Disposing Classical Field Theory, Part IV

Inspection of Gauge Invariance And its Relation with Mass Conservation

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2013-8-14

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Abstract

It is shown a.o. that a gauge invariant scalar (classical and quantum theoretical) electrodynamical field is a trivial field theory; in fact, it is shown that a non-zero scalar gauge field will not be charge/mass conservating, unless it is zero.

The classical action of a jet of charged and neutral jets of particles is calculated. It is shown that this action is a spinor field Ψ which satisfies $\Box\Psi = 0$, i.e.: the action of neutral and charged currents of particles spreads at the speed of light, a result which was already shown in [3] by other means. The fact that both solutions differ only by a constant factor γ^0, suggests that electromagnetic and gravitational field are of the same nature.

Now, why is the gravitational force so much weaker than electromagnetic one? A hint can perhaps be given with [4].

1 Preliminaries

In the appendix of [3] I proved that the space of (complex valued) tempered distributions $\mathcal{S}'(\mathbb{R}^4)$ is the toplogical direct sum $\mathcal{S}'(\mathbb{R}^4) = range(\Box) \oplus ker(\Box)$, where $ker(\Box)$ is the subspace $\{T \in \mathcal{S}'(\mathbb{R}^4) | \Box T = 0\}$ and $range(\Box)$ is its complementary subspace. The elements of $ker(\Box)$ are (generalized) plane waves.

2 Gauge invariance

In classical electrodynamics the gauge invariance is stated as the invariance of field equations w.r.t. the addition $A^\mu \mapsto A^\mu + \partial^\mu\Lambda$, where Λ is an arbitrary function on $\mathbb{R}^4$, in quantum electrodynamics it is the invariance of the state transformation $\Phi \mapsto e^{i\Lambda}\Phi$ (see e.g.: [2, II-18.10] and [6, Ch. 7]).

Both statements have in common that they don't define exactly as to what set of functions for Λ they refer to, and that will show up to be relevant, but to make a point, let us demand the Λ to be (3 times continuously differentiable) real-valued, scalar functions of utmost polynomial growth. (It will be shown in the next section, below, that in practically all physically relevant situations the 4-vector current components j^μ and with them also the Λ are supposed to be infinitely differentiable functions of utmost polynomial growth; given that, there is no big loss of generality to impose that condition on Λ.)

So, the Λ are elements of $\mathcal{S}'_{\mathbb{R}}(\mathbb{R}^4)$. Then $F := \Box\Lambda$ and $j^\mu := \partial^\mu F$, with $(0 \leq \mu \leq 3)$, are tempered distributions, and, if the j^μ are supposed to be currents, then they must obey the charge/mass conservation principle, i.e.: $\partial^0 j^0 + \cdots + \partial^3 j^3 = 0$ must hold. Then $(\partial_1^2 + \cdots + \partial_3^2)F = 0$, which implies $F \equiv Const$, and therefore $j^\mu = \partial^\mu F \equiv 0$ for all μ.

So, as to classical electrodynamics, the conclusion is ***not*** the gauge invariance, but: In order that mass/charge conservation holds, the 4-vector current j must not contain any non-zero additive part $(\partial^0 F, \cdots, \partial F)$ for some (non-constant) real-valued function F. Now, given that the j^μ are continously differentiable (on the simply connected region $\mathbb{R}^4$, a necessary and sufficient condition for the existence of F with $\partial^\mu F = j^\mu$ is that the matrix $(\partial^\mu j^\nu)_{0\leq\mu,\nu\leq 3}$ is to be symmetrical. This is a special case of Poincaré's lemma (see e.g.: [1, Sec. 2-12 to 2-13]). The proof is uncomplicated: Because $\partial^\mu\partial^\nu F = \partial^\nu\partial^\mu F$, the condition is necessary, and, the other way round, given the symmetry, the path integration of j^μ along a path from start to end point is dependent only of the two points, and the existence of such F follows.

Armed with that, the goal then is to exclude from j just that additive part, that spoils charge/mass conservation, and to see, what that rest then is.

