not know their behaviour regarding spacetime. This is the case of ^{6}He, ^{8}He, ^{14}Be... For example, ^{11}Li has a core with $3p6n$ and a halo of $2n$. Since we do not know the penetration of spacetime inside these nuclei, these volumes can not be classified in any of the preceding categories.

5. Can we Replace Mass by Volume?

We must bear in mind there it does not exist only one kind of volume, but five. This leads to the following questions:

1/ Can we replace mass by volume?

No, because the word "Volume" is undefined.

2/ Can we replace mass by apparent volumes?

No, because apparent volumes are a combination of closed volumes, with mass, and massless open volumes. Only closed volumes (with mass) are significant.

3/ Can we replace mass by closed or hermetic volumes?

Yes, because there is a relation between closed or hermetic volumes and mass (see Eq. 2, Section 6). It is obvious that if we have $V = f_{(M)}$ and reciprocally $M = f_{(V)}$, we can use indifferently V or M in our GR calculus.

6. Expression of Mass

Mass is not a basic quantity per se but the consequence of the following mechanism:

– A closed or hermetic volume curves spacetime,
– Spacetime exerts a pressure on the surface of this kind of volume,
– The pressure of spacetime increases the resistance to movement of the volume,
– Which leads to a *mass effect*.

The mass component [M] can be extracted from the pressure [M/LT^2] by simple basic operations (see appendix C, section 12). This leads to a 4D expression for the mass, which is more accurate than the SEMF empirical formulation $R = A^{1/3} r_0$:

$$M \; = \; \frac{c^2}{G_0} \epsilon_v \frac{V}{S} \; \delta = f_{(x,y,z,t)} \qquad (2)$$

where

M = mass effect (kg),
V = volume of the closed volume (m^3),
S = surface of the closed volume (m^2),
c = speed of light (m/s),
ϵ_v = elasticity of spacetime,
G_0 = universal constant of gravitation,
δ = density of spacetime, relative to a flat spacetime.

For spherical objects or particles, if the radius is R, Eq. (2) becomes:

$$M \; = \; \frac{c^2}{G_0} \epsilon_v \frac{R}{3} \; \delta \qquad (3)$$

Here, $\epsilon_v \delta / 3$ is a simple coefficient. Putting $\Delta R = (\epsilon_v \delta / 3) R$ gives:

$$\Delta R \; = \; \frac{M G_0}{c^2} \qquad (4)$$

As we see, Eq. (3), which is a particular case of Eq. (2), leads to the main term of the Schwarzschild metric (Eq. 4)[17][18].

7. Expansion of the Universe, Dark Matter

The pressure of spacetime on a point located on the periphery of a galaxy is a function of the amount of spacetime over this point. Consider a galaxy “A” located at the center of the universe. This galaxy is static because the pressure of spacetime on its periphery is isostatic. Consider now a similar galaxy “B” located near the boundaries of the universe. Contrary to galaxy “A”, the pressure of spacetime around galaxy “B” is unbalanced. Indeed, the amount of spacetime towards the center is greater to that towards the boundary of the universe. This difference of amount of spacetime leads to a difference of pressure which produces a movement of galaxy “B” towards the boundaries of the universe This could explain the increase of expansion of the universe.

8. What is Gravitation?

The mechanism for gravitation becomes clear and easy to understand if we think "closed or hermetic volume” instead of "mass” (Fig. 4) [19]:

> *Gravitation is not an attractive force between masses, but a pressure force exerted by the curvature of spacetime on closed or hermetic volumes, which tends to bring them closer to each other.*

The result is identical. For example, a pressure on one side of a sheet of paper produces the same effects as an attractive force on the other side. About the replacement of mass by closed or hermetic volumes, the result is also identical (see Eq. 2, Section 6).

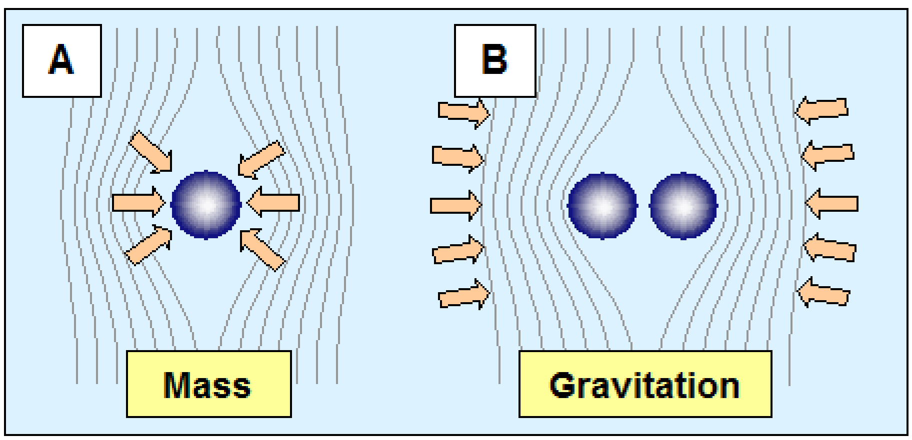

Figure 4. Mass and gravitation are two similar phenomena. They are the consequence of the elasticity of spacetime (Einstein) which exerts a pressure on one (mass) or more (gravity) closed or hermetic volumes.

9. Curvature of spacetime vs. Pressure

Figure 5a (next page) shows the curvature of spacetime produced by the closed and/or hermetic volumes of two spheres.

– **Point L** (Left side): The curvatures of spacetime of the two spheres are added.

– **Point R** (Right side): The curvature of spacetime of the red sphere is subtracted from that of the blue one because the two curvatures are in opposition.

The difference of the curvature of spacetime on each side of the two spheres leads to a difference of pressure (black and orange arrows on Fig. 5b). Fig. 5a and 5b are two different views of the same phenomenon: *curvature of spacetime vs. pressure*.

The same principle also applies if the two spheres are the Earth and the Moon. In the Lagrangian Point (green area), the curvature of spacetime produced by the Earth is canceled by that produced by the Moon. The curvature of spacetime is null at this point, and the gravity disappears. This well-known phenomenon confirms our explanation of gravitation: *Curvature of spacetime* $\equiv$ *Pressure* (Fig. 5).

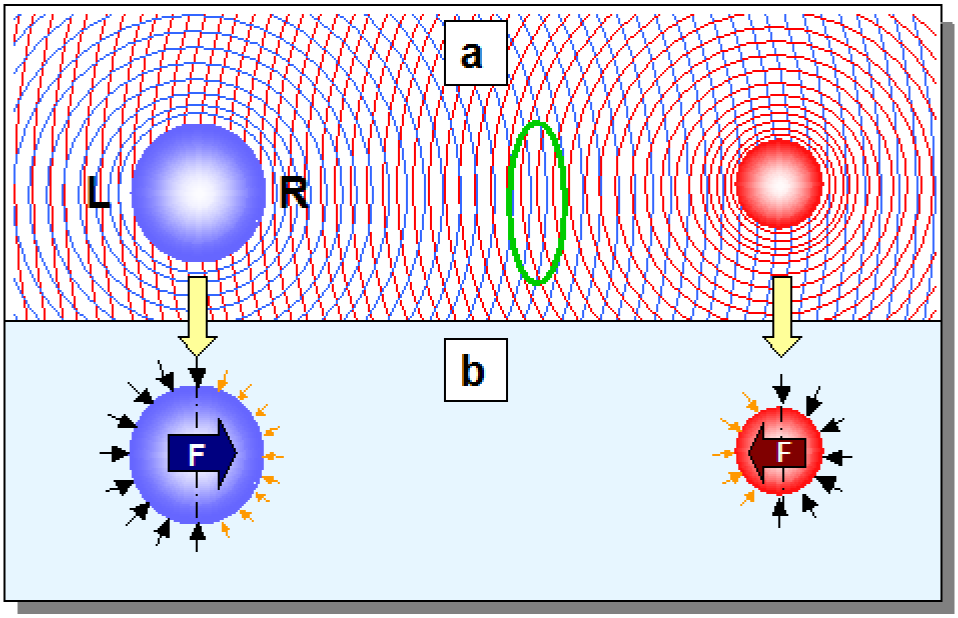

Figure 5. (a) The curvature of spacetime produced by the red sphere interacts with that produced by the blue one and vice-versa. Therefore, the curvature of spacetime in not constant on each point of the surface of the two spheres. (b) This difference of curvature of spacetime leads to a difference of pressure on each point of the surface of the two spheres (black and orange arrows). This explains gravitation.

10. Discussion

Solving Inconsistencies #1 and #2

The fact that Einstein built his EFE from fluid mechanics has two consequences:

1: Spacetime does not exert an attractive force but a pressure force on closed volumes;

2: The curvature of spacetime is not concave but convex.

We can combine these two consequences in the following assertion:

Concave curvature + attractive force $\equiv$ *convex curvature + pressure force*.

If we assign by convention a positive sign to a convex curvature and to a pressure force, the above assertion gives $(--) = (++)$. As we see here, this modern view of GR explains the mechanism of mass and gravity without changing anything in the EFE.

Solving Inconsistency #3

Today, the "pudding model" is obsolete and our traditional definition of only one kind of volume must be dropped. We must replace it by a new definition closer to reality: *"Any volume must be closed, open, apparent, hermetic, or special"*. Since a relation exists between closed/hermetic volumes and mass, it becomes possible to replace mass by closed /hermetic volumes: $M = f_{(V)}$ (Eq. 2). This solves inconsistency #3.

Solving Inconsistency #4

If we include the expression of mass $m = f_{(x,y,z,t)}$ (Eq. 2) in the EFE, the latter becomes homogeneous: $[L^3][T] = [L^3][T]$. This solves inconsistency #4.

11. Conclusions

This article highlights that there is not only one kind of volume but five. Some volumes curve spacetime (closed/hermetic volumes), whereas others do not.

Since the curvature of spacetime is at the heart of GR, we must drop our traditional view of a unique volume and consider each volume separately. As a result, there is a relation $M = f_{(V)}$ between mass and closed/hermetic volumes. Therefore, mass can be replaced by this kind of volumes in GR calculus. This substitution explains the genesis of the curvature of spacetime and shows that it is convex, not concave.

On the other hand, the stress and energy–momentum tensors also highlight that the attractive force of gravitation must be replaced by a pressure force.

The originality of this paper lies in the association of these two simple observations:
1/ Replacement of mass in the EFE by closed/hermetic volumes $M = f_{(V)}$,
2 / Replacement of the attractive force of gravitation by a pressure force.

This association explains the phenomena of mass and gravitation, and solves several enigmas of modern physics such as *"How can we explain $E = mc^2$?"*. It also suggests important solutions to astrophysics enigmas such as dark matter or dark energy.

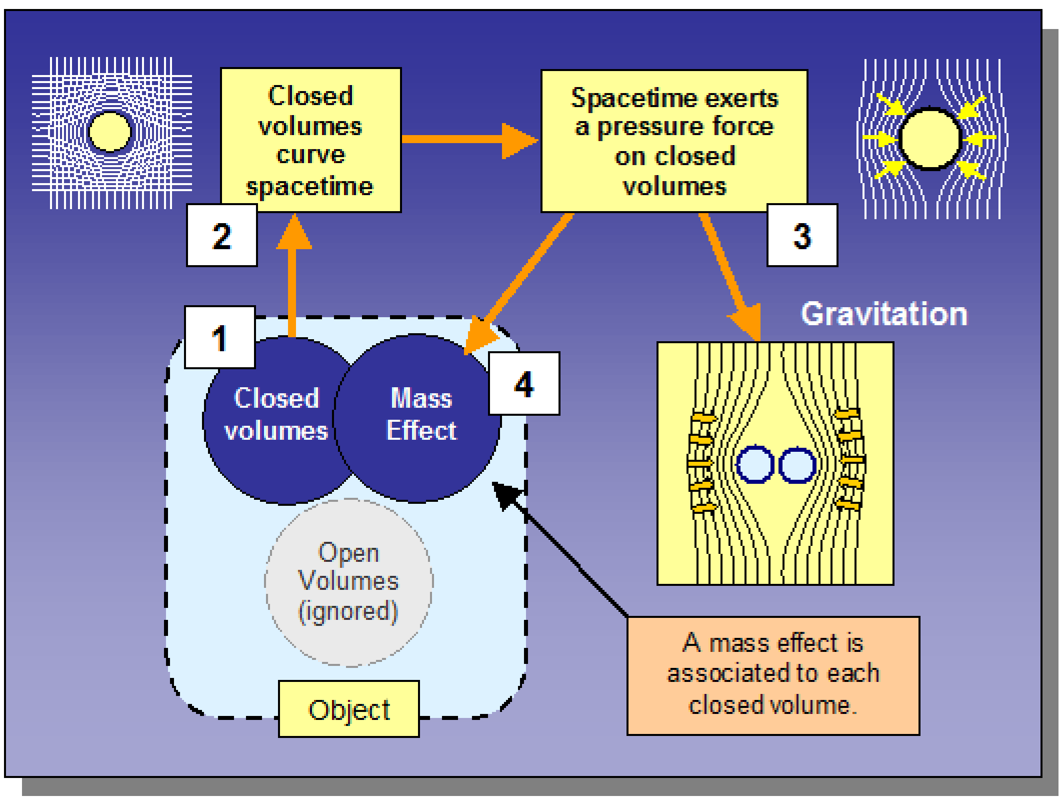

Figure 6. Relation between the curvature of spacetime, (closed/hermetic) volumes, mass, and gravitation. In order to simplify the whole mechanism, here we have taken only closed volumes.

Another interesting application is to the Higgs theory [50]. The hypothetical Higgs field could be nothing but a Newtonian field in spacetime. The curvature of spacetime producing the Higgs field is created by the closed/hermetic volume of the particle which is expected to get mass.

12. Appendices

The second part of this article is made of seven appendices which cover:

A – New Version of the Newton's Law calculated from closed/hermetic volumes,
B – The Schwarzschild metric calculated from closed/hermetic volumes,
C – Expression of the mass effect in 4D: $m = f_{(x,y,z,t)}$,
D – Explanation of the increase of the mass of relativistic particles,
E – New formulation of the EFE: $R_{\mu\nu} - 1/2 g_{\mu\nu} R = (8\pi\delta_v)/S \, . \, J_{\mu\nu}$ [20][21][22],
F – Connexion with the Von Laue Geodesics,
G – New version of the Equivalence Principle.

13. References

The bibliography is available at the end of the Appendices.

[1] This paper has been registered at INPI under the following references: 238268, 238633, 244221, 248427, 258796, 261255, 268327, 297706, 297751, 297811, 297928, 298079, 298080, 329638, 332647, 335152, 335153, 339797, 12-01112,

Mass and Gravitation

Mathematical Demonstrations

The following appendices cover:

New Version of the Newton's Law

A-1 Introduction

The Newton's Law is obtained from the Einstein Field Equations (EFE). It is a particular solution for a spherical static symmetry object using the weak field approximation. Here we show that the Newton's Law can be easily obtained replacing mass by closed volume. This method is much more simpler than calculating the Newton's Law from the EFE.

A-2 Bulk Modulus

The bulk modulus K_B of a substance measures the substance's resistance to uniform compression. It is defined as the pressure increase needed to cause a given relative decrease in volume (fig. A-1).

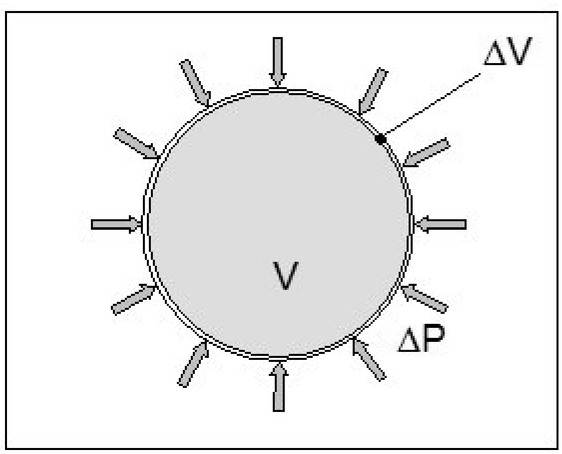

Figure A-1: Bulk modulus

$$K_B = -V\frac{\Delta P}{\Delta V} \qquad (1)$$

Starting with the Navier-Stokes Equations of the Fluid Mechanics, Einstein, helped by Grossman, demonstrated in the 1910's that spacetime:

- Can be identified to a newtonian fluid,
- Returns to its rest shape after having applied a stress (properties of elasticity).

Therefore, the Bulk Modulus (equation 1), which is a version of the Cauchy Tensor of the Fluid Mechanics, can also be applied to spacetime. It means that the displacement of spacetime made by a closed volume exerts a pressure on the surface of the latter (fig. A-1).

Note: This subject is also covered in Appendix E.

A-3 Elasticity Law

Elasticity phenomena follow the well-known logarithmic law:

$$\epsilon = ln\left(\frac{R - \Delta R}{R}\right) \qquad (2)$$

with ϵ = coefficient of elasticity.

The Schwarzschild Metric gives an order of magnitude of the curvature of spacetime, which is infinitesimal. For example, the ratio curvature of spacetime/radius, or $\Delta R/R = GM/Rc^2$, is 1.4166E-39 for the proton, with M = 1.672E-27 kg, R = 8.768E-16 m, G = 6.674E-11, c = 8.987E+16. See Appendix B for the meaning of $\Delta R/R$ and of the factor 2 in $2GM/Rc^2$.

Under these conditions, whatever the formula used, logarithmic or not, the curvature of spacetime can be considered as a linear function since we are working on an infinitesimal segment near to the point zero. So, equation (2) becomes in first order approximation:

$$\epsilon_R \approx \frac{\Delta R}{R} \qquad (3)$$

or, with volumes:

$$\epsilon_V \approx \frac{\Delta V}{V} \qquad (4)$$

For the moment, the linear ϵ_R and volumetric ϵ_v coefficients of elasticity of spacetime are unknown.

