Rates of approximation for general sampling-type operators in the setting of filter convergence

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Abstract

We investigate the order of approximation of a real-valued function by means of suitable families of sampling type operators, which include both discrete and integral ones. We give a unified approach, by means of which it is possible to consider several kinds of classical operators, for instance Urysohn integral operators, in particular Mellin-type convolution integrals, and generalized sampling series. We deal with filter convergence, obtaining proper extensions of classical results.

We investigate, in the context of modular spaces, the rates of approximation of a real-valued function f by means of a family of operators of the type

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))\, d\mu_w(t), \quad w \in W, \quad s \in G,$$

where $W \subset \mathbb{R}$ is a suitable directed set, $(G,+)$ is a locally compact abelian Hausdorff topological group endowed with its Borel σ-algebra $\mathcal{B}$, $(H_w)_w$ is a sequence of nonempty closed sets of $\mathcal{B}$ with $G = \overline{\bigcup_{w \in W} H_w}$, μ_w is a regular measure defined on the Borel σ-algebra $\mathcal{B}_w$ of H_w and f belongs to the domain of the operators T_w for each $w \in W$. We deal with the rates of approximation of $T_w f$ in the setting of modular spaces, which contain as particular cases L^p, Orlicz and Musielak-Orlicz spaces. A particular case of filter convergence is the statistical convergence. As applications of our results, our theory includes both integral operators of Urysohn type, like Mellin convolution operators (in particular, Mellin-Gauss-Weierstrass, Mellin-Poisson-Cauchy and moment kernels), and discrete generalized sampling operators, which are useful in the reconstruction of signals, images and videos.

Let $G = (G,+)$ be a locally compact abelian Hausdorff topological group with neutral element θ. Let $\mathcal{B}$ be the σ-algebra of all Borel subsets of G, $\mu : \mathcal{B} \to \mathbb{R}$ be a positive σ-finite regular measure, and $\mathcal{U}$ be a base of μ-measurable symmetric neighborhoods of θ. Let us denote by $L^0(G,\mathcal{B},\mu) = L^0(G)$ the space of all real-valued μ-measurable functions with identification up to sets of measure μ zero.

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A filter $\mathcal{F}$ of W is said to be *free* iff it contains $\mathcal{F}_{\text{cofin}}$.

A family x_w, $w \in W$, in $\mathbb{R}$ is *$\mathcal{F}$-convergent* to $x \in \mathbb{R}$ (and we write $x = (\mathcal{F}) \lim\limits_{w \in W} x_w$) iff

$$\{w \in W : |x_w - x| \leq \varepsilon\} \in \mathcal{F} \quad \text{for every } \varepsilon > 0.$$

Given two functions f_1, $f_2 : W \to \mathbb{R}$ and a filter $\mathcal{F}$ of W, we say that $f_1(w) = O(f_2(w))$ *with respect to $\mathcal{F}$* iff there exists a $D > 0$ with

$$\{w \in W : |f_1(w)| \leq D\,|f_2(w)|\} \in \mathcal{F}.$$

From now on we suppose that $W = (W, \succeq)$ is a directed set, and $\mathcal{F}$ is a free filter of W. Some examples used frequently in the literature are $(W, \succeq) = (\mathbb{N}, \geq)$, or $W \subset [a, w_0[\subset \mathbb{R}$ endowed with the usual order, where $w_0 \in \mathbb{R} \cup \{+\infty\}$ is a limit point of W. We also will consider the above set G endowed with the filter $\mathcal{H}_\theta$ of all neighborhoods of its neutral element θ.

For each $w \in W$, let H_w be a nonempty closed set of $\mathcal{B}$, with $\overline{\bigcup\limits_{w \in W} H_w} = G$, and μ_w be a regular measure defined on the Borel σ-algebra $\mathcal{B}_w$ generated by the family $\{A \cap H_w : A$ is an open subset of $G\}$. For every $w \in W$ let $\mathcal{L}_w$ be the set of all measurable non-negative functions $L_w : G \times G \to \mathbb{R}$, and suppose that L_w is *$\mathcal{F}$-homogeneous uniformly with respect to $w \in W$*, namely there is a set $F^* \in \mathcal{F}$ with

$$L_w(\sigma + s, u + s) = L_w(\sigma, u) \quad \text{for every } \sigma, s, u \in G \text{ and } w \in F^* \tag{1}$$