To do so, let j be an arbitrary quadrupel of (real-valued) functions on $\mathbb{R}^4$. Then the matrix $j^{\mu\nu} := (\partial^\mu j^\nu)_{\mu,\nu}$ splits into the sum of a symmetric matrix $(g^{\mu\nu})_{\mu\nu}$, which is defined by $g^{\mu\nu} := \frac{1}{2}(j^{\mu\nu} + j^{\nu\mu})$ for $\mu \neq \nu$ and $g^{00} = \cdots = g^{33} = 0$, and the matrix $(h^{\mu\nu}) := (j^{\mu\nu} - g^{\mu\nu})$. $(g^{\mu\nu})$ then is symmetric with zero diagonal elements, wheras $(h^{\mu\nu})$ is anti-symmetric in its off-diagonal elements. Let's tweak the terminology and call $(h^{\mu\nu})$ "skew-symmetric" for short. Integrating these matrices in each of the four components then gives us two continuously differentiable quadruples of functions g and h, such that $j = g + h$, where $(\partial^\mu g^\nu)$ is symmetric and $(\partial^\mu h^\nu)$ is skew-symmetric (in the above defined sense).

So, we arrive at a convenient classification: If the j^μ are continuously differentiable, in order that the current is charge/energy conserving, the matrix $(\partial^\mu j^\nu)$ must be skew-symmetric.

Let's now work out, to what degree that condition also is sufficient: If $(\partial^\mu j^\nu)$ is skew-symmetric, then $(\gamma^\mu\partial^\mu\gamma^\nu j^\nu)$ is symmetric. That is, the quadruple $(\gamma^0 j^0, \cdots, \gamma^3 j^3)$ can be integrated w.r.t. the (Lorentz invariant) differential 1-form

$$d := \sum_{0\leq\mu\leq 3} (\partial/\partial(\gamma_\mu x_\mu))d(\gamma^\mu x^\mu).$$

The result is a spinor-valued function Ψ on $\mathbb{R}^4$ which satisfies: $(\partial/\partial(\gamma_\mu x_\mu))\Psi =$

$\gamma^\mu j^\mu$, $(0 \leq \mu \leq 3)$. Then the charge conservation for the 4-vector current $(\gamma^0 j^0, \cdots, \gamma^3 j^3)$ holds, if and only if $\Box\Psi = (\partial^2/\partial(\gamma_0 x_0)^2 - \cdots - \partial^2/\partial(\gamma_3 x_3)^2)\Psi \equiv 0$.

But wait: The j^μ were supposed to be scalar (real-valued) functions, and this makes $(\gamma^0 j^0, \cdots, \gamma^3 j^3)$ to be a charged current. Its neutral counterpart is $(\gamma^0\gamma^0 j^0, \cdots, \gamma^3\gamma^0 j^3)$, and, because that is a constant factor γ^0 away from the charged current, it is likewise integrable w.r.t. the above Lorentz invariant differential form, and the integral is $\Psi\gamma^0$. And again, $\Box\Psi \equiv 0$ is enforced by the mass conservation.

It is clear, what the result means: Every 4-vector current (be it neutral or charged) is the source of a zero-mass action field at every space-time coordinate $x \in \mathbb{R}^4$, spreading (as a wave) at the speed of light. And this action field in turn exactly determines the state of the current j at its retarded times (by calculating its Lorentz invariant gradient). That is exactly the relativistic extension of the conception of the gravitational field in classical mechanics.

The equation $\Box\Psi = 0$ in itself has a physical meaning: it says that the "emission" of Ψ by the 4-current j does not change anything for j, which means two things: Firstly, the emission of Ψ is gratis, and, secondly, the self-interaction, i.e. the interaction of j with is own emitted field Ψ is zero.

3 Why 4-vector currents are supposed to be smooth functions

In quantum theory it is often lectured that the concept of indistinguishability of spacially separated masses or charges was a new quantum theoretical concept in physics.

I beg to differ: That conception already is the heart of the mechanics of fluids, in which the singular molecules loose their singular state and instead, the superposition of all states results into the fluidal state (see [3, Introduction]). The concept of current densities j in classical electrodynamics is exactly that of a fluid of charges in motion, and the basic law that drives the dynamics the fluidal mechanics is the mass/charge conservation. In particular, the momentum density of the fluid is to be defined not as the momentum of n particles passing a point at some time t, but as the negative flux of energy through an infinitesimally small surface containing that point at some time t.

In other words, in order to get at the energy and momentum density of that fluid, we have to take the Fourier transform of j. And because the Fourier transform is an isomorphism on the space of tempered distributions on $\mathbb{R}^4$, that space is a convenient space to base discussions on.

In particular, the Fourier transform of j is a 4-vector $\hat{j} \in \mathcal{S}'(\mathbb{R}^4)$, which is j in terms of energy and momentum coordinates. Of this $\hat{j}$ we can make three restrictions:

First, we demand that the (absolute) value of energy of $\hat{j}$ is to be bounded above by a constant $E_0 > 0$. That means that $\hat{j}$ has compact support in $\mathbb{R}$, i.e. it vanishes outside a closed and bounded set $X \subset \mathbb{R}^4$, and therefore j is an infinitely differentiable function on $\mathbb{R}^4$ itself.