A-4 Curvature of Spacetime

A closed volume V inserted into spacetime pushes it to make room (Fig. A-2). So the following volumes are identical:

$$V = V_1 = V_2 ... = V_n \qquad (5)$$

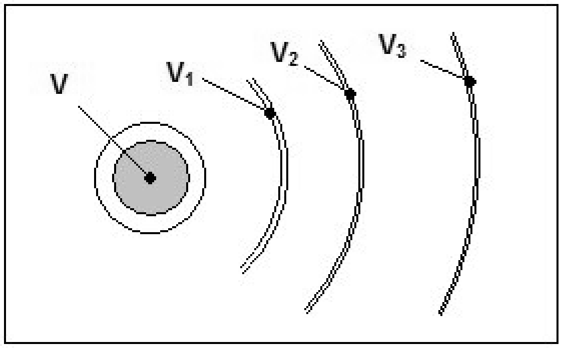

Figure A-2: Volumes V, V1, V2, V3... are identical

Since the curvature is infinitesimal, the coefficient of elasticity of spacetime ϵv can be considered constant. So, combining (4) and (5) gives:

$$\Delta V = \Delta V_1 = \Delta V_2 ... = \Delta V_n \qquad (6)$$

However, there should not be any confusion between a simple displacement of spacetime, Vx, produced by the insertion of a closed volume into a flat spacetime, and the curvature $\Delta V_x = \epsilon_v V_x$ due to the elasticity of spacetime (fig. A-3).

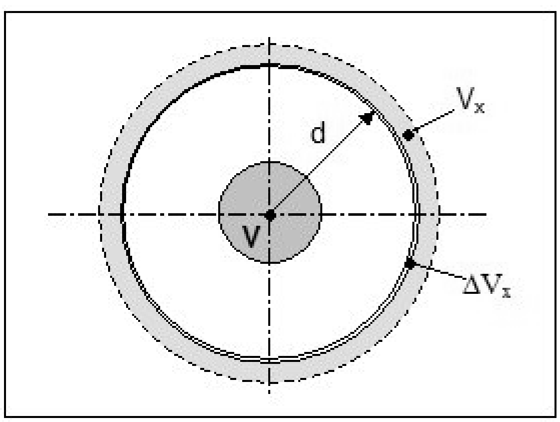

Figure A-3: Simple displacement (V_x) vs curvature of spacetime (ΔV_x)

A-5 Solving $\Delta\mathbf{x} = \mathbf{f}_{(\Delta\mathbf{R})}$

Since the ΔV's are infinitesimal, the volume ΔVx is simply the product of Δx by the surface Sx (fig. A-4):

$$\Delta V_x = \; S_x \; \Delta x \;\; = \;\; 4\pi d^2 \Delta x \qquad (7)$$

The volume ΔV_R is also the product of ΔR by the surface S_R:

$$\Delta V_R = \; S_R \; \Delta R \;\; = \;\; 4\pi R^2 \Delta R \qquad (8)$$

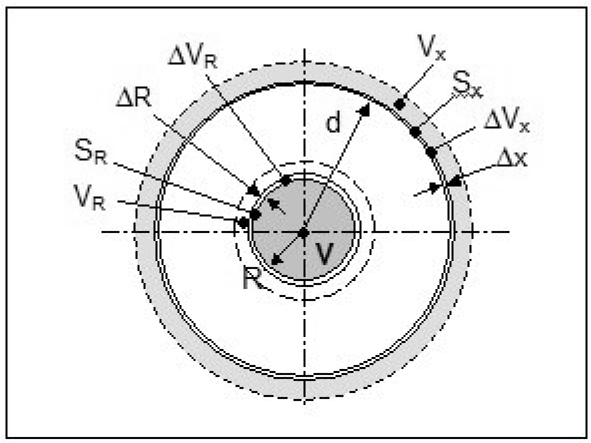

Figure A-4: Displacement and curvature at distances R and d

From (6) we have:

$$\Delta V_R = \Delta V_x \qquad (9)$$

Combining (7), (8) and (9) gives:

$$4\pi R^2 \Delta R = 4\pi d^2 \Delta x \qquad (10)$$

Finally, we get:

$$\Delta x = \frac{R^2}{d^2} \Delta R \qquad (11)$$

Where:
- R is the radius of the closed volume V_R,
- ΔR is the curvature of spacetime on the surface of the closed volume V_R,
- d is the distance of the point of measurement,
- Δx is the curvature of spacetime at distance d.

A-6 Curvature Δx vs Mass M

As explained in the main text, a relation exists between the curvature of spacetime, ΔR (or Δx at a distance "d" from R), and the "mass effect" of the object:

$$\Delta R = f_{(M)} \qquad (12)$$

It is the pressure that produces the mass effect. This suggests that the latter is inversely proportional to the surface S, or [1/L^2], as any pressure does. On the other hand, it is obvious that the mass effect is also proportional to the volume, or [L^3]. Therefore, the dimensional quantity of the mass effect is [1/L^2][L^3] = [L]. In other words, $[M] \equiv [L]$.

At this point, we don't know the relation between ΔR and M but, in referring to Einstein's works, we have good reasons to believe that this relation is a simple linear function like:

$$\Delta R = KM \qquad (13)$$

...where K is an constant having the dimensional quantity of [L/M].

Here we show that [K] = [L/M], but we will see later that K = G/c^2. This result is in line with the dimensional quantity of some terms of the Schwarzschild Metric: $2\epsilon = 2\Delta r/r = 2GM/rc^2$ (see Appendix B). The challenge, now, is to calculate K to get the Newton's Law.

A-7 The Newton Law

Porting (13) in (11) gives:

$$\Delta x = \frac{R^2}{d^2}\, KM \qquad (14)$$

or

$$\frac{\Delta x}{R^2} = K\frac{M}{d^2} \qquad (15)$$

Since x = ct, replacing R^2 by c^2t^2 gives:

$$\frac{\Delta x}{c^2t^2} = K\frac{M}{d^2} \qquad (16)$$

or :

$$\frac{\Delta x}{t^2} = c^2K\frac{M}{d^2} \qquad (17)$$

The value $\Delta x/t^2$ has the dimensional quantity of an acceleration $[L/T^2]$. So, replacing this fraction by the acceleration symbol "a" (see the note below), we get:

$$a = c^2 K \frac{M}{d^2} \qquad (18)$$

Notes: Δx is an infinitesimal quantity, not a differential quantity such as dx. Moreover, we are working in a linear segment of the elasticity of spacetime. In such a situation, $\Delta x/\Delta t \approx x/t$.

On the other hand, the multiplication of a constant c^2 by a second constant K gives another constant. So, we can replace the product c^2K by a new and unknown constant for the moment, G for example:

$$c^2 K \ = \ G \qquad (19)$$

or *(this equation isn't necessary here but will be used for the calculation of the Schwarzschild Metric in Appendix B.)*

$$K \ = \ \frac{G}{c^2} \qquad (20)$$

Porting (19) in (18) gives:

$$a = G\frac{M}{d^2} \qquad (21)$$

To be consistent, this unknown constant G must have the same dimensional quantity of the product c^2K (equation 19):

- c^2 : Dimensional quantity $\Rightarrow [L^2/T^2]$
- K : Dimensional quantity $\Rightarrow [L/M]$ (see paragraph A-6)

So,

The dimensional quantity of this new constant G is
$$\mathbf{[c^2K] = [L^2/T^2][L/M] = [L^3/MT^2].}$$

On the other hand, we know that a force is the product of an acceleration by a mass, here "m". Therefore, equation (21) can be written as follows:

$$\boxed{F = G\,\frac{Mm}{d^2}} \qquad (22)$$

For the moment, G is unknown but we must note that:

- G is a constant,
- Its dimensional quantity is $[L^3/MT^2]$.

So, we can identify G to the constant of gravitation issued from experimentation:

$$G = 6,67428.E - 11 \qquad (23)$$

In other words,

Equation (22) can be identified to the Newton Law of Universal Gravitation.

New Version of the Schwarzschild Metric

B-1 Introduction

Here we show that the Schwarzschild Metric can be easily obtained using closed volumes instead of masses. The following demonstration doesn't require tensor knowledge.

B-2 The Minkowski Metric

The expression of the Minkowski Metric, in spherical coordinates, is:

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1)$$

The Schwarzschild Metric refers to a static object with a spherical symmetry. It is built from a Minkowski Metric, in spherical coordinates, with two unknown functions: A(r) and B(r):

$$ds^2 = -B_{(r)}c^2dt^2 + A_{(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (2)$$

The Minkowski Equation must follow the Lorentz Invariance in Special Relativity (SR) or General Relativity (GR). To get this invariance, we must set $A_{(r)} = 1/B_{(r)}$. Details of calculus are described in Appendix E and in books concerning GR. So:

$$B_{(r)}A_{(r)} = 1 \qquad (3)$$

B-3 The Schwarzschild Metric

To calculate the Schwarzschild Metric, we can start with fig. B-1 (next page), which is issued from the theory described in the main article, where:

- d_{rout} is an elementary differential radial variation outside of any mass,
- d_{rin} is an elementary differential radial variation inside a Schwarzschild spacetime,
- r is the point of measurement.

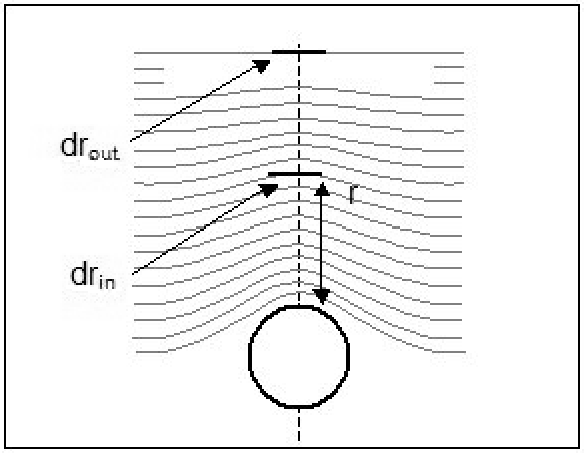

Figure B-1: Spacetime has been reduced to 1D

Supplementary Information A gives:

$$\epsilon = ln\left(\frac{r - \Delta R}{r}\right) \qquad (4)$$

where:

- ϵ is a coefficient of the increase of spacetime curvature at distance r,
- ΔR is the initial curvature of spacetime produced by the closed volume,

The order of magnitude of ϵ is 10E-39. So, we can use the first order approximation from Supplementary Information A, equation (3):

$$\epsilon \approx \frac{\Delta R}{r} \qquad (5)$$

Since ϵ is a simple coefficient, we can calculate the relation between two differential elementary radius dr(out) and dr(in), out and in a gravitational field:

$$dr_{in} = (1 + \epsilon)\ dr_{out} \qquad (6)$$

Since $\epsilon \ll 1$, equation (6) becomes:

$$dr_{in} = \frac{1}{(1 - \epsilon)}\ dr_{out} \qquad (7)$$

or, elevating in square:

$$dr_{in}^2 = \frac{1}{(1-\epsilon)^2}\, dr_{out}^2 \qquad (8)$$

Developing the denominator $(1-\epsilon)^2 = 1-2\epsilon+\epsilon^2$ and ignoring the last term ϵ^2, we obtain:

$$dr_{in}^2 = \frac{1}{(1-2\epsilon)}\, dr_{out}^2 \qquad (9)$$

This result is nothing but the radial component of the Schwarzschild Metric, that is to say the function A(r) of dr^2 in (2). Then, the calculation of B(r) is immediate, taking into account that $A(r)B(r) = 1$ from equation (3). So:

$$A_{(r)} = \frac{1}{(1-2\epsilon)} \qquad (10)$$

$$B_{(r)} = (1-2\epsilon) \qquad (11)$$

Thus, equation (2) becomes:

$$ds^2 = -(1-2\epsilon)c^2dt^2 + \frac{1}{(1-2\epsilon)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (12)$$

In Appendix A "The Newton Law", we have got ΔR = KM (equation 13). Since K = G/c^2 (Appendix A equation 20), equation (5) can be rewritten as:

$$\epsilon = \frac{\Delta R}{r} = \frac{KM}{r} \;=\; \frac{GM}{rc^2} \qquad (13)$$

Finally, porting this expression in equation (12) gives:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)}\, dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (14)$$

B-4 Conclusions

This new calculus of the Schwarzschild Metric, which is exclusively based on closed volumes, gives identical results than developing the EFE in the special case of a static spherical symmetry. This is due to the fact that the origin of EFE is the Fluid Mechanics, which is itself based on volumes, not on masses (see Appendix E).

Expression of the Mass Effect in 4D

C-1 Expression of "m"

We have seen that the displacement of spacetime V_R is equal to that of the closed volume, V, which produces this displacement (fig. C-1):

$$V_R = V \qquad (1)$$

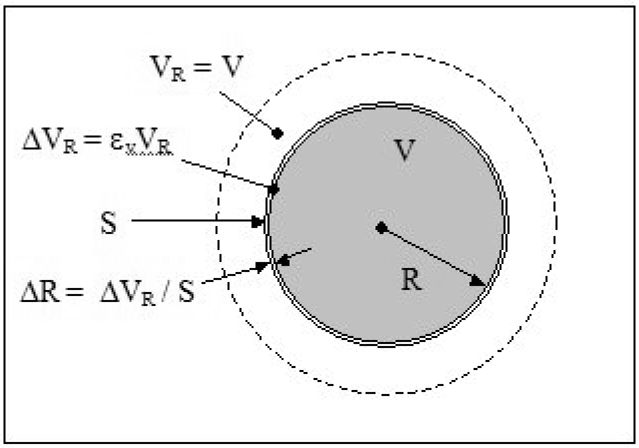

Figure C-1: The curvature of spacetime

On the other hand, the curvature of spacetime is:

$$\Delta V_R = \epsilon_v \; V_R \qquad (2)$$

Porting (1) in (2) gives:

$$\Delta V_R = \epsilon_v \; V \qquad (3)$$

The radial curvature of spacetime, ΔR, at the surface of M, is calculated dividing the volume by the surface:

$$\Delta R = \frac{\Delta V_R}{S} \qquad (4)$$

Replacing ΔV_R by its expression (3) gives:

$$\Delta R = \epsilon_v \frac{V}{S} \qquad (5)$$

On the other hand, we have calculated the Newton's Law starting with equation (13) of Appendix A:

$$\Delta R = KM \qquad (6)$$

...where K is an unknown constant having the dimensional quantity of [L/M]. Porting (6) in (5) gives:

$$KM = \epsilon_v \frac{V}{S} \qquad (7)$$

or

$$M = \frac{\epsilon_v}{K} \frac{V}{S} \qquad (8)$$

Porting in equation (8) the expression of K given in equation (20) of Appendix A gives the expression of the "mass effect". Here, we have added a new coefficient, ρ (see explanation below):

$$\boxed{M = \frac{c^2}{G_0} \epsilon_v \frac{V}{S} \rho} \qquad (9)$$

with:

M = Mass effect (kg)
V = Volume of the closed volume (m^3)
S = Surface of the closed volume (m^2)
ϵ_v = Coefficient of volumetric elasticity of spacetime in a flat spacetime. This parameter is unknown but can be calculated from the mass/diameter of spherical particles such as some leptons or "magic" nuclei. See the next sections.
c = Speed of the light (m/s)
G_0 = Universal constant of gravitation
ρ = Density of surrounding spacetime relative to a flat spacetime. This parameter is equal to 1 in a Minkowsky Spacetime.

It seems useful to differentiate ϵ_v, the coefficient of elasticity of spacetime in a flat spacetime, and ρ, the density of surrounding spacetime. We could merge these two parameters in one common parameter since we are faced with two coefficients. In both cases, result is the same. However, we must note that the *"proper coefficient of elasticity of spacetime"*, as proper length in special relativity, must be measured in a flat spacetime. This is why the two parameters have been separated. For example, the particle may be located in a riemanian spacetime, i.e. in a field produced by another particle. In this case, since the mass effect is a function of the curvature of spacetime, we need to know the latter before any calculation.

C-2 Case of a sphere

In the particular case of a sphere, we have $V = 4/3\pi R^3$ and $S = 4\pi R^2$. Thus, equation (9) becomes:

$$\boxed{M = \frac{\epsilon_v R c^2}{3\ G_0}\ \rho} \qquad (10)$$

C-3 Nuclei

Nuclei aren't spherical generally. Since we don't know exactly their shape, it is not possible to apply equation (10) to calculate their mass effect.

The semi-empirical mass formula (SEMF), sometimes called the Bethe-Weizsacker mass formula, is used to approximate various properties of an atomic nucleus. It is based partly on the liquid drop model proposed by George Gamow, and partly on empirical measurements. From the SEMF formula, the radius R is defined as $R = 1.2A^{1/3}$, A being the mass number. In reality, the right member does not mean that the mass vs. the radius follows a $A^{1/3}$ law. It is the arrangement of nucleons inside the nucleus that follows this rule. The result is equivalent but the significance is different.

We must also note that a surface component also exists in the Bethe-Weizsacker expression. It means that, early in 1937, Bethe and Weizsacker predicted the equation (9) here demonstrated.

We must keep in mind that nuclei are made of open and closed volumes. The space between nucleons may vary from one nucleus to another. Thus, it is necessary to know the arrangement of nucleons with accuracy before any calculations. A particular case are magic nuclei (nuclei having a null quadripolar moment) because they are supposed to be spherical. It will be interesting to make accurate experiments on the relation volume/surface/mass effect of magic nuclei.

On the other hand, some nuclei have a halo made of open volumes that are not relevant in mass calculation. This is the case for example of the ^{11}Li (3p8n), which has open volumes between the ^{9}Li and the 2n orbitals. These exceptions highlight the difficulty to make accurate calculations of the mass effect. In all cases, before any calculation, we must know exactly the geometry of closed and open volumes inside the nucleus. It means that the calculus of the mass from the geometry of the nucleus (equation 9) is not as simple as it sounds.

As a direct consequence of the proposed theory, it could be possible that a relationship exists between the sphericity of particles or nuclei and the accuracy of measurements. This deduction suggests that leptons could be spherical since their mass effect is known with an excellent accuracy. This is also the case of some particles such as the proton, neutron, or π meson. Inversely, this is not the case of quarks. It means that quarks could have a non-spherical or complex shape.