Let Ψ be the class of all functions $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\psi(t, \cdot)$ is continuous, nondecreasing, $\psi(t, 0) = 0$ and $\psi(t, u) > 0$, for every $t \in G$ and $u > 0$. We consider a family $(\psi_w)_w \subset \Psi$, with the property that there exist two constants E_1, $E_2 \geq 1$ and measurable functions $\phi_w : G \times G \to \mathbb{R}_0^+$, $w \in W$, with

$$\psi_w(t, u) \leq E_1 \psi_w(t - s, E_2\, u) + \phi_w(t, s - t) \quad \text{for all } u \in \mathbb{R}_0^+,\ s,\ t \in G, w \in F^*. \tag{2}$$

Let $\mathcal{K}$ be the class of all families of functions $K_w : G \times H_w \times \mathbb{R} \to \mathbb{R}$, $w \in W$, satisfying the following conditions:

- $K_w(\cdot, \cdot, u)$ is measurable on $G \times H_w$ for each $w \in W$ and $u \in \mathbb{R}$;
- $K_w(s, t, 0) = 0$ for every $w \in W$, $s \in G$ and $t \in H_w$;
- for each $w \in W$ there are $L_w \in \mathcal{L}_w$ and $\psi_w \in \Psi$, with

  $$|K_w(s, t, u) - K_w(s, t, v)| \leq L_w(s, t)\psi_w(t, |u - v|) \tag{3}$$

  for all $s \in G$, $t \in H_w$ and u, $v \in \mathbb{R}$.

Let $\mathbb{K} = (K_w)_w \in \mathcal{K}$ and $\mathbf{T} = (T_w)_{w\in W}$ be a net of operators defined by

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))\, d\mu_w(t), \quad s \in G, \tag{4}$$

where $f \in \text{Dom}\, \mathbf{T} = \bigcap_{w\in W} \text{Dom}\, T_w$, and for every $w \in W$, Dom T_w is the subset of $L^0(G)$ on which $T_w f$ is well-defined as a μ-measurable function of $s \in G$.

For $s \in G$, $w \in W$ and L_w as in (3), set $l_w(s) := L_w(\theta, s)$, and suppose that

- l_w is a μ-measurable function with $l_w(\cdot - s) \in L^1(H_w)$ for every $s \in G$;
- there are $D^* > 0$ and $\overline{F} \in \mathcal{F}$ with

$$\int_{H_w} l_w(t-s)\, d\mu_w(t) \le D^* \tag{5}$$

  for each $s \in G$ and $w \in \overline{F}$.

Let now Ξ be the class of all functions $\xi : W \to \mathbb{R}_0^+$ such that $(\mathcal{F}) \lim_{w\in W} \xi(w) = 0$.

Definition 0.1 Let $\xi \in \Xi$, $\mathbb{K} \in \mathcal{K}$, l_w be as before and $\pi_w : G \to \mathbb{R}_0^+$, $w \in W$, be μ-measurable functions. We say that $\mathbb{K}$ is *$(\mathcal{F}, \xi)$-singular with respect to l_w and π_w* iff

0.1.1) for each $U \in \mathcal{U}$,

$$\int_{G\setminus U} l_w(s)\,(\pi_w(s)+1) d\mu(s) = O(\xi(w)) \quad \text{with respect to } \mathcal{F};$$

0.1.2) If $r^w(s) := \sup_{u\in\mathbb{R}\setminus\{0\}} \left| \frac{1}{u} \int_{H_w} K_w(s,t,u)\, d\mu_w(t) - 1 \right|$, $s \in G$, then $\sup_{s\in G} r^w(s) = O(\xi(w))$ with respect to $\mathcal{F}$;

0.1.3) there exist $F^* \in \mathcal{F}$ and $D' > 0$ such that for every $s \in G$ and $w \in F^*$ we get $r^w(s) \le D'$ and

$$\int_G l_w(s)\, d\mu(s) \le D'. \tag{6}$$

A family $m_w : G \times \mathcal{B}_w \to \mathbb{R}_0^+$, $w \in W$, is said to be *$\mathcal{F}$-regular* iff it is of the type

$$m_w(s, A) = \int_A \gamma_w(s,t)\, d\mu_w(t), \quad s \in G,\, w \in W,\, A \in \mathcal{B}_w,$$

where $\gamma_w : G \times G \to \mathbb{R}$ is measurable and the following properties are fulfilled:

- there is a constant $D_1 > 0$ such that, if $b_w^*(s) := m_w(s, H_w)$ for any $w \in W$ and $s \in G$, then

$$\left\{ w \in W : 0 < b_w^*(s) \le D_1 \text{ for all } s \in G \right\} \in \mathcal{F};$$