Second, the fluid is made of particles that at least have the rest energy of the electron, which is $|q_e| > 0$. So, the support of the $\hat{j}^\mu$ does neither contain the light cone $\{(p_0, \cdots, p_3) \in \mathbb{R}^4 | p_0^2 - \cdots - p_3^2 = 0\}$, nor the zero-energy hyperplane $\{(p_0, \cdots, p_3) \in \mathbb{R}^4 | p_0 = 0\}$. The support of the $\hat{j}^\mu$ therefore is contained in 4 disjoint (bounded) regions: the interior of the forward and backward light cones, and the two spacelike regions of the upper and lower hemisphere, i.e. of positive and negative parity. In particular, the j^μ are contained in $range \Box$, which means that the $A^\mu := \Box^{-1} j^\mu$ are well-defined.

Third, in many situations - but maybe not all - as a theory of fluids, one would demand smoothness of the $\hat{j}^\mu$. And, given that the $\hat{j}^\mu$ are n-times continuously differentiable, the j^μ go to zero with n-th order as $|x| \to \infty$. (In general, as the Fourier inverses of tempered distributions with compact support, the j^μ have a polynomial growth as $|x| \to \infty$.)

4 Remark on Dirac Distributions

One might object that, restricting to j as continuous functions, will exclude Dirac distributions. Frankly put, Dirac distributions appear to be inappropriate in the context of the j:

In a fluid j, a single particle can be traced by picking its particular time path $\lambda : \mathbb{R} \ni x_0 \mapsto \mathbb{R}^3$, and if one associates a mass/charge with that particle, it's giving us its kinetic description. (In particular, the square of the energy momentum can be calculated, something which is not possible with Dirac distributions, and so forth.) But the path $\mathbb{R} \ni x_0 \mapsto j(x_0, \lambda(x_0)) \in \mathbb{R}^{\not 4}$, the objective that comes by applying the Dirac distribution $\delta(\cdot - x)$ to j, has a different physical meaning as the (statistic) superposition of the particles along, the chosen path.

However, there is a connection between the two different conceptions: It's Feynman's path integration: Suppose we have a huge set of particles and understand their mutual interaction, their forces will superimpose, these will make it into a superimposed motion of the particles, and we should end up with a dynamics of the fluid (which is just j). The application of path integration to classical physics is not a new idea, let alone to Feynman himself: see e.g [5].

5 Conclusion

In all, we saw that the covariant Maxwell equations extend extend the original ones beyond a theory of charges to a relativistic theory of neutral masses either. Its general solution is a spinor-valued action function Psi, which is the integral of the 4-vector current j, taken w.r.t. the Lorentz metrics, and satisfies $\Box \Psi \equiv 0$. Also, the neutral action function differs from the neutral one by just a constant factor γ_0.

Now we have already a physical concept for the field equation $\Box \Psi = 0$, namely that of the electromagnetic field. So, why invent new, different ones which follow the same equation and hence wouldn't be distinguishable from the electromagnetic field?

Much speaks in favour of a generalized, more abstract view of the covariant Maxwell equations as the underlying field theory for any Lorentz invariant theory of matter. That way, every physical theory would finally have to make it into that equation, in order to become Lorentz invariant. Interestingly, the general, covariant Maxwell equations solve in terms of 4×4-matrix functions, which means their solution has place for 4 "orthogonal" components, each coming with their own coupling force. And we currently have 4 such theories: the nuclear, strong SU(3)-theory, the $U(2)$-theory of weak interactions, the electrodynamics, which proved to be a $U(2)$-theory either, plus the theory of gravitation, which we based on the absolute value of energy of the three others. (That makes gravitation a $U(1)$-theory, and we took its dimension from the electromagnetic $U(2) \cong SU(2) \oplus U(1)$-group.)

It is known that the coupling forces of the 4 theories are temperature dependent, that they appear to converge towards eachother at high temperature, and that at low temperatures the relation of electromagnetic force by gravitational force is extremely high (approx. 10^{36}).

This makes gravitational waves very difficult to discover, especially since at high temperatures, the electromagnetic action field (of the charges in the environment) will hide the gravitational, neutral one. For now one cannot even exclude the possibility of mistaking gravitational waves with electromagnetic ones.

References

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