Explanation of the Increase of the Mass of Relativistic Particles

D-1 Introduction

The increase of the mass of relativistic particles is covered by special relativity. However, this phenomenon remains particularly obscure and to date, we are still unable to explain with simple words, i.e. without using mathematics, why does the mass increase with velocity. The proposed theory gives a simple and rational explanation of this strange phenomenon.

D-2 Length contraction

Special relativity states that, at relativistic speed, times *expand,* lengths *contract* and angles *are modified.* A simple demonstration is given in 1923 by Einstein himself in his book "The Theory of Special and General Relativities". The length contraction is defined by the formula

$$l_m \;=\; l_0 \sqrt{1-\frac{v^2}{c^2}} \qquad (1)$$

with

- l_m = Measured length
- l_0 = Proper length
- v = Speed of object
- c = Speed of light

D-3 Mass increase

Lets consider a particle at rest (fig. D-1a next page). Its closed volume produces a curvature of the spacetime. Geodesics of spacetime are spaced of l_0.

If this particle moves at a relativistic speed "v" (fig. D-1b), spacetime geodesics seems to shrink. This is the well-known phenomenon of length contraction. The closer the geodesics are to each other, the more important spacetime density is, according to equation (9) of Appendix C. In other words, the curvature of spacetime is inversely proportional to the space between two geodesics (see note 1). So, relation (1) becomes:

$$\Delta R_m \;=\; \frac{\Delta R_0}{\sqrt{1-v^2/c^2}} \qquad (2)$$

with

- ΔR_m = Infinitesimal element of the curvature of spacetime measured (external observer)
- ΔR_0 = Infinitesimal element of the proper curvature of spacetime
- v = Speed of the particle
- c = Speed of light

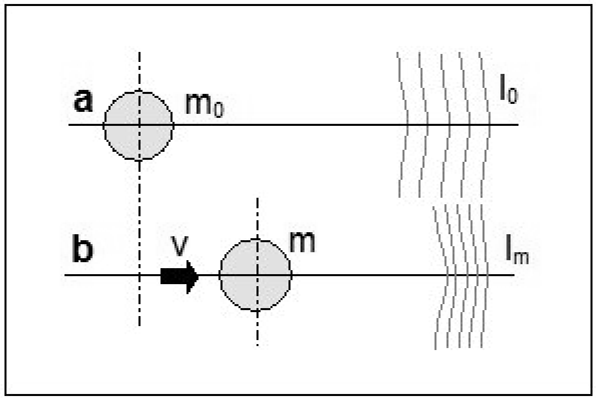

Figure D-1: Since the spacetime is more dense in (b), the mass effect increases.

Since the curvature is a function of mass $\Delta R = KM$ (see Appendix A, equations 13 and 20), we can replace the curvature of spacetime ΔR_m of equation (2) by the mass effect m, and the proper curvature of spacetime ΔR_0 by the proper mass effect m_0 (see notes 2 and 3).

$$\boxed{m = \frac{m_0}{\sqrt{1 - v^2/c^2}}} \qquad (3)$$

Thus, the theory presented in this paper based on closed volumes instead of masses, and figure D-1, give a very simple and rational explanation of the mass increase of relativistic particles.

Note 1: The spacetime curvature is the difference of displacement ΔR of a geodesic vs. to the same geodesic in a Minkowski space. As shown in this article, the Schwarzschild metric gives an order of magnitude of this spacetime curvature: 1.4166 E-39 meters for the proton on its surface. This value is much smaller at distance r. Thus, regardless of the function used, the portion of the curve on which we work is linear. Taking this linearity into account, there is no objection to consider that the curvature is inversely proportional to the space between two geodesics.

Note 2: The nature of expression $\Delta R = KM$ *is not relevant because this section covers exclusively the calculation of the coefficient to be applied to a proper value to get the measured value. This coefficient is noted* γ *(or* $1/\gamma$*) in scientific literature. It means that the relationship between the spacetime curvature and the mass effect is not affected by this study. For example, if we make 5 measurements of the curvature of spacetime at different speeds, we will not have 5 different relationships between* ΔR *and M, but only one applicable in all cases, ...but we will have 5 different coefficients* γ*.*

Note 3: The principles of special relativity state that the measurement of the mass of a relativistic particle increases. However, the converse is also true if we swap the reference systems. If we could pick up a measuring device on a particle in movement, this device would indicate that our spacetime, that in which we live, is much more dense as we see it. Thus, a section of the LHC for example, with a mass of 3 tons, measured from a device located on the particle in motion, would have a mass of 3000 tons if $\gamma = 1000$*. From our point of view, the mass of a relativistic particle increases, but from the particle's point of view, it is our world that increases. In all cases, the proper mass of the particle or that of our world remains unchanged. This "relative view" is often misunderstood.*

Note 4: Many physicists think that the mass, so the "volume", of relativistic particles really increases. In reality, it is the mass effect due to th apparent compression of spacetime that increases. The volume remains unchanged. On Earth, we consider that "mass" is an intrinsic value of a particle, such as the volume. It is not true. Since the mass effect comes from the pressure of spacetime on the particle, it is a virtual effect, such as pressure, speed, force, energy... On the other hand, the mass effect depends of the surrounding density of spacetime. Thus, for example, if the spacetime density was two times higher, the mass effect would be twice as important as well, but the intrinsic characteristics of the particle would remain unchanged. This explanation is shown in the graphic of fig. D-1.

Partial Rewriting of the Einstein Field Equations (EFE)

E-1 General Relativity Origins

In the 1910s, Einstein studied gravity. Following the reasoning of Faraday and Maxwell, he thought that if two objects are attracted to each other, there would be some medium. The only medium he knew in 1910 was spacetime. He then deduced that the gravitational force is an indirect effect carried by spacetime. He concluded that any mass perturbs spacetime, and that the spacetime, in turn, has an effect on mass, which is gravitation. So, when an object enters in the volume of the curvature of spacetime made by a mass, i.e. the volume of a gravitational field, it is subject to an attracting force. In other words, Einstein assumed that the carrier of gravitation is the *curvature of spacetime*. Thus, he tried to find an equation connecting:

1. The curvature of spacetime. This mathematical object, called the “Einstein tensor”, is the left hand side of the EFE (Eq. 1).

2. The properties of the object that curves spacetime. This quantity, called the “Energy–Momentum tensor”, is $T_{\mu\nu}$ in the right hand side of the EFE.

$$Curvature\ of\ spacetime \quad \equiv \quad Object\ producing\ this\ curvature$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad (1)$$

Einstein Tensor (curvature of spacetime)

Thus, Einstein early understood that gravity is a consequence of the curvature of spacetime. Without knowing the mechanism of this curvature, he posed the question of the special relativity in a curved space. He left aside the flat Minkowski space to move to a Gaussian curved space. The latter leads to a more general concept, the “Riemannian space”. On the other hand, he identified the gravitational acceleration to the inertial acceleration (see the Appendix G “New Version of the Equivalent Principle”).

The curvature of a space is not a single number, though. It is described by “tensors”, which are a kind of matrices. For a 4D space, the curvature is given by the Riemann-Christoffel tensor which becomes the Ricci Tensor after reductions. From here, Einstein created another tensor called ”Einstein Tensor” (left member of equation 1) which combines the Ricci Curvature Tensor $R_{\mu\nu}$, the metric tensor $g_{\mu\nu}$ and the scalar curvature R (see the explanations below).

Fluid Mechanics

In fluid mechanics, the medium has effects on objects. For example, the air (the medium) makes pressures on airplanes (objects), and also produces perturbations around them. So, Einstein thought that the fluid mechanics could be adapted to gravity. He found that the Cauchy-Stress Tensor was close to what he was looking for. Thus, he identified 1/ the **"volume"** in fluid mechanics to **"mass"**, and 2/ the **"fluid"** to **"spacetime"**.

Energy-Momentum Tensor

The last thing to do is to include the characteristics of the object that curves spacetime in the global formulation. To find the physical equation, Einstein started with the elementary volume $dx.dy.dz$ in fluid mechanics. The tensor that describes the forces on the surface of this elementary volume is the Cauchy Tensor, often called Stress Tensor. However, this tensor is in 3D. To convert it to 4D, Einstein used the "Four-Vectors" in Special Relativity. More precisely, he used the "Four-Momentum" vector Px, Py, Pz and Pt. The relativistic "Four-Vectors" in 4D (x, y, z and t) are an extension of the well-known non-relativistic 3D (x, y and z) spatial vectors. Thus, the original 3D Stress Tensor of the fluid mechanics became the 4D Energy-Momentum Tensor of EFE.

Einstein Field Equations (EFE)

Finally, Einstein identified its tensor that describes the curvature of spacetime to the Energy-Momentum tensor that describes the characteristics of the object which curves spacetime. He added the coefficient $8\pi G/c^4$ to homogeize the two members of the EFE. This coefficient is calculated to get back the Newton's Law from EFE in the case of a static sphere in a weak field. If no matter is present, the energy-momentum tensor vanishes, and we come back to a flat spacetime without gravitational field.

The Proposed Theory

However, some unsolved questions exist in the EFE, despite the fact that they work perfectly. For example, Einstein built the EFE without knowing 1/ what is mass, 2/ the mechanism of gravity, 3/ the mechanism by which spacetime is curved by mass ... To date, these enigmas remain. Considering that "mass curves spacetime" does not explain anything. No one knows by which strange phenomenon a mass can curve spacetime. It seems obvious that *if a process makes a deformation of spacetime, it may reasonably be expected to provide information about the nature of this phenomenon.* Therefore, the main purpose of the present paper is to try to solve these enigmas, i.e. to give a rational explanation of mass, gravity and spacetime curvature. The different steps to achieve this goal are:

- ***Special Relativity (SR).*** This section gives an overview of SR.
- ***Einstein Tensor.*** Explains the construction of the Einstein Tensor.
- ***Energy-Momentum Tensor.*** Covers the calculus of this tensor.
- ***Einstein Constant.*** Explains the construction of the Einstein Constant.
- ***EFE.*** This section assembles the three precedent parts to build the EFE.
- ***EFE Inconsistencies.*** Shows and solves four inconsistencies of EFE.
- ***Metrics.*** Explains how to build special metrics using the new theory.

E-2 Special Relativity (Background)

Lorentz Factor

$$\beta = \frac{v}{c} \qquad (2)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}} \qquad (3)$$

Minkowski Metric with signature $(-, +, +, +)$:

$$\eta_{\mu\nu} \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4)$$

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \qquad (5)$$

Minkowski Metric with signature $(+, -, -, -)$:

$$\eta_{\mu\nu} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (6)$$

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \qquad (7)$$

Time Dilatation
"τ" is the proper time.

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2 d\tau^2 \qquad (8)$$

$$d\tau^2 = \frac{ds^2}{c^2} \quad \Rightarrow \quad d\tau^2 = dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}\right) \qquad (9)$$

$$v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2} \qquad (10)$$

$$d\tau = dt\ \sqrt{1-\beta^2} \qquad (11)$$

Lenght Contractions
"dx'" is the proper lenght.

$$dx' = \frac{dx}{\sqrt{1-\beta^2}} \qquad (12)$$

Lorentz Transformation

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \qquad (13) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\,t \\ x \\ y \\ z \end{bmatrix} \qquad (14)$$

Inverse transformation on the x-direction:

$$\begin{aligned} ct &= \gamma(ct' + \beta x') \qquad (15) \\ x &= \gamma(x' + \beta ct') \\ y &= y' \\ z &= z' \end{aligned}$$

$$\begin{bmatrix} c\,t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\,t' \\ x' \\ y' \\ z' \end{bmatrix} \qquad (16)$$

Four-Position
Event in a Minkowski space:

$$\mathbf{X} = x_\mu = (x_0, x_1, x_2, x_3) = (ct, x, y, z) \qquad (17)$$

Displacement:

$$\Delta X_\mu = (\Delta x_0, \Delta x_1, \Delta x_2, \Delta x_3) = (c\Delta t, \Delta x, \Delta y, \Delta z) \qquad (18)$$

$$dx_\mu = (dx_0, dx_1, dx_2, dx_3) = (cdt, dx, dy, dz) \qquad (19)$$

Four-Velocity

$v_x, v_y, v_z =$ Traditional speed in 3D.

$$\mathbf{U} = u_\mu = (u_0, u_1, u_2, u_3) = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right) \qquad (20)$$

from (7):

$$ds^2 = c^2 dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2} \right) \qquad (21)$$

$$v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2} \qquad (22)$$

$$ds^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2} \right) \quad \Rightarrow \quad ds = cdt\,\sqrt{1 - \beta^2} \qquad (23)$$

Condensed form:

$$u_\mu = \frac{dx_\mu}{d\tau} = \frac{dx_\mu}{dt}\frac{dt}{d\tau} \qquad (24)$$

Thus:

$$u_0 = \frac{dx_0}{d\tau} = \frac{cdt}{dt\,\sqrt{1-\beta^2}} = \frac{c}{\sqrt{1-\beta^2}} = \gamma c \qquad (25)$$

$$u_1 = \frac{dx_1}{d\tau} = \frac{dx}{dt\,\sqrt{1-\beta^2}} = \frac{v_x}{\sqrt{1-\beta^2}} = \gamma v_x \qquad (26)$$

$$u_2 = \frac{dx_2}{d\tau} = \frac{dy}{dt\,\sqrt{1-\beta^2}} = \frac{v_y}{\sqrt{1-\beta^2}} = \gamma v_y \qquad (27)$$

$$u_3 = \frac{dx_3}{d\tau} = \frac{dz}{dt\,\sqrt{1-\beta^2}} = \frac{v_z}{\sqrt{1-\beta^2}} = \gamma v_z \qquad (28)$$

Four-Acceleration

$$a_i = \frac{du_i}{d\tau} = \frac{d^2 x_i}{d\tau^2} \qquad (29)$$

Four-Momentum

$p_x, p_y, p_z =$ Traditional momentum in 3D

U = Four-velocity

$$\mathbf{P} = m\mathbf{U} = m(u_0, u_1, u_2, u_3) = \gamma(mc, p_x, p_y, p_z) \qquad (30)$$

(p = mc = E/c)

$$\|\mathbf{P}\|^2 = \frac{E^2}{c^2} - |\vec{p}|^2 = m^2c^2 \qquad (31)$$

$$p_\mu = mu_\mu = m\frac{dx_\mu}{d\tau} \qquad (32)$$

hence

$$p_0 = mc\frac{dt}{d\tau} = mu_0 = \gamma mc = \gamma E/c \qquad (33)$$

$$p_1 = m\frac{dx}{d\tau} = mu_1 = \gamma mv_x \qquad (34)$$

$$p_2 = m\frac{dy}{d\tau} = mu_2 = \gamma mv_y \qquad (35)$$

$$p_3 = m\frac{dz}{d\tau} = mu_3 = \gamma mv_z \qquad (36)$$

Force

$$\mathbf{F} = F_\mu = \frac{dp_\mu}{d\tau} = \left(\frac{d(mu_0)}{d\tau}, \frac{d(mu_1)}{d\tau}, \frac{d(mu_2)}{d\tau}, \frac{d(mu_3)}{d\tau}\right) \qquad (37)$$

E-3 Einstein Tensor

Since the Einstein Tensor is not affected by the presented theory, one could think that it is not useful to study it in the framework of this document. However, the knowledge of the construction of the Einstein Tensor is necessary to fully understand the four inconsistencies highlighted and solved in this document. A more accurate development of EFE can be obtained on books or on the Internet.

The Gauss Coordinates

Consider a curvilinear surface with coordinates u and v (Fig. E-1A). The distance between two points, $M(u, v)$ and $M'(u + du, v + dv)$, has been calculated by Gauss. Using the g_{ij} coefficients, this distance is:

$$ds^2 = g_{11}du^2 + g_{12}dudv + g_{21}dvdu + g_{22}dv^2 \qquad (38)$$

The Euclidean space is a particular case of the Gauss Coordinates that reproduces the Pythagorean theorem (Fig. E-1B). In this case, the Gauss coefficients are $g_{11} = 1$, $g_{12} = g_{21} = 0$, and $g_{22} = 1$.

$$ds^2 = du^2 + dv^2 \qquad (39)$$

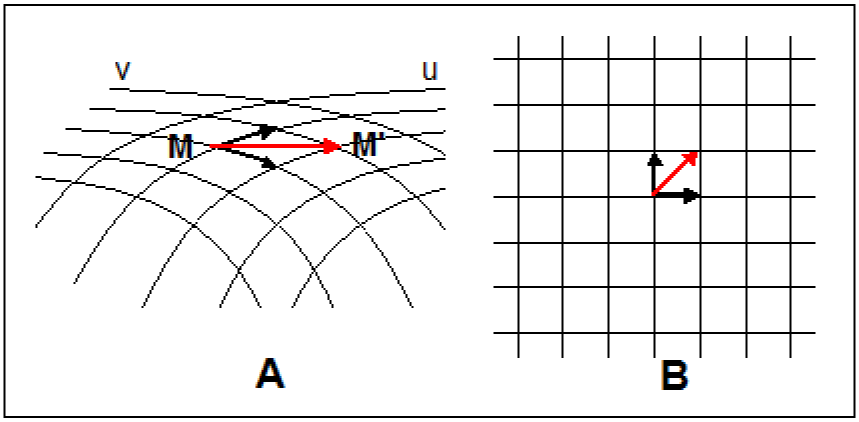

Figure E-1: Gauss coordinates in a curvilinear space (A) and in an Euclidean space (B).

Equation (39) may be condensed using the Kronecker Symbol δ, which is 0 for $i \neq j$ and 1 for $i = j$, and replacing du and dv by du_1 and du_2. For indexes $i, j = 0$ and 1, we have:

$$ds^2 = \delta_{ij} du_i du_j \qquad (40)$$

The Metric Tensor

Generalizing the Gaussian Coordinates to "n" dimensions, equation (38) can be rewritten as:

$$ds^2 = g_{\mu\nu} du_\mu du_\nu \qquad (41)$$

or, with indexes μ and ν that run from 1 to 3 (example of x, y and z coordinates):

$$ds^2 = g_{11} du_1^2 + g_{12} du_1 du_2 + g_{21} du_2 du_1 \cdots + g_{32} du_3 du_2 + g_{33} du_3^2 \qquad (42)$$

This expression is often called the "Metric" and the associated tensor, $g_{\mu\nu}$, the "Metric Tensor". In the spacetime manifold of RG, μ and ν are indexes which run from 0 to 3 (t, x, y and z). Each component can be viewed as a multiplication factor which must be placed in front of the differential displacements. Therefore, the matrix of coefficients $g_{\mu\nu}$ are a tensor 4×4, i.e. a set of 16 real-valued functions defined at all points of the spacetime manifold.

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} \qquad (43)$$

However, in order for the metric to be symmetric, we must have:

$$g_{\mu\nu} = g_{\nu\mu} \qquad (44)$$

...which reduces to 10 independent coefficients, 4 for the diagonal in bold face in equation (45), g_{00}, g_{11}, g_{22}, g_{33}, and 6 for the half part above - or under - the diagonal, i.e. $g_{01} = g_{10}$, $g_{02} = g_{20}$, $g_{03} = g_{30}$, $g_{12} = g_{21}$, $g_{13} = g_{31}$, $g_{23} = g_{32}$. This gives:

$$g_{\mu\nu} = \begin{bmatrix} \mathbf{g_{00}} & g_{01} & g_{02} & g_{03} \\ (g_{10} = g_{01}) & \mathbf{g_{11}} & g_{12} & g_{13} \\ (g_{20} = g_{02}) & (g_{21} = g_{12}) & \mathbf{g_{22}} & g_{23} \\ (g_{30} = g_{03}) & (g_{31} = g_{13}) & (g_{32} = g_{23}) & \mathbf{g_{33}} \end{bmatrix} \qquad (45)$$

To summarize, the metric tensor $g_{\mu\nu}$ in equations (43) and (45) is a matrix of functions which tells how to compute the distance between any two points in a given space. The metric components obviously depend on the chosen local coordinate system.

The Riemann Curvature Tensor

The Riemann curvature tensor $R^{\alpha}_{\beta\gamma\delta}$ is a four-index tensor. It is the most standard way to express curvature of Riemann manifolds. In spacetime, a 2-index tensor is associated to each point of a 2-index Riemannian manifold. For example, the Riemann curvature tensor represents the force experienced by a rigid body moving along a geodesic.

The Riemann tensor is the only tensor that can be constructed from the metric tensor and its first and second derivatives. These derivatives must exist if we are in a Riemann manifold. They are also necessary to keep homogeneity with the right member of EFE which can have first derivative such as the velocity dx/dt, or second derivative such as an acceleration d^2x/dt^2.

Christoffel Symbols

The Christoffel symbols are tensor-like objects derived from a Riemannian metric $g_{\mu\nu}$. They are used to study the geometry of the metric. There are two closely related kinds of Christoffel symbols, the first kind Γ_{ijk}, and the second kind Γ^k_{ij}, also known as "affine connections" or "connection coefficients".

At each point of the underlying n-dimensional manifold, the Christoffel symbols are numerical arrays of real numbers that describe, in coordinates, the effects of parallel transport in curved surfaces and, more generally, manifolds. The Christoffel symbols may be used for performing practical calculations in differential geometry. In particular, the Christoffel symbols are used in the construction of the Riemann Curvature Tensor.

In many practical problems, most components of the Christoffel symbols are equal to zero, provided the coordinate system and the metric tensor possess some common symmetries.

Comma Derivative

The following convention is often used in the writing of Christoffel Symbols. The components of the gradient dA are denoted $A_{,k}$ (a comma is placed before the index) and are given by:

$$A_{,k} = \frac{\partial A}{\partial x^k} \qquad (46)$$

Christoffel Symbols in spherical coordinates

The best way to understand the Christoffel symbols is to start with an example. Let's consider vectorial space $\mathbb{E}_3$ associated to a punctual space in spherical coordinates $\mathcal{E}_3$. A vector **OM** in a fixed Cartesian coordinate system $(0, e_i^0)$ is defined as:

$$\mathbf{OM} = x^i e_i^0 \qquad (47)$$

or

$$\mathbf{OM} = r\ sin\theta\ cos\varphi\ e_1^0 + r\ sin\theta\ sin\varphi\ e_2^0 + r\ cos\theta\ e_3^0 \qquad (48)$$

Calling e_k the evolution of **OM**, we can write:

$$e_k = \partial_k (x^i e_i^0) \qquad (49)$$

We can calculate the evolution of each vector e_k. For example, the vector e_1 (equation 50) is simply the partial derivative regarding r of equation (48). It means that the vector e_1 will be supported by a line OM oriented from *zero* to infinity. We can calculate the partial derivatives for θ and φ by the same manner. This gives for the three vectors e_1, e_2 and e_3:

$$e_1 = \partial_1 \mathbf{M} = sin\theta\ cos\varphi\ e_1^0 + sin\theta\ sin\varphi\ e_2^0 + cos\theta\ e_3^0 \qquad (50)$$

$$e_2 = \partial_2 \mathbf{M} = r\ cos\theta\ cos\varphi\ e_1^0 + r\ cos\theta\ sin\varphi\ e_2^0 - r\ sin\theta\ e_3^0 \qquad (51)$$

$$e_3 = \partial_3 \mathbf{M} = -r\ sin\theta\ sin\varphi\ e_1^0 + r\ sin\theta\ cos\varphi\ e_2^0 \qquad (52)$$

The vectors e_1^0, e_2^0 and e_3^0 are constant in module and direction. Therefore the differential of vectors e_1, e_2 and e_3 are:

$$\begin{aligned} de_1 = {} & (cos\theta\ cos\varphi\ e_1^0 + cos\theta\ sin\varphi\ e_2^0 - sin\theta\ e_3^0) d\theta \ldots \\ & \cdots + (-sin\theta\ sin\varphi\ e_1^0 + sin\theta\ cos\varphi\ e_2^0) d\varphi \end{aligned} \qquad (53)$$

$$\begin{aligned} de_2 = &(-r\ sin\theta\ cos\varphi\ e_1^0 - r\ sin\theta\ sin\varphi\ e_2^0 - r\ cos\theta\ e_3^0)d\theta \ldots \\ &\cdots + (-r\ cos\theta\ sin\varphi\ e_1^0 + r\ cos\theta\ cos\varphi\ e_2^0)d\varphi \ldots \\ &\cdots + (cos\theta\ cos\varphi\ e_1^0 + cos\theta\ sin\varphi\ e_2^0 - sin\theta\ e_3^0)dr \end{aligned} \quad (54)$$

$$\begin{aligned} de_3 = &(-r\ cos\theta\ sin\varphi\ e_1^0 + r\ cos\theta\ cos\varphi\ e_2^0)d\theta \ldots \\ &\cdots + (-r\ sin\theta\ cos\varphi\ e_1^0 - r\ sin\theta\ sin\varphi\ e_2^0)d\varphi \ldots \\ &\cdots + (-sin\theta\ sin\varphi\ e_1^0 + sin\theta\ cos\varphi\ e_2^0)dr \end{aligned} \quad (55)$$

We can remark that the terms in parenthesis are nothing but vectors e_1/r, e_2/r and e_3/r. This gives, after simplifications:

$$de_1 = (d\theta/r)e_2 + (d\varphi/r)e_3 \quad (56)$$

$$de_2 = (-r\ d\theta)e_1 + (dr/r)e_2 + (cotang\theta\ d\varphi)e_3 \quad (57)$$

$$de_3 = (-r\ sin^2\theta\ d\varphi)e_1 + (-sin\theta\ cos\theta\ d\varphi)e_2 + ((dr/r) + cotang\theta\ d\theta)e_3 \quad (58)$$

In a general manner, we can simplify the writing of this set of equation writing ω_i^j the contravariant components vectors de_i. The development of each term is given in the next section. The general expression, in 3D or more, is:

$$de_i = \omega_i^j e_j \quad (59)$$

Christoffel Symbols of the second kind

If we replace the variables r, θ and φ by u^1, u^2, and u^3 as follows:

$$u^1 = r; \quad u^2 = \theta; \quad u^3 = \varphi \quad (60)$$

...the differentials of the coordinates are:

$$du^1 = dr; \quad du^2 = d\theta; \quad du^3 = d\varphi \quad (61)$$

... and the ω_i^j components become, using the Christoffel symbol Γ_{ki}^j:

$$\omega_i^j = \Gamma_{ki}^j\ du^k \quad (62)$$

In the case of our example, quantities Γ^j_{ki} are functions of r, θ and φ. These functions can be explicitly obtained by an identification of each component of ω^j_i with Γ^j_{ki}. The full development of the precedent expressions of our example is detailed as follows:

$$\begin{cases} \omega_1^1 = 0 \\ \omega_1^2 = 1/r\ d\theta \\ \omega_1^3 = 1/r\ d\varphi \\ \omega_2^1 = -r\ d\theta \\ \omega_2^2 = 1/r\ dr \\ \omega_2^3 = cotang\ \theta\ d\varphi \\ \omega_3^1 = -r\ sin^2\theta\ d\theta \\ \omega_3^2 = -sin\theta\ cos\theta\ d\varphi \\ \omega_3^3 = 1/r\ dr + cotang\theta\ d\theta \end{cases} \tag{63}$$

Replacing dr by du^1, $d\theta$ by du^2, and $d\varphi$ by du^3 as indicated in equation (61), gives:

$$\begin{cases} \omega_1^1 = 0 \\ \omega_1^2 = 1/r\ du^2 \\ \omega_1^3 = 1/r\ du^3 \\ \omega_2^1 = -r\ du^2 \\ \omega_2^2 = 1/r\ du^1 \\ \omega_2^3 = cotang\ \theta\ du^3 \\ \omega_3^1 = -r\ sin^2\theta\ du^2 \\ \omega_3^2 = -sin\theta\ cos\theta\ du^3 \\ \omega_3^3 = 1/r\ du^1 + cotang\ \theta\ du^2 \end{cases} \tag{64}$$

On the other hand, the development of Christoffel symbols are:

$$\begin{cases} \omega_1^1 = \Gamma^1_{11}\ du^1 + \Gamma^1_{21}\ du^2 + \Gamma^1_{31}\ du^3 \\ \omega_1^2 = \Gamma^2_{11}\ du^1 + \Gamma^2_{21}\ du^2 + \Gamma^2_{31}\ du^3 \\ \omega_1^3 = \Gamma^3_{11}\ du^1 + \Gamma^3_{21}\ du^2 + \Gamma^3_{31}\ du^3 \\ \omega_2^1 = \Gamma^1_{12}\ du^1 + \Gamma^1_{22}\ du^2 + \Gamma^1_{32}\ du^3 \\ \omega_2^2 = \Gamma^2_{12}\ du^1 + \Gamma^2_{22}\ du^2 + \Gamma^2_{32}\ du^3 \\ \omega_2^3 = \Gamma^3_{12}\ du^1 + \Gamma^3_{22}\ du^2 + \Gamma^3_{32}\ du^3 \\ \omega_3^1 = \Gamma^1_{13}\ du^1 + \Gamma^1_{23}\ du^2 + \Gamma^1_{33}\ du^3 \\ \omega_3^2 = \Gamma^2_{13}\ du^1 + \Gamma^2_{23}\ du^2 + \Gamma^2_{33}\ du^3 \\ \omega_3^3 = \Gamma^3_{13}\ du^1 + \Gamma^3_{23}\ du^2 + \Gamma^3_{33}\ du^3 \end{cases} \tag{65}$$

Finally, identifying the two equations array (64) and (65) gives the 27 Christoffel Symbols.

$$\begin{array}{l|l|l}
\Gamma^1_{11} = 0 & \Gamma^1_{21} = 0 & \Gamma^1_{31} = 0 \\
\Gamma^2_{11} = 0 & \Gamma^2_{21} = 1/r & \Gamma^2_{31} = 0 \\
\Gamma^3_{11} = 0 & \Gamma^3_{21} = 0 & \Gamma^3_{31} = 1/r \\
\Gamma^1_{12} = 0 & \Gamma^1_{22} = -r & \Gamma^1_{32} = 0 \\
\Gamma^2_{12} = 0 & \Gamma^2_{22} = 0 & \Gamma^2_{32} = 0 \\
\Gamma^3_{12} = 0 & \Gamma^3_{22} = 0 & \Gamma^3_{32} = cotang\ \theta \\
\Gamma^1_{13} = 0 & \Gamma^1_{23} = -r\ sin^2\theta & \Gamma^1_{33} = 0 \\
\Gamma^2_{13} = 0 & \Gamma^2_{23} = 0 & \Gamma^2_{33} = -sin\theta\ cos\theta \\
\Gamma^3_{13} = 1/r & \Gamma^3_{23} = cotang\ \theta & \Gamma^3_{33} = 0
\end{array} \qquad (66)$$

These quantities Γ^j_{ki} are the Christoffel Symbols of the second kind. Identifying equations (59) and (62) gives the general expression of the Christoffel Symbols of the second kind:

$$\boxed{de_i = \omega^j_i e_j = \Gamma^j_{ki}\ du^k\ e_j} \qquad (67)$$

Christoffel Symbols of the first kind

We have seen in the precedent example that we can directly get the quantities Γ^j_{ki} by identification. These quantities can also be obtained from the components g_{ij} of the metric tensor. This calculus leads to another kind of Christoffel Symbols.

Lets write the covariant components, noted ω_{ji}, of the differentials de_i:

$$\omega_{ji} = e_j\ de_i \qquad (68)$$

The covariant components ω_{ji} are also linear combinations of differentials du^i that can be written as follows, using the Christoffel Symbol of the first kind Γ_{kji} :

$$\omega_{ji} = \Gamma_{kji}\ du^k \qquad (69)$$

On the other hand, we know the basic relation:

$$\omega_{ji} = g_{jl}\ \omega^l_i \qquad (70)$$

Porting equation (69) in equation (70) gives:

$$\Gamma_{kji}\ du^k = g_{jl}\ \omega^l_i \qquad (71)$$

Let's change the name of index j to l of equation (62):

$$\omega_i^l = \Gamma_{ki}^l \; du^k \qquad (72)$$

Porting equation (117) in equation (116) gives the calculus of the Christoffel Symbols of the first kind from the Christoffel Symbols of the second kind:

$$\boxed{\Gamma_{kji} \; du^k = g_{jl} \; \Gamma_{ki}^j \; du^k} \qquad (73)$$

Geodesic Equations

Let's take a curve M_0-C-M_1. If the parametric equations of the curvilinear abscissa are $u^i(s)$, the length of the curve will be:

$$l = \int_{M_0}^{M_1} \left(g_{ij} \frac{du^i}{ds} \frac{du^j}{ds} \right)^{1/2} ds \qquad (74)$$

If we pose $u'^i = du^i/ds$ and $u'^j = du^j/ds$ we get:

$$l = \int_{M_0}^{M_1} \left(g_{ij} u'^i u'^j \right)^{1/2} ds \qquad (75)$$

Here, the u'^j are the direction cosines of the unit vector supported by the tangent to the curve. Thus, we can pose:

$$f(u^k, u'^j) = g_{ij} u'^i u'^j = 1 \qquad (76)$$

The length l of the curve defined by equation (74) has a minimum and a maximum that can be calculated by the Euler-Lagrange Equation which is:

$$\frac{\partial \mathcal{L}}{\partial f_i} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial f_i'} \right) = 0 \qquad (77)$$

In the case of equation (74), the Euler-Lagrange equation gives:

$$\frac{d}{ds}(g_{ij} u'^j) - \frac{1}{2} \partial_i g_{jk} \; u'^j \; u'^k = 0 \qquad (78)$$

or

$$g_{ij}u'^j + (\partial_k g_{ij} - \frac{1}{2}\partial_i g_{jk})\ u'^j\ u'^k = 0 \qquad (79)$$

After developing derivative and using the Christoffel Symbol of the first kind, we get:

$$g_{ij}\frac{du'^j}{ds} + \Gamma_{jik}\ u'^j\ u'^k = 0 \qquad (80)$$

The contracted multiplication of equation (79) by g^{il}gives, with $g_{ij}g^{il}$ and $g^{il}\Gamma_{ijk} = \Gamma^l_{jk}$:

$$\frac{d^2u^l}{ds^2} + \Gamma^l_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 0 \qquad (81)$$

Parallel Transport

Figure E-2 shows two points M and M' infinitely close to each other in polar coordinates.

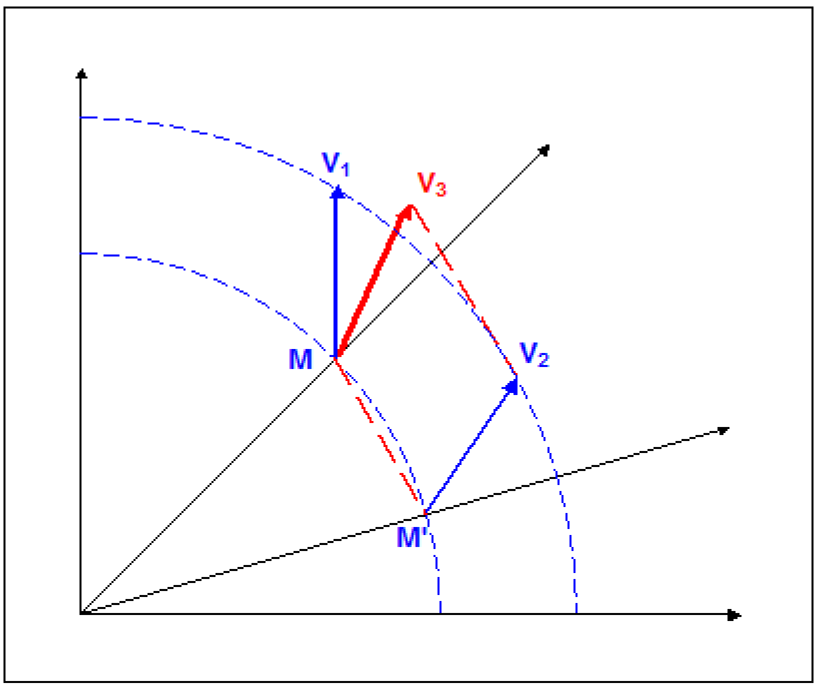

Figure E-2: Parallel transport of a vector.

In polar coordinates, the vector V_1 will become V_2. To calculate the difference between two vectors V_1 and V_2, we must before make a parallel transport of the vector V_2 from point M' to point M. This gives the vector V_3. The absolute differential is defined by:

$$dV = V_3 - V_1 \qquad (82)$$

Variations along a Geodesic

The variation along a 4D geodesic follows the same principle. For any curvilinear coordinates system y^i, we have (from equation 81):

$$\frac{d^2y^i}{ds^2} + \Gamma^i_{kj}\frac{dy^j}{ds}\frac{dy^k}{ds} = 0 \qquad (83)$$

where s is the abscissa of any point of the straight line from an origin such as M in figure E-2.

Let's consider now a vector $\vec{v}$ having covariant components v_i. We can calculate the scalar product of $\vec{v}$ and $\vec{n} = dy^k/ds$ as follows:

$$\vec{v}\cdot\vec{n} = v_i\frac{dy^i}{ds} \qquad (84)$$

During a displacement from M to M' (figure E-2), the scalar is subjected to a variation of:

$$d\left(v_i\frac{dy^i}{ds}\right) = dv_k\frac{dy^k}{ds} + v_i\ d\left(\frac{d^2y^i}{ds^2}\right) \qquad (85)$$

or:

$$d\left(v_i\frac{dy^i}{ds}\right) = dv_k\frac{dy^k}{ds} + v_i\frac{d^2y^i}{ds^2}ds \qquad (86)$$

On one hand, the differential dv_k can be written as:

$$dv_k = \partial_j v_k\frac{dy^j}{ds}ds \qquad (87)$$

On the other hand, the second derivative can be extracted from equation (83) as follows:

$$\frac{d^2y^i}{ds^2} = -\Gamma^i_{kj}\frac{dy^j}{ds}\frac{dy^k}{ds} \qquad (88)$$

Porting equations (87) and (88) in (86) gives:

$$d\left(v_i \frac{dy^i}{ds}\right) = \partial_j v_k \frac{dy^j}{ds}\frac{dy^k}{ds}ds - v_i \Gamma^i_{kj}\frac{dy^j}{ds}\frac{dy^k}{ds}ds \qquad (89)$$

or:

$$d\left(v_i \frac{dy^i}{ds}\right) = (\partial_j v_k - v_i \Gamma^i_{kj})\ \frac{dy^j}{ds}\frac{dy^k}{ds}ds \qquad (90)$$

Since $(dy^j/ds)ds = dy^j$, this expression can also be written as follows:

$$d(\vec{v}\cdot\vec{n}) = (\partial_j v_k - v_i \Gamma^i_{kj})\ dy^j \frac{dy^k}{ds} \qquad (91)$$

The absolute differentials of the covariant components of vector $\vec{v}$ are defined as:

$$D v_k \frac{dy^k}{ds} = (\partial_j v_k - v_i \Gamma^i_{kj})\ dy^j \qquad (92)$$

Finally, the quantity in parenthesis is called "affine connection" and is defined as follows:

$$\boxed{\nabla_j v_k = \partial_j v_k - v_i \Gamma^i_{kj}} \qquad (93)$$

Some countries in the world use ";" for the covariant derivative and "," for the partial derivative. Using this convention, equation (138) can be written as:

$$v_{k;j} = v_{k,j} - v_i \Gamma^i_{kj} \qquad (94)$$

To summarize, given a function f, the covariant derivative $\nabla_v f$ coincides with the normal differentiation of a real function in the direction of the vector $\vec{v}$, usually denoted by $\vec{v}f$ and $df(\vec{v})$.

Second Covariant Derivatives of a Vector

Remembering that the derivative of the product of two functions is the sum of partial derivatives, we have:

$$\nabla_a(t_b r_c) = r_c\ .\ \nabla_a t_b + t_b\ .\ \nabla_a r_c \qquad (95)$$

Porting equation (93) in equation (95) gives:

$$\nabla_a(t_b r_c) = r_c\ (\partial_a t_b - t_l \Gamma^l_{ab}) + t_b\ (\partial_a r_c - r_l \Gamma^l_{ac}) \qquad (96)$$

or:

$$\nabla_a(t_b r_c) = r_c \partial_a t_b - r_c t_l \Gamma^l_{ab} + t_b \partial_a r_c - t_b r_l \Gamma^l_{ac} \qquad (97)$$

Hence:

$$\nabla_a(t_b r_c) = r_c \partial_a t_b + t_b \partial_a r_c - r_c t_l \Gamma^l_{ab} - t_b r_l \Gamma^l_{ac} \qquad (98)$$

Finally:

$$\nabla_a(t_b r_c) = \partial_a(t_b r_c) - r_c t_l \Gamma^l_{ab} - t_b r_l \Gamma^l_{ac} \qquad (99)$$

Posing $t_b r_c = \nabla_j v_i$ gives:

$$\boxed{\nabla_k(\nabla_j v_i) = \partial_k(\nabla_j v_i) - (\nabla_j v_r)\Gamma^r_{ik} - (\nabla_r v_i)\Gamma^r_{jk}} \qquad (100)$$

Porting equation (93) in equation (100) gives:

$$\nabla_k(\nabla_j v_i) = \partial_k(\partial_j v_i - v_l \Gamma^l_{ji}) - (\partial_j v_r - v_l \Gamma^l_{jr})\Gamma^r_{ik} - (\partial_r v_i - v_l \Gamma^l_{ri})\Gamma^r_{jk} \qquad (101)$$

Hence:

$$\nabla_k(\nabla_j v_i) = \partial_{kj} v_i - (\partial_k \Gamma^l_{ji}) v_l - \Gamma^l_{ji} \partial_k v_l - \Gamma^r_{ik} \partial_j v_r + \Gamma^r_{ik} \Gamma^l_{jr} v_l - \Gamma^r_{jk} \partial_r v_i + \Gamma^r_{jk} \Gamma^l_{ri} v_l \qquad (102)$$

The Riemann-Christoffel Tensor

In expression (102), if we make a swapping between the indexes j and k in order to get a differential on another way (i.e. a parallel transport) we get:

$$\nabla_j(\nabla_k v_i) = \partial_{jk} v_i - (\partial_j \Gamma^l_{ki}) v_l - \Gamma^l_{ik} \partial_j v_l - \Gamma^r_{ij} \partial_k v_r + \Gamma^r_{ij} \Gamma^l_{kr} v_l - \Gamma^r_{kj} \partial_r v_i + \Gamma^r_{kj} \Gamma^l_{ri} v_l \qquad (103)$$

A subtraction between expressions (102) and (103) gives, after a rearrangement of some terms:

$$\nabla_k(\nabla_j v_i) - \nabla_j(\nabla_k v_i) = (\partial_{kj} - \partial_{jk}) v_i + (\partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji}) v_l + (\Gamma^l_{ik} \partial_j - \Gamma^l_{ji} \partial_k) v_l \ldots$$
$$\cdots + (\Gamma^r_{ij} \partial_k - \Gamma^r_{ik} \partial_j) v_r + (\Gamma^r_{ik} \Gamma^l_{jr} - \Gamma^r_{ij} \Gamma^l_{kr}) v_l + (\Gamma^r_{kj} - \Gamma^r_{jk}) \partial_r v_i + (\Gamma^r_{jk} \Gamma^l_{ri} - \Gamma^r_{kj} \Gamma^l_{ri}) v_l \quad (104)$$

On the other hand, since we have:

$$\Gamma^r_{jk} = \Gamma^r_{kj} \qquad (105)$$

Some terms of equation (104) are canceled:

$$\partial_{kj} - \partial_{jk} = 0 \qquad (106)$$

$$\Gamma^r_{kj} - \Gamma^r_{jk} = 0 \qquad (107)$$

$$\Gamma^r_{jk}\Gamma^l_{ri} - \Gamma^r_{kj}\Gamma^l_{ri} = 0 \qquad (108)$$

And consequently:

$$\nabla_k(\nabla_j v_i) - \nabla_j(\nabla_k v_i) = (\partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji})v_l + (\Gamma^l_{ik}\partial_j - \Gamma^l_{ji}\partial_k)v_l \dots$$
$$\dots + (\Gamma^r_{ij}\partial_k - \Gamma^r_{ik}\partial_j)v_r + (\Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{ij}\Gamma^l_{kr})v_l \quad (109)$$

Since the parallel transport is done on small portions of geodesics infinitely close to each other, we can take the limit:

$$\partial_j v_l, \quad \partial_k v_l, \quad \partial_j v_r, \quad \partial_k v_r \;\to\; 0 \qquad (110)$$

This means that the velocity field is considered equal in two points of two geodesics infinitely close to each other. Then we can write:

$$\nabla_k(\nabla_j v_i) - \nabla_j(\nabla_k v_i) \cong (\partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji} + \Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{ij}\Gamma^l_{kr})v_l \quad (111)$$

As a result of the tensorial properties of covariant derivatives and of the components v_l, the quantity in parenthesis is a four-order tensor defined as:

$$\boxed{R^l_{i,jk} = \partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji} + \Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{ij}\Gamma^l_{kr}} \qquad (112)$$

In this expression, the comma in Christoffel Symbols means a partial derivative. The tensor $R^l_{i,jk}$ is called *Riemann-Christoffel Tensor* or *Curvature Tensor* which characterizes the curvature of a Riemann Space.

The Ricci Tensor

The contraction of the Riemann-Christoffel Tensor $R^l_{i,jk}$ defined by equation (112) relative to indexes l and j leads to a new tensor:

$$\boxed{R_{ik} = R^{l}_{i,lk} = \partial_l \Gamma^{l}_{ki} - \partial_k \Gamma^{l}_{li} + \Gamma^{r}_{ik}\Gamma^{l}_{lr} - \Gamma^{r}_{il}\Gamma^{l}_{kr}} \qquad (113)$$

This tensor R_{ik} is called the *"Ricci Tensor"*. Its mixed components are given by:

$$R^{jk} = g^{ji} R_{ik} \quad (114)$$

The Scalar Curvature

The *Scalar Curvature*, also called the *"Curvature Scalar"* or *"Ricci Scalar"*, is given by:

$$\boxed{R = R^{i}_{i} = g^{ij} R_{ij}} \qquad (115)$$

The Bianchi Second Identities

The Riemann-Christoffel tensor verifies a particular differential identity called the *"Bianchi Identity"*. This identity involves that the Einstein Tensor has a null divergence, which leads to a constraint. The goal is to reduce the degrees of freedom of the Einstein Equations. To calculate the second Bianchi Identities, we must derivate the Riemann-Christoffel Tensor defined in equation (112):

$$\nabla_t R^{l}_{i,rs} = \partial_{rt} \Gamma^{l}_{si} - \partial_{st} \Gamma^{l}_{ri} \qquad (116)$$

A circular permutation of indexes r, s and t gives:

$$\nabla_r R^{l}_{i,st} = \partial_{sr} \Gamma^{l}_{ti} - \partial_{tr} \Gamma^{l}_{si} \qquad (117)$$

$$\nabla_s R^{l}_{i,tr} = \partial_{ts} \Gamma^{l}_{ri} - \partial_{rs} \Gamma^{l}_{ti} \qquad (118)$$

Since the derivation order is interchangeable, adding equations (116), (117) and (118) gives:

$$\nabla_t R^{l}_{i,rs} + \nabla_r R^{l}_{i,st} + \nabla_s R^{l}_{i,tr} = 0 \qquad (119)$$

The Einstein Tensor

If we make a contraction of the second Bianchi Identities (equation 119) for $t = l$, we get:

$$\nabla_l R^{l}_{i,rs} + \nabla_r R^{l}_{i,sl} + \nabla_s R^{l}_{i,lr} = 0 \qquad (120)$$

Hence, taking into account and the definition of the Ricci Tensor of equation (113) and that $R^l_{i,sl} = -R^l_{i,ls}$, we get:

$$\nabla_l R^l_{i,rs} + \nabla_s R_{ir} - \nabla_r R_{is} = 0 \qquad (121)$$

The variance change with g_{ij} gives:

$$\nabla_s R_{ir} = \nabla_s (g_{ik} R^k_r) \qquad (122)$$

or

$$\nabla_s R_{ir} = g_{ik} \nabla_s R^k_r \qquad (123)$$

Multiplying equation (121) by g^{ik} gives:

$$g^{ik} \nabla_l R^l_{i,rs} + g^{ik} \nabla_s R_{ir} - g^{ik} \nabla_r R_{is} = 0 \qquad (124)$$

Using the property of equation (123), we finally get:

$$\nabla_l R^{kl}_{,rs} + \nabla_s R^k_r - \nabla_r R^k_s = 0 \qquad (125)$$

Let's make a contraction on indexes k and s:

$$\nabla_k R^{kl}_{,rk} + \nabla_k R^k_r - \nabla_r R^k_k = 0 \qquad (126)$$

The first term becomes:

$$\nabla_k R^k_r + \nabla_k R^k_r - \nabla_r R^k_k = 0 \qquad (127)$$

After a contraction of the third term we get:

$$2 \nabla_k R^k_r - \nabla_r R = 0 \qquad (128)$$

Dividing this expression by two gives:

$$\nabla_k R^k_r - \frac{1}{2} \nabla_r R = 0 \qquad (129)$$

or:

$$\nabla_k \left(R^k_r - \frac{1}{2}\delta^k_r R \right) = 0 \qquad (130)$$

A new tensor may be written as follows:

$$G^k_r = R^k_r - \frac{1}{2}\delta^k_r R \qquad (131)$$

The covariant components of this tensor are:

$$G_{ij} = g_{ik}\ G^k_j \qquad (132)$$

or

$$G_{ij} = g_{ik} \left(R^k_j - \frac{1}{2}\delta^k_j R \right) \qquad (133)$$

Finally get the Einstein Tensor which is defined by:

$$\boxed{G_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R} \qquad (134)$$

The Einstein Constant

The Einstein Tensor G_{ij} of equation (134) must match the Energy-Momentum Tensor T_{uv} defined later. This can be done with a constant κ so that:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \qquad (135)$$

This constant κ is called *"Einstein Constant"* or *"Constant of Proportionality"*. To calculate it, the Einstein Equation (134) must be identified to the Poisson's classical field equation, which is the mathematical form of the Newton Law. So, the weak field approximation is used to calculate the Einstein Constant. Three criteria are used to get this "Newtonian Limit":

1 - The speed is low regarding that of the light c.
2 - The gravitational field is static.
3 - The gravitational field is weak and can be seen as a weak perturbation $h_{\mu\nu}$ added to a flat spacetime $\eta_{\mu\nu}$ as follows:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (136)$$

We start with the equation of geodesics (83):

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \qquad (137)$$

This equation can be simplified in accordance with the first condition:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{00}\left(\frac{dx^0}{ds}\right)^2 = 0 \qquad (138)$$

The two other conditions lead to a simplification of Christoffel Symbols of the second kind as follows:

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g00) \qquad (139)$$

Or, considering the second condition:

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda\ g_{00} \qquad (140)$$

And also considering the third condition:

$$\Gamma^\mu_{00} \approx -\frac{1}{2}(\eta^{\mu\lambda} + h^{\mu\lambda})(\partial_\lambda\eta_{00} + \partial_\lambda h_{00}) \qquad (141)$$

In accordance with the third condition, the term $\partial_\lambda\eta_{00}$ is canceled since it is a flat space:

$$\Gamma^\mu_{00} \approx -\frac{1}{2}(\eta^{\mu\lambda} + h^{\mu\lambda})\partial_\lambda h_{00} \qquad (142)$$

Another simplification due to the approximation gives:

$$\Gamma^\mu_{00} \approx -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \qquad (143)$$

The equation of geodesics then becomes:

$$\frac{d^2x^\mu}{dt^2} - \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dx^0}{dt}\right)^2 = 0 \qquad (144)$$

Reduced to the time component ($\mu = 0$), equation (144) becomes:

$$\frac{d^2x^\mu}{dt^2} - \frac{1}{2}\eta^{0\lambda}\partial_\lambda h_{00}\left(\frac{dx^0}{dt}\right)^2 = 0 \qquad (145)$$

The Minkowski Metric shows that $\eta_{0\lambda} = 0$ for $\lambda > 0$. On the other hand, a static metric (third condition) gives $\partial_0 h_{00} = 0$ for $\lambda = 0$. So, the 3x3 matrix leads to:

$$\frac{d^2x^i}{dt^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dx^0}{dt}\right)^2 = 0 \qquad (146)$$

Replacing dt by the proper time $d\tau$ gives:

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0 \qquad (147)$$

Dividing by $(dx_0/d\tau)^2$ leads to:

$$\frac{d^2x^i}{d\tau^2}\left(\frac{d\tau}{dx^0}\right)^2 = \frac{1}{2}\partial_i h_{00} \qquad (148)$$

$$\frac{d^2x^i}{(dx^0)^2} = \frac{1}{2}\partial_i h_{00} \qquad (149)$$

Replacing x^0 by ct gives:

$$\frac{d^2x^i}{d(ct)^2} = \frac{1}{2}\partial_i h_{00} \qquad (150)$$

or:

$$\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\partial_i h_{00} \qquad (151)$$

Let us pose:

$$h_{00} = -\frac{2}{c^2}\Phi \qquad (152)$$

or

$$\triangle h_{00} = -\frac{2}{c^2}\triangle\Phi \qquad (153)$$

Since the approximation is in an euclidean space, the Laplace operator can be written as:

$$\triangle h_{00} = -\frac{2}{c^2}\nabla^2\Phi \qquad (154)$$

$$\triangle h_{00} = -\frac{2}{c^2}\left(4\pi G_0\rho\right) \qquad (155)$$

$$\triangle h_{00} = -\frac{8\pi G_0}{c^2}\rho \qquad (156)$$

On the other hand, the element T_{00} defined later is:

$$T_{00} = \rho c^2 \qquad (157)$$

or

$$\rho = \frac{T_{00}}{c^2} \qquad (158)$$

Porting equation (158) in equation (156) gives:

$$\triangle h_{00} = -\frac{8\pi G_0}{c^2}\frac{T_{00}}{c^2} \qquad (159)$$

or:

$$\triangle h_{00} = -\frac{8\pi G_0}{c^4}T_{00} \qquad (160)$$

The left member of equation (160) is the perturbation part of the Einstein Tensor in the case of a static and weak field approximation. It directly gives a constant of proportionality which also verifies the homogeneity of the EFE (equation 1). This equation will be fully explained later in this document. Thus:

$$\boxed{Einstein\ Constant = \frac{8\pi G_0}{c^4}} \qquad (161)$$

E-4 Movement Equations in a Newtonian Fluid

The figure E-3 (next page) shows an elementary parallelepiped of dimensions dx, dy, dz which is a part of a fluid in static equilibrium. This cube is generally subject to volume forces in all directions, as the Pascal Theorem states. The components of these forces are oriented in the three orthogonal axis. The six sides of the cube are: A-A', B-B' and C-C'.

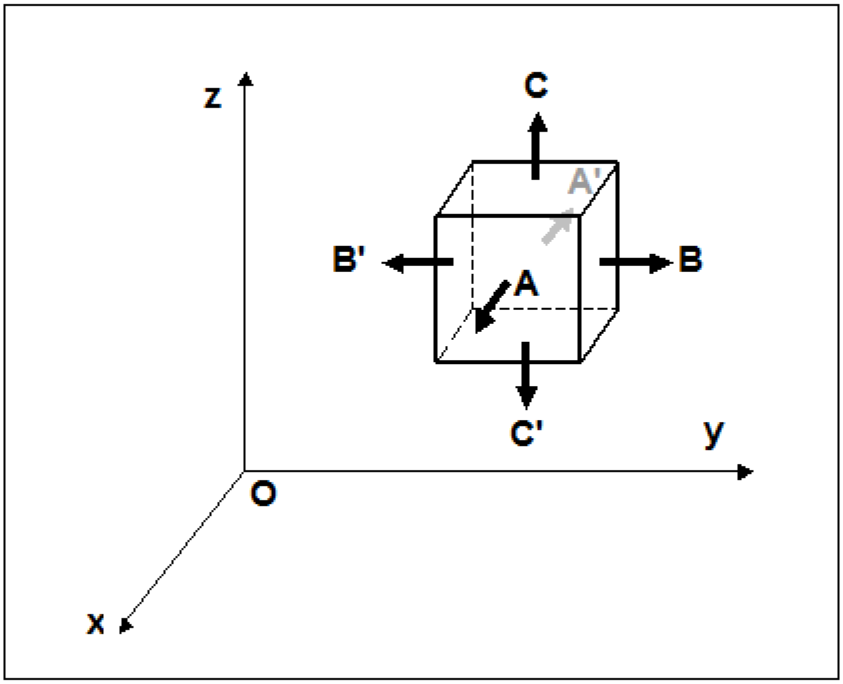

Figure E-3: Elementary parallelepiped dx, dy, dz

E-5 Normal Constraints

On figure E-4 (next page), the normal constraints to each surface are noted "σ". The tangential constraints to each surface are noted "τ". Since we have six sides, we have six sets of equations. In the following equations, 'σ" and "τ' are constraints (a constraint is a pressure), dF is an elementary force, and dS is an elementary surface:

$$\frac{d\vec{F}_{+x}}{dS} = \vec{\sigma}_x + \vec{\tau}'_{yx} + \vec{\tau}_{xz} \qquad (162)$$

$$\frac{d\vec{F}_{-x}}{dS} = \vec{\sigma}'_x + \vec{\tau}'_{xy} + \vec{\tau}_{zx} \qquad (163)$$

$$\frac{d\vec{F}_{+y}}{dS} = \vec{\sigma}_y + \vec{\tau}_{yz} + \vec{\tau}''_{yx} \qquad (164)$$

$$\frac{d\vec{F}_{-y}}{dS} = \vec{\sigma}'_y + \vec{\tau}_{zy} + \vec{\tau}''_{xy} \qquad (165)$$

$$\frac{d\vec{F}_{+z}}{dS} = \vec{\sigma}_z + \vec{\tau}''_{xz} + \vec{\tau}'_{yz} \qquad (166)$$

$$\frac{d\vec{F}_{-z}}{dS} = \vec{\sigma}'_z + \vec{\tau}''_{zx} + \vec{\tau}'_{zy} \qquad (167)$$

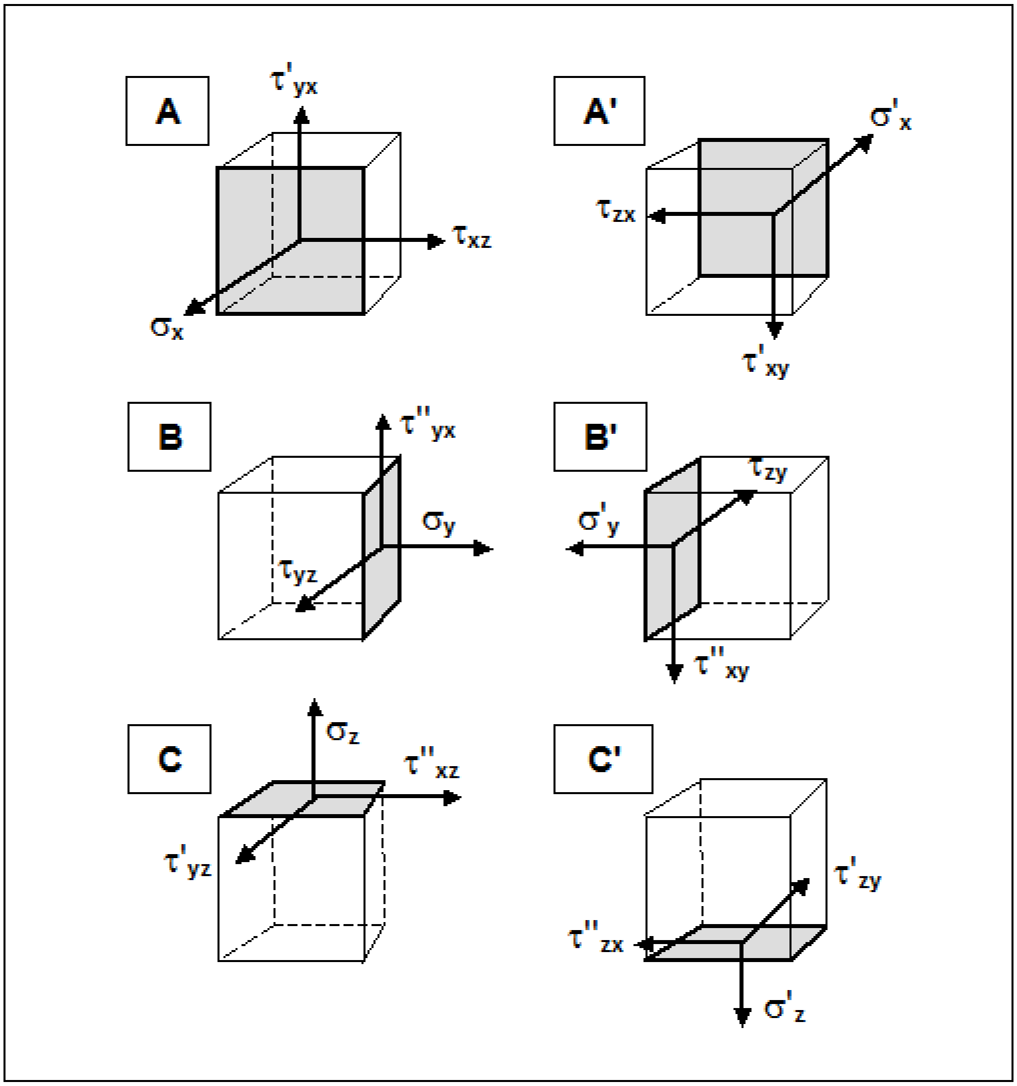

Figure E-4: Forces on the elementary parallelepiped sides.

We can simplify these equations as follows:

$$\vec{\sigma}_x + \vec{\sigma}'_x = \vec{\sigma}_X \qquad (168)$$

$$\vec{\sigma}_y + \vec{\sigma}'_y = \vec{\sigma}_Y \qquad (169)$$

$$\vec{\sigma}_z + \vec{\sigma}'_z = \vec{\sigma}_Z \qquad (170)$$

So, only three components are used to define the normal constraint forces, i.e. one per axis.

E-6 Tangential Constraints

If we calculate the force's momentum regarding the gravity center of the parallelepiped, we have 12 tangential components (two per side). Since some forces are in opposition to each other, only 6 are sufficient to describe the system. Here, we calculate the three momenta for each plan, XOY, XOZ and YOZ, passing through the gravity center of the elementary parallelepiped:

For the XOY plan:

$$\mathcal{M}_{XOY} = (\vec{\tau}_{zy}dzdx)\frac{dy}{2} + (\vec{\tau}_{zx}dydz)\frac{dx}{2} + (\vec{\tau}_{yz}dxdz)\frac{dy}{2} + (\vec{\tau}_{xz}dzdy)\frac{dx}{2} \qquad (171)$$

$$\mathcal{M}_{XOY} = \frac{1}{2}\left[(\vec{\tau}_{zy} + \vec{\tau}_{yz}) + (\vec{\tau}_{zx} + \vec{\tau}_{xz})\right] \qquad (172)$$

$$\mathcal{M}_{XOY} = \frac{1}{2}dV\left[(\vec{\tau}_{ZY} + \vec{\tau}_{ZX})\right] \qquad (173)$$

$$\mathcal{M}_{XOY} = \frac{1}{2}dV\vec{\tau}_{XOY} \qquad (174)$$

For the XOZ plan:

$$\mathcal{M}_{XOZ} = (\vec{\tau}'_{yx}dydz)\frac{dx}{2} + (\vec{\tau}'_{yz}dxdy)\frac{dz}{2} + (\vec{\tau}'_{xy}dzdy)\frac{dx}{2} + (\vec{\tau}'_{zy}dydx)\frac{dz}{2} \qquad (175)$$

$$\mathcal{M}_{XOZ} = \frac{1}{2}dV\left[(\vec{\tau}'_{yx} + \vec{\tau}'_{xy}) + (\vec{\tau}'_{yz} + \vec{\tau}'_{zy})\right] \qquad (176)$$

$$\mathcal{M}_{XOZ} = \frac{1}{2}dV\left[(\vec{\tau}_{YX} + \vec{\tau}_{YZ})\right] \qquad (177)$$

$$\mathcal{M}_{XOZ} = \frac{1}{2}dV\vec{\tau}_{XOZ} \qquad (178)$$

For the ZOY plan:

$$\mathcal{M}_{ZOY} = (\vec{\tau}''_{xz}dxdy)\frac{dz}{2} + (\vec{\tau}''_{yx}dxdz)\frac{dy}{2} + (\vec{\tau}''_{zx}dydx)\frac{dz}{2} + (\vec{\tau}''_{xy}dzdx)\frac{dy}{2} \qquad (179)$$

$$\mathcal{M}_{ZOY} = \frac{1}{2}dV\left[(\vec{\tau}''_{xz} + \vec{\tau}''_{zx}) + (\vec{\tau}''_{yx} + \vec{\tau}''_{xy})\right] \qquad (180)$$

$$\mathcal{M}_{ZOY} = \frac{1}{2}dV\left[(\vec{\tau}_{ZX} + \vec{\tau}_{XY})\right] \qquad (181)$$

$$\mathcal{M}_{ZOY} = \frac{1}{2}dV\vec{\tau}_{ZOY} \qquad (182)$$

So, for each plan, only one component is necessary to define the set of momentum forces. Since the elementary volume dV is equal to a*a*a (a = dx, dy or dz), we can come back to the constraint equations dividing each result (174), (178) and (182) by dV/2.

E-7 Constraint Tensor

Finally, the normal and tangential constraints can be reduced to only 6 terms with $\vec{\tau}_{xy} = \vec{\tau}_{yx}$, $\vec{\tau}_{xz} = \vec{\tau}_{zx}$, and $\vec{\tau}_{zy} = \vec{\tau}_{yz}$:

$$\vec{\sigma}_X = \vec{\sigma}_{xx} \tag{183}$$
$$\vec{\sigma}_Y = \vec{\sigma}_{yy} \tag{184}$$
$$\vec{\sigma}_Z = \vec{\sigma}_{zz} \tag{185}$$
$$\vec{\tau}_{XOY} = \vec{\tau}_{xy} \tag{186}$$
$$\vec{\tau}_{XOZ} = \vec{\tau}_{xz} \tag{187}$$
$$\vec{\tau}_{ZOY} = \vec{\tau}_{zy} \tag{188}$$

Using a matrix representation, we get:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} \tag{189}$$

The constraint tensor at point M becomes:

$$T_{(M)} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \tag{190}$$

This tensor is symmetric and its meaning is shown in Figure E-5.

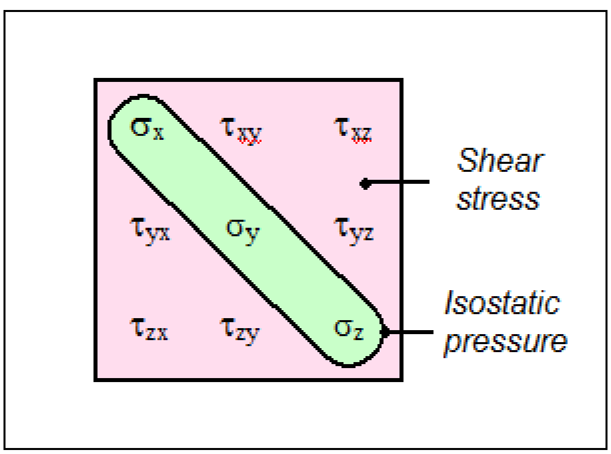

Figure E-5: Meaning of the Constraint Tensor.

Since all the components of the tensor are pressures (more exactly constraints), we can represent it by the following equation where F_i are forces and s_i are surfaces:

$$\boxed{F_i = \sigma_{ij} s_j = \sum_j \sigma_{ij} s_j} \qquad (191)$$

E-8 Energy-Momentum Tensor

We can write equation (191) as:

$$\sigma_{ij} = \frac{F_i}{s_j} = \frac{\Delta F_i}{\Delta s_j} = \frac{\Delta(m a_i)}{\Delta s_j} \qquad (192)$$

Here we suppose that only volumes (x, y and z) and time (t) make the force vary. Therefore, the mass (m) can be replaced by volume (V) using a constant density (ρ):

$$m = \rho V \qquad (193)$$

Porting (193) in (192) gives:

$$\sigma_{ij} = \frac{\Delta(m.a_i)}{\Delta s_j} = \frac{\Delta(\rho V.a_i)}{\Delta s_j} = \frac{\rho V}{\Delta s_j} \Delta a_i \qquad (194)$$

As shown in figure E-3, V and S concern an elementary parallelepiped.

$$V = (\Delta X_j)^3 \quad and \quad S_j = (\Delta X_j)^2 \qquad (195)$$

Thus:

$$\frac{V}{S_j} = \frac{(\Delta X_j)^3}{(\Delta X_j)^2} = \Delta X_j \qquad (196)$$

Hence

$$\sigma_{ij} = \rho \ \Delta X_j \Delta a_i \qquad (197)$$

$$\sigma_{ij} = \rho \Delta X_j \frac{v_i}{\Delta t} = \rho \frac{\Delta X_j}{\Delta t} v_i \qquad (198)$$

Finally

$$\sigma_{ij} = \rho v_j v_i \qquad (199)$$

This tensor comes from the Fluid Mechanics and uses traditional variables v_x, v_y and v_z. We can extend these 3D variables to 4D in accordance with Special Relativity (see above). The new 4D tensor created, called the "Energy-Momentum Tensor', has the same properties as the old one, in particular symmetry.

To avoid confusion, let's replace σ_{ij} by $T_{\mu\nu}$ as follows:

$$\boxed{T_{\mu\nu} = \rho u_\mu u_\nu} \qquad (200)$$

or

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \rho u_0 u_0 & \rho u_0 u_1 & \rho u_0 u_2 & \rho u_0 u_3 \\ \rho u_1 u_0 & \rho u_1 u_1 & \rho u_1 u_2 & \rho u_1 u_3 \\ \rho u_2 u_0 & \rho u_2 u_1 & \rho u_2 u_2 & \rho u_2 u_3 \\ \rho u_3 u_0 & \rho u_3 u_1 & \rho u_3 u_2 & \rho u_3 u_3 \end{bmatrix} \qquad (201)$$

...with u_μ and u_ν as defined in equations (25) to (28).

This tensor may be written in a more explicit form, using the "traditional" velocity v_x, v_y and v_z instead of the relativistic velocities, as shown in equations (25) to (28):

$$T_{\mu\nu} = \begin{bmatrix} \rho\gamma^2 c^2 & \rho\gamma^2 c v_x & \rho\gamma^2 c v_y & \rho\gamma^2 c v_z \\ \rho\gamma^2 c v_x & \rho\gamma^2 v_x v_x & \rho\gamma^2 v_x v_y & \rho\gamma^2 v_x v_z \\ \rho\gamma^2 c v_y & \rho\gamma^2 v_y v_x & \rho\gamma^2 v_y v_y & \rho\gamma^2 v_y v_z \\ \rho\gamma^2 c v_z & \rho\gamma^2 v_z v_x & \rho\gamma^2 v_z v_y & \rho\gamma^2 v_z v_z \end{bmatrix} \qquad (202)$$

For low velocities, we have $\gamma = 1$:

$$T_{\mu\nu} = \begin{bmatrix} \rho c^2 & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \rho v_x v_x & \rho v_x v_y & \rho v_x v_z \\ \rho c v_y & \rho v_y v_x & \rho v_y v_y & \rho v_y v_z \\ \rho c v_z & \rho v_z v_x & \rho v_z v_y & \rho v_z v_z \end{bmatrix} \qquad (203)$$

Replacing the spatial part of this tensor by the old definitions (equation 189) gives:

$$T_{\mu\nu} = \begin{bmatrix} \rho c^2 & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \sigma_x & \tau_{xy} & \tau_{xz} \\ \rho c v_y & \tau_{yx} & \sigma_y & \tau_{yz} \\ \rho c v_z & \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad (204)$$

Replacing ρ by mc^2 (equation 193) and mc^2 by E leads to one of the most commons form of the Energy-Momentum Tensor, V being the volume:

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} E/V & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \sigma_x & \tau_{xy} & \tau_{xz} \\ \rho c v_y & \tau_{yx} & \sigma_y & \tau_{yz} \\ \rho c v_z & \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad (205)$$

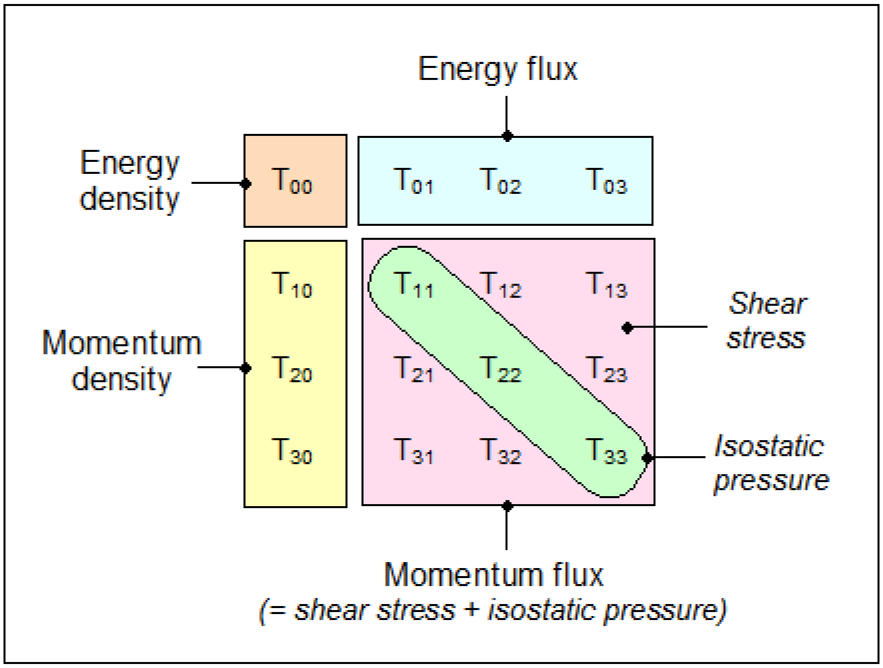

Figure E-6: Meaning of the Energy-Momentum Tensor.

E-9 Einstein Field Equations

Finally, the association of the Einstein Tensor $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R$ (equation 134), the Einstein Constant $8\pi G_0/c^4$ (equation 161), and the Energy-Momentum Tensor $T_{\mu\nu}$ (equations 204/205), gives the full Einstein Field Equations, excluding the cosmological constant Λ which is not proven:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_0}{c^4} T_{\mu\nu}} \qquad (206)$$

To summarize, the Einstein field equations are 16 nonlinear partial differential equations that describe the curvature of spacetime, i.e. the gravitational field, produced by a given mass. As a result of the symmetry of $G_{\mu\nu}$ and $T_{\mu\nu}$, the actual number of equations are reduced to 10, although there are an additional four differential identities (the Bianchi identities) satisfied by $G_{\mu\nu}$, one for each coordinate.

E-10 Dimensional Analysis

The dimensional analysis verifies the homogeneity of equations. The most common dimensional quantities used in this chapter are:

Speed V $\Rightarrow [L/T]$
Energy E $\Rightarrow [ML^2/T^2]$
Force F $\Rightarrow [ML/T^2]$
Pressure P $\Rightarrow [M/LT^2]$
Momentum M $\Rightarrow [ML/T]$
Gravitation constant G $\Rightarrow [L^3/MT^2]$

Here are the dimensional analysis of the Energy-Momentum Tensor:

$\mathbf{T_{00}}$ is the energy density, i.e. the amount of energy stored in a given region of space per unit volume. The dimensional quantity of E is $[ML^2/T^2]$ and V is $[L^3]$. So, the dimensional quantity of E/V is $[M/LT^2]$. Energy density has the same physical units as pressure which is $[M/LT^2]$.

$\mathbf{T_{01}}, \mathbf{T_{02}}, \mathbf{T_{03}}$ are the energy flux, i.e. the rate of transfer of energy through a surface. The quantity is defined in different ways, depending on the context. Here, ρ is the density $[M/V]$, and c and v_i (i = 1 to 3) are velocities $[L/T]$. So, the dimensional quantity of T_{0i} is $[M/L^3][L/T][L/T]$. It is that of a pressure $[M/LT^2]$.

$\mathbf{T_{10}}, \mathbf{T_{20}}, \mathbf{T_{30}}$ are the momentum density, which is the momentum per unit volume. The dimensional quantity of T_{i0} (i = 1 to 3) is identical to T_{0i}, i.e. that of a pressure $[M/LT^2]$.

$\mathbf{T_{12}}, \mathbf{T_{13}}, \mathbf{T_{23}}, \mathbf{T_{21}}, \mathbf{T_{31}}, \mathbf{T_{32}}$ are the shear stress, or a pressure $[M/LT^2]$.

$\mathbf{T_{11}}, \mathbf{T_{22}}, \mathbf{T_{33}}$ are the normal stress or isostatic pressure $[M/LT^2]$.

Note: The Momentum flux is the sum of the shear stresses and the normal stresses.

As we see, all the components of the Energy-Momentum Tensor have a pressure-like dimensional quantity:

$$\boxed{T_{\mu\nu} \Rightarrow \text{Pressure } [M/LT^2]} \qquad (207)$$

E-11 Inconsistencies of EFE

Here we have demonstrated that the Energy-Momentum Tensor is nothing but the extension in spacetime of the Stress Tensor of the Fluid Mechanics. We also have demonstrated that all elements of the Energy-Momentum Tensor have a pressure-like dimension. This is not abnormal since the Energy-Momentum Tensor is built on the Stress Tensor, which describes pressures. This is why the Energy-Momentum Tensor is often called "Stress-Energy-Momentum Tensor" and sometimes is also used in the Fluid Mechanics. The generalization of the Stress Tensor to the Energy-Momentum Tensor is shown in figure E-7 (next page). The related mathematics have been described in the precedent sections.

The Stress Tensor $\mathbb{P}$ in Fluid Mechanics is a part of the NavierStokes equations that describe the motion of fluid substances such as liquids and gases: $\rho Dv/Dt = \nabla \cdot \mathbb{P} + \rho \mathbf{f}$. This tensor represents the external pressures, more exactly normal and tangential constraints, acting on the fluid, as shown in figure E-8 (next page). For pedagogical purposes, only the upper normal pressure is shown on this figure. Other pressures, as defined in figure E-4, are not represented.

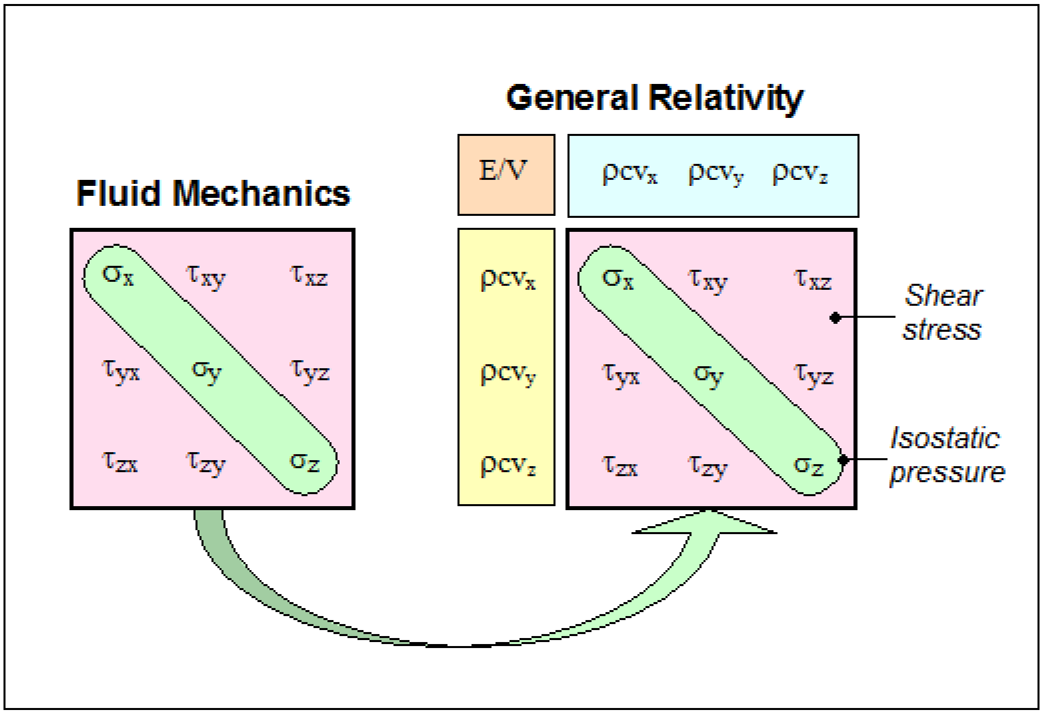

Figure E-7: Stress Tensor vs. Energy-Momentum Tensor.

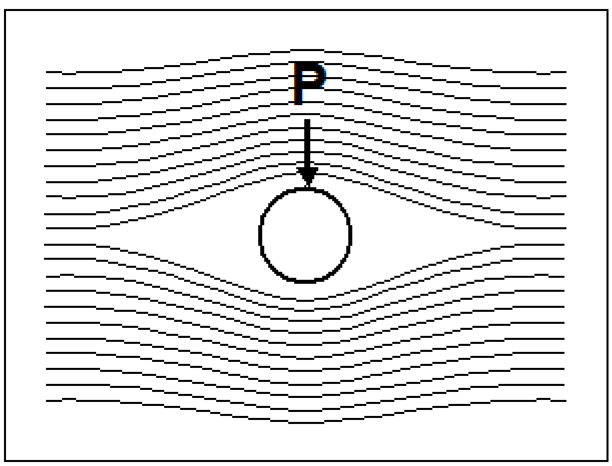

Figure E-8: External pressures on a volume.

QUESTION #1

Since the Energy-Momentum Tensor represents a set of pressures made by spacetime on objects, and since the normal component (i.e. gravity) is an isostatic pressure, why is gravitation considered an ATTRACTIVE force instead of a PRESSURE force?

QUESTION #2
Since the Fluid Mechanics exerts a pressure on the VOLUME of the object, why has EFE replaced the volume by MASS?

QUESTION #3
In GR, the curvature of spacetime is supposed to be concave, as shown by Figure E-9. This figure is taken from one of the thousands of graphics representing the curvature of spacetime. On the contrary, as shown on Figure E-8, the curvature of the fluid in the Fluid Mechanics is convex. So, why has GR replaced the CONVEX curvature by a CONCAVE curvature?

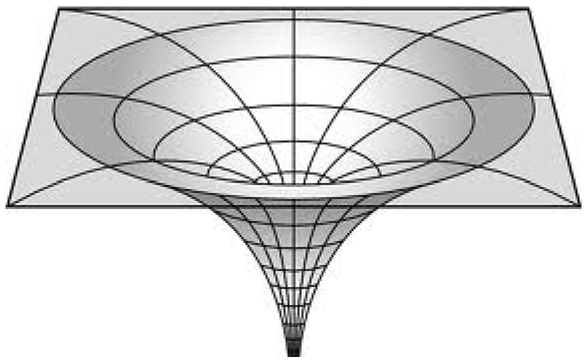

Figure E-9: Curvature of spacetime by a black hole.

QUESTION #4
What is mass? To date, no one knows. So, we are faced with another inconsistency due to an unknown variable, "m", present in the right side of EFE (equation 193) but not in the left side. This leads to the following alternative:

1/ The mass variable "m" is not defined in 4D. In this case, we need an additional dimension to define it. In other words, the 4D Energy-Momentum tensor must be converted in a 5D tensor including a new dimension, "m". The problem is that the expression of the left member of EFE is in 4D:

$$f_{(x,y,z,t)} = g_{(x,y,z,t,m)} \qquad (208)$$

As we see, the homogeneity of the EFE is not respected since the number of dimensions of the left member is different to that of the right member.

2/ The mass variable "m" is fully defined in 4D. In this case, the EFE should explicitly include the nature of mass and its expression: $m = f_{(x,y,z,t)}$. This information is missing inside the EFE. In other words, the EFE uses a variable "m" without knowing its meaning nor its expression.

Summary
Since the Energy-Momentum Tensor is built from the Stress Tensor, it is obvious that these two tensors must share the same reasoning and principle of construction. As we see, this is not the case. Einstein and Grossman brought some modifications to the Stress Tensor on several points. This is why the EFE have these inconsistencies. To summarize, the questions are:

1/ Why has the Attractive force replaced the original Pressure force?
2/ Why has mass replaced volume?
3/ Why a concave curvature replaced the convex curvature of the fluid mechanics?
4/ Why has the problem of the mass variable been ignored for the last century?

E-12 Explanations to these Enigmas

The theory here presented gives the solution to these four enigmas.

ENIGMA #1
In the 1910s and even today, physicists thought that gravitation was an attractive force. However, in this paper we have shown that an alternative exists. *"Why would Gravity not be a pressure force ?"*. Indeed, a pressure on one side of a sheet of paper produces the same effects as an attractive force on the other side. Here we have demonstrated that the mechanism of pressure of spacetime on closed volumes is much more credible than an attractive force between masses that no one can explain. Moreover, this pressure has a scientific origin: the Stress Tensor.

ENIGMA #2
In the 1910s, the constitution of atoms was unknown. The proton was discovered in 1918 by Rutherford and the neutron in 1932 by Chadwick. In the 1910s, physicists thought that the atom was made only of electrons, and that these electrons were distributed like raisins in a pudding. Therefore, is was impossible for Einstein to separate closed to open volumes, i.e. massive volumes (protons, neutrons, electrons) to massless volumes (orbitals). This is probably why Einstein took mass instead of closed volumes in the construction of its EFE.

Here we have demonstrated that spacetime is not curved by masses but by closed volumes. Please note that *"Closed volumes"*, *"Open volumes"*, *"Standard apparent volumes"*, *"Hermetic apparent volumes"*, and *"Special apparent volumes"* are five different definitions of "Volumes" (see the main article). Replacing Mass $m = f_{(???)}$ by the Mass Effect $m = f_{(x,y,z,t)}$ defined in Appendix C gives identical results in EFE while solving this second enigma.

ENIGMA #3
Figure E-8 shows that the curvature of the medium in Fluid Mechanics is CONVEX, not CONCAVE. Therefore, the figure E-9 is wrong, despite its popularity. Since the Energy-Momentum Tensor is built from the Stress Tensor, it is obvious that we must keep the basic principles of construction, as shown in figure E-7. To date, no one can explain why has Einstein and Grossman replaced the original CONVEX curvature by a CONCAVE curvature. The most likely explanation is given on the next page: *"Consistency of ENIGMAS #1 and #3"*.

ENIGMA #4
Each member of EFE is built on a 4D space. Since EFE works perfectly, it means that only the second possibility of the precedent "QUESTION #4" can be accepted. The first possibility, i.e. the Energy-Momentum Tensor must be converted to a 5D space, must be rejected. As we know, in any expression, it is not acceptable to have a member in 4D and the other in 5D. Replacing the unknown variable of the mass $m =$??? by the Mass Effect $m = f(x, y, z, t)$ gives identical results and solves this fourth enigma. The expression is given in Appendix C:

$$M \ = \ \frac{c^2}{G_0}\frac{V}{S}\delta_v = f_{(x,y,z,t)} \qquad (209)$$

M = Mass effect (kg)
V = Volume of the closed volume (m^3)
S = Surface of the closed volume (m^2)
δ_v = Coefficient of volumetric elasticity of spacetime
c = Speed of the light (m/s)
G_0 = Universal constant of gravitation

Note: To calculate the mass effect as shown in Appendix C, it is simpler to have two constants, 1/ the coefficient of volumetric elasticity of spacetime ϵ_v, and 2/ the density of the surrounding spacetime ρ. On the contrary, in EFE, it is simpler to have only one constant defined as: $\delta_v = \epsilon_v + \rho$. In both cases, the result is identical.

Consistency of ENIGMAS #1 and #3

In physics, a normal attracting constraint is positive by convention, and a pressure constraint is negative. We can also consider that a concave curvature has the "-" sign, and a convex curvature has the "+" sign. Since the signs are given "by convention", this leads to four combinations:

1 - Gravitation is an attractive force in a concave curvature of spacetime: (+ -)
2 - Gravitation is an attractive force in a convex curvature of spacetime: (+ +)
3 - Gravitation is a pressure force in a concave curvature of spacetime: (- -)
4 - Gravitation is a pressure force in a convex curvature of spacetime: (- +)

These four combinations can be interpreted as:

1 - (+-) This combination is that of Newton-Einstein. It works perfectly and doesn't need validation. However, despite its popularity, this combination does not explain the origin of the mass, gravitation, and curvature of spacetime,
2 - (++) This combination does not explain anything and must be rejected,
3 - (- -) This combination does not explain anything and must be rejected,
4 - (-+) This combination is that of the proposed theory. It conducts to an identical result as the first combination because (+ -) = (- +). However, this combination is much more credible than the first one because, for identical results, it gives a rational explanation of the preceding enigmas #1 and #3.

E-13 Objections

Simplicity or Complexity?

We could think that the basic laws of physics are extremely complex since the mathematics of physics are. Such is not the case.

Let us consider, for example, a drum. A 5 years old child intuitively knows the principle, namely that by striking it he makes noise. On the other hand, the mathematical description of the surface waves requires the knowledge of Bessel Functions. It is thus advisable to distinguish the principle, **always very simple**, from the laws governing it, which may be extremely complex.

Modern physics shares the same principle. The basic laws of the universe are not embedded in increasingly complex theories but, on the contrary, in simplicity. Most scientists throughout the world agree with this point of view. For example, to detect any trace of life on Mars, the biologists will not seek complex living organisms but elementary molecules like H_2O.

This view must also be applied to mass, gravitation, and the curvature of spacetime. Here we show that the basic principles of these phenomena are very simple but their formulation complex.

Association of basic ideas

A new theory may be issued from a genius idea, such as GR, or simply the association of some basic ideas that everybody knows. This is the case of the theory here described, which is the association of three basic and well-known ideas:

1/ Differentiating closed volumes (with mass) to open volumes (without mass),
2/ Considering that the the origin of the curvature of spacetime is not mass but a closed volume. Consequently, the concave curvature becomes a convex curvature (see the stress tensor),
3/ Replacing the attractive force of gravitation by a pressure force exerted by the curvature of spacetime on closed volumes.

About Einstein

One might wonder why Einstein did not think to the four enigmas discussed above. The main reason is that in the 1910s physicists thought that the atom was built on the "pudding model". Therefore, the volume definition was wrong. However, other reasons can also be retained.

Science is not the matter of only one man such as Einstein, but by thousand scientists. GR was also indirectly built by Gauss, Faraday, Maxwell, Riemann, Christoffel, Ricci, Hooke, Cauchy, Navier, Stokes, Minkowski, Lorentz, Poincarre, Michelson, Morlay, Grosmann, Schwarzschild...

Why in 1915 Einstein did not think to solve the EFE in the case of a static sphere? No one knows... This very simple solution was devised by Schwarzschild in 1916, one year after the publishing of the EFE by Einstein.

This example, like many others, shows that a genius like Einstein can build great theories but, for different reasons, can also miss very simple ideas such as the Schwarzschild solution. The theory described here follows the same principle. It is built on basic ideas that many physicists know but do not apply, such as separating closed to open volumes. Why since the 1910s no one thought to these simple ideas? ...No one knows.

Multiple Solutions

We can also think that, since GR works perfectly, any other theories must be rejected. Imagine, for example, a theory with an equation such as $y = x^2$. This theory is verified if the result is 4. It means that $x = 2$ is the solution. This result may be widely accepted because it is the most obvious solution, but another solution that could be a better solution also exists: $x = -2$.

This example shows that we must not systematically reject new ideas, even if they are contrary to theories already established. This is the case of gravitation which may be a pressure force instead of an attractive force.

E-14 New version of the EFE

In equation (208), the time component t is present in c^2, in $[L^2/T^2]$, and G_0, in $[L^3/MT^2]$. However, it disappears in the ratio c^2/G_0, or $[L^2/T^2][L^3/MT^2]$, which gives $[L/M]$ after simplification. In this expression, the quantity M is not a part of the basic four dimensions but is an expression of the kind $m = f_{(x,y,z,t)}$. It means that in the energy-momentum tensor, in the case of non-relativistic objects, the density of matter ρ should be replaced by:

$$\rho = \frac{m}{V} = \frac{c^2}{G_0}\frac{V}{S}\ \delta_v \cdot \frac{1}{V} \qquad (210)$$

After simplification by V, we get:

$$\rho = \frac{c^2}{G_0\ S}\delta_v \qquad (211)$$

From these three equations (209), (210) and (211), we can calculate the elements of the Energy-Momentum Tensor. For teaching purpose, the symmetric elements have been duplicated here.

Element $\mathbf{T_{00}}$.

$$T_{00} = \frac{E}{V} = \frac{mc^2}{V} = \frac{c^2}{G_0}\frac{V}{S}\ \delta_v \cdot \frac{c^2}{V} \qquad (212)$$

After simplification by V:

$$T_{00} = \frac{c^4}{G_0 S}\ \delta_v \qquad (213)$$

Elements $\mathbf{T_{01}, T_{02}, T_{03}, T_{10}, T_{20}, T_{30}}$.
For $\mu\nu$ = 01, 02, 03, 10, 20 and 30:

$$T_{\mu\nu} = \rho\ c\ v_{\mu\nu} = \frac{c^2}{G_0\ S}\delta_v \cdot cv_{\mu\nu} \qquad (214)$$

or:

$$T_{\mu\nu} = \frac{c^4}{G_0 S}\ \delta_v \cdot \frac{v_{\mu\nu}}{c} \qquad (215)$$

Elements $\mathbf{T_{11}, T_{22}, T_{33}}$.
σ is a pressure. For $\mu\nu$ = 11, 22 and 33 we have:

$$\sigma_{\mu\nu} = \frac{F_{\mu\nu}}{S} = \frac{ma_{\mu\nu}}{S} = \frac{c^2}{G_0}\frac{V}{S}\ \delta_v \cdot \frac{a_{\mu\nu}}{S} \qquad (216)$$

or:

$$\sigma_{\mu\nu} = \frac{c^4}{G_0\ S}\ \delta_v \cdot \frac{V}{S\ c^2}a_{\mu\nu} \qquad (217)$$

Elements $\mathbf{T_{12}, T_{13}, T_{23}, T_{21}, T_{31}, T_{32}}$.
τ is a pressure. For $\mu\nu$ = 12, 13, 23, 21, 31 and 32 we have:

$$\tau_{\mu\nu} = \frac{F_{\mu\nu}}{S} = \frac{ma_{\mu\nu}}{S} = \frac{c^2}{G_0}\frac{V}{S}\ \delta_v \cdot \frac{a_{\mu\nu}}{S} \qquad (218)$$

or:

$$\tau_{\mu\nu} = \frac{c^4}{G_0\ S}\ \delta_v \cdot \frac{V}{S\ c^2}a_{\mu\nu} \qquad (219)$$

New Energy-Momentum Tensor

A new Energy-Momentum tensor can be built with the elements of equations (213), (215), (217), and (219). However, we can extract the quantity from each equation:

$$\frac{c^4}{G_0\ S}\ \delta_v \qquad (220)$$

To avoid confusion with the traditional Energy-Momentum tensor $T_{\mu\nu}$, this new tensor is called $J_{\mu\nu}$. Including the quantity (220) in the Einstein Constant gives:

$$J_{\mu\nu} = \begin{bmatrix} 1 & v_{01}/c & v_{02}/c & v_{03}/c \\ v_{10}/c & (V/Sc^2)a_{11} & (V/Sc^2)a_{12} & (V/Sc^2)a_{13} \\ v_{20}/c & (V/Sc^2)a_{21} & (V/Sc^2)a_{22} & (V/Sc^2)a_{23} \\ v_{30}/c & (V/Sc^2)a_{31} & (V/Sc^2)a_{32} & (V/Sc^2)a_{33} \end{bmatrix} \qquad (221)$$

Note: Accelerations a_{11}, a_{22}, and a_{33}, are normal accelerations regarding the surface. The other accelerations are tangential.

The coefficient (220) may be merged with the Einstein Constant (161) as follows:

$$\frac{8\pi G_0}{c^4} \cdot \frac{c^4}{G_0\ S}\ \delta_v = \frac{8\pi\delta_v}{S} \qquad (222)$$

Finally, this new version of EFE can be written as:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi\delta_v}{S} J_{\mu\nu}} \qquad (223)$$

Important note:

As shown in this equation, the two members of this “modified EFE” are in 4D. The variable “m” disappears.

Case of a static sphere

This particular case can be identified to the criteria of construction of the Schwarzschild Metric. We have:

$$V = \frac{4}{3}\pi r^3 \qquad (224)$$

and

$$S = 4\pi r^2 \qquad (225)$$

hence

$$\frac{V}{S} = \frac{4\pi r^3}{3} \frac{1}{4\pi r^2} = \frac{r}{3} \qquad (226)$$

Porting this expression in the tensor $J_{\mu\nu}$ of equation(221) gives, after simplifications:

$$J'_{\mu\nu} = \begin{bmatrix} 1 & v_{01}/c & v_{02}/c & v_{03}/c \\ v_{10}/c & (r/3c^2)a_{11} & (r/3c^2)a_{12} & (r/3c^2)a_{13} \\ v_{20}/c & (r/3c^2)a_{21} & (r/3c^2)a_{22} & (r/3c^2)a_{23} \\ v_{30}/c & (r/3c^2)a_{31} & (r/3c^2)a_{32} & (r/3c^2)a_{33} \end{bmatrix} \qquad (227)$$

In this particular case, the new version of EFE can be written as:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{2\delta_v}{r^2} J'_{\mu\nu}} \qquad (228)$$

E-15 Conclusions

From a mathematical point of view, the principles of inheritance can be applied: *"If the Energy-Momentum Tensor is built from the Stress Tensor, automatically it inherits all its properties"*. It means that if the Stress Tensor applies to **volumes**, **pressures** and **convex curvature**, the Energy-Momentum Tensor must do likewise. The only difference is the number of dimensions: 3D for the Stress Tensor, 4D for the Energy-Momentum Tensor.

On the other hand, adding an unknown variable, the mass "m", only in the right member of the EFE creates a lack of homogeneity between the two members.

This is why the traditional EFE has been reconsidered and rewritten.

To use the new version of EFE, we must know the volume and surface of the closed volumes which produce the curvature of spacetime, and the coefficient of curvature of spacetime δ_v. We could get the information in an indirect way, but it is more convenient to continue using the intermediate variable m in the Energy-Momentum Tensor as usual. In other words, this new version of EFE is only interesting to provide a correct formulation of the mass and gravitation enigmas. Its advantages regarding the actual EFE are:

- **Curvature of Spacetime** . . . Produced by Closed Volumes,
- **Nature of the Curvature** . . . Convex, not concave,
- **Gravitation** . . . Pressure force, not an attractive force,
- **Mass Enigma** . . . Comes from the pressure of spacetime on closed volumes,
- **Mass Variable** . . . Fully defined $m = f_{(x,y,z,t)}$ (equation 209),
- **Homogeneity** . . . $4D \equiv 4D$ instead of $4D \equiv 5D$ as in EFE. The unknown variable m has been converted in a 4D variable $m = f_{(x,y,z,t)}$ (equation 209).

E-16 Special Metrics

We could think that the spacetime curvature ”C” depends on mass ”m” since the expression of the energy-momentum tensor is C = f(m):

Mass → Spacetime curvature

Since the mass is unknown, this expression is incomplete and should be written as

??? → Mass → Spacetime curvature

The present theory explains the mass and proposes to replace ”???” by the following group:

$$\left[\begin{array}{c} \textit{Closed Volumes} \Rightarrow \\ \textit{...curve Spacetime} \Rightarrow \\ \textit{...that exerts a Pressure on the Body} \Rightarrow \\ \textit{...which leads to a “Mass Effect”} \Rightarrow \end{array}\right]$$

⇒ Mass ⇒ Spacetime curvature

We see that the spacetime curvature is redundant. So, it is not necessary to specify the last line since the spacetime curvature has already been calculated (second line in italics).

It is important to note that this scheme is static. Its purpose is nothing but to calculate the mass effect in a flat spacetime, as shown in Appendix C.

If we need to know the dynamic spacetime curvature in a particular situation, the first thing to do is to calculate the static mass effect in a flat spacetime. Then, we must use the following scheme:

Step 1

$$\left[\begin{array}{c} \textit{Closed Volumes} \Rightarrow \\ \textit{...curve Spacetime} \Rightarrow \\ \textit{...that exerts a Pressure on the Body} \Rightarrow \\ \textit{...which leads to a “Mass Effect”} \Rightarrow \end{array}\right]$$

Step 2

⇒ Dynamic spacetime curvature

Step 1: Calculation of the mass effect of the body from the static curvature of spacetime, as shown in Appendix C.

Step 2: The knowledge of the static mass effect ”m” allows the calculation of the spacetime curvature in a particular dynamic context. To do that, we must use EFE with correct parameters or one of its known solutions.

For example, for a rotating sphere, the process is to:

- Step 1: Calculate the static mass effect ”m” from closed volumes (equation 9, Appendix C),
- Step 2: Calculate the dynamic spacetime curvature as usual, applying the Kerr Metric.

However, it is simpler to continue to calculate special metrics using the traditional EFE as done since the 1920’s, even if the mass variable m is undefined.

E-17 The Schwarzschild Metric

To calculate the spacetime curvature with a static body having a spherical symmetry, we must use the Schwarzschild Metric. In that particular case, step 2 isn’t necessary since the spacetime curvature has already been calculated from closed volumes in step 1 (see Appendix B).

Using this principle of calculation in two steps, here we show that the theory presented in this paper is in perfect accordance with EFE but, as stated above, this method is not the simplest. For any calculation, we can continue to use the unknown variable ”m”, even if this calculus is not “academic”.

Von Laue Geodesics

F-1 Introduction

A set of concentric circles is drawn in (fig. F-1a). These lines represent the geodesics of spacetime far from any mass, in a Minkowski spacetime. If a static spherical symmetry closed volume is inserted in the centre (fig. F-1b), spacetime will be curved, as explained in the main text. This figure F-1b has been duplicated in fig. F-2.

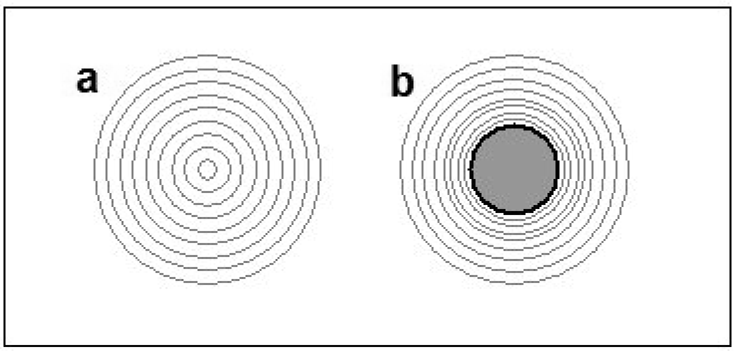

Figure F-1: Curvature of spacetime

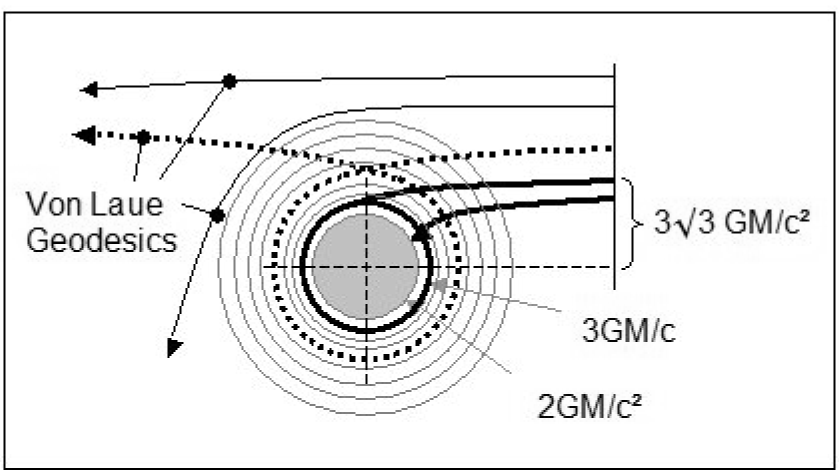

Figure F-2: Von Laue Geodesics

The Von Laue Geodesics has been drawn over the concentric circles of fig. F-1b.

We see that the Von Laue Geodesics match EXACTLY the concentric circles.

In other words, it seems that Von Laue, early as 1927's, predicted the theory presented here. It is obvious that **his diagram shows volumes, not masses,** even if the Von Laue Formulas are related to the mass of the body.

New Version of the Equivalence Principle

G-1 Demonstration

Lets consider a static object on Earth (fig. G-1). Closed volumes of this object cause a curvature of spacetime that exerts a gravitational force on this object. $g = 9.81 m.s^{-2}$ on the surface of Earth.

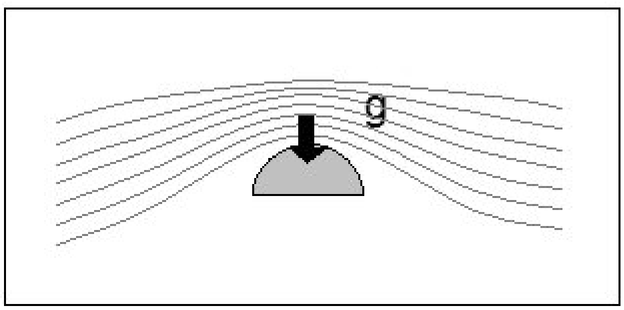

Figure G-1: Gravitationnal acceleration

Lets now consider the same object accelerated out of any gravitational field. We can represent this object in two different views (fig. G-2a and G-2b).

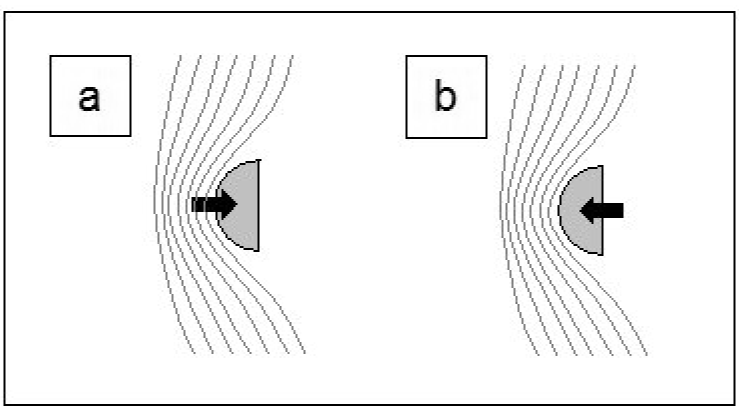

Figure G-2: Inertial acceleration

In both cases, acceleration "a" is supposed to be identical to g:

$$a = g = 9.81 m.s^{-2}$$

Without any reference, a local observer cannot know if the curvature of spacetime is due to a pressure on the object (fig. G-2a) or to its acceleration (fig. G-2b). In fact, these two figures are identical and depend on where the observer stands, as described in Special Relativity.

Since:

• By definition, g = 9.81 m.s^{-2} (fig. G-1) is identical to a = 9.81 m.s^{-2} (fig. G-2a and G-2b).
• These examples use the same object. Therefore, the curvature of spacetime produced by the closed volume of this object is identical.
• The two precedent points show that the "mass effect" produced by these curvatures will be necessarily identical in both figures.

We deduce that the "gravitational mass effect" (fig. G-1) is identical to the "inertial mass effect" (fig. G-2):

Gravitational mass effect
=
Inertial mass effect
=
Effect from spacetime curvature

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