- putting

$$\omega_w^t(A) := \int_A \gamma_w(t, s+t)\, d\mu(s), \quad w \in W,\, t \in H_w,\, A \in \mathcal{B}_w,$$

  there is a family of measures ω_w, $w \in W$, such that

$$\{w \in W : \omega_w^t(A) \leq \omega_w(A) \text{ for all } t \in H_w \text{ and } A \in \mathcal{B}(G)\} \in \mathcal{F}. \tag{7}$$

observe that, by virtue of our assumptions and taking into account (5), the family l_w, $w \in W$, generates a family of $\mathcal{F}$-regular measures m_w. Indeed, it is enough to set

$$\begin{aligned} \gamma_w(s,t) &= l_w(t-s), \\ m_w(s,A) &= \int_A l_w(t-s)\, d\mu_w(t), \quad s \in G,\, w \in W,\, A \in \mathcal{B}_w, \\ \omega_w^t(A) &= \omega_w(A) = \int_A l_w(s)\, d\mu(s), \quad w \in W,\, t \in H_w,\, A \in \mathcal{B}(G). \end{aligned} \tag{8}$$

Given a modular ρ on $L^0(G)$, the space

$$L^\rho(G) := \{f \in L^0(G) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0\}$$

is the *modular space* associated with ρ.

A family $(f_w)_w$ of functions in $L^\rho(G)$ is said to be $\mathcal{F}$*-modularly convergent* to $f \in L^\rho(G)$ iff there exists $\lambda > 0$ with $(\mathcal{F}) \lim_{w \in W} \rho[\lambda(f_w - f)] = 0$.

For $w \in W$, let ρ_w, η_w be modulars on $L^0(H_w, \mathcal{B}_w, \mu_w) = L^0(H_w)$. We denote by $L^{\rho_w}(H_w)$, $L^{\eta_w}(H_w)$ the spaces of all functions $f \in L^0(G)$, whose restriction $f_{|H_w}$ belongs to the modular spaces generated by ρ_w, η_w respectively.

An $\mathcal{F}$-regular family $(m_w)_w$ is $\mathcal{F}$*-compatible* with the pair (ρ, ρ_w) with respect to a net $(b_w)_w$ in $\mathbb{R}$ iff there are two positive real numbers N, Q and a set $F_1 \in \mathcal{F}$ with

$$\rho\Big(\int_{H_w} g(t,\cdot)\, dm_w^{(\cdot)}(t)\Big) \leq Q \int_G \rho_w(N\, g(\cdot, s+\cdot))\, d\omega_w(s) + b_w \tag{9}$$

for every measurable function $g : G \times G \to \mathbb{R}_0^+$ and for each $w \in F_1$.

Let $\Xi = (\psi_w)_w \subset \Psi$ be as in (2). The triple (ρ_w, ψ_w, η_w), $w \in W$, is said to be $\mathcal{F}$*-properly directed* with respect to a net $(c_w)_w$ in $\mathbb{R}$, iff for every $\lambda \in (0,1)$ there are $C_\lambda \in (0,1)$ and $F_2 \in \mathcal{F}$ with

$$\rho_w(C_\lambda \psi_w(s, g(\cdot))) \leq \eta_w(\lambda\, g(\cdot)) + c_w \quad \text{whenever } w \in W,\, s \in G,\, 0 \leq g \in L^0(G). \tag{10}$$

Let $\mathcal{T}$ be the class of all measurable functions $\tau : G \to \mathbb{R}_0^+$, continuous at θ, with $\tau(\theta) = 0$ and $\tau(t) > 0$ for all $t \neq \theta$. For a fixed $\tau \in \mathcal{T}$, let $Lip(\tau)$ be the class of all functions $f \in L^0(G)$ such that there are $\lambda > 0$ and $\widetilde{F} \in \mathcal{F}$ with

$$\sup_{w \in \widetilde{F}} [\eta_w(\lambda\, |f(\cdot) - f(\cdot + t)|)] = O(\tau(t)) \tag{11}$$

with respect to the filter $\mathcal{H}_\theta$ of all neighborhoods of θ.

A family of modulars η_w, $w \in W$, is *$\mathcal{F}$-subbounded* iff there are $C \geq 1$, $\pi_w : G \to \mathbb{R}_0^+$, $\widehat{F} \in \mathcal{F}$ and a non-trivial linear subspace Y_η of $L^0(G)$, with

$$\eta_w(f(s+\cdot)) \leq \eta_w(C\,f) + \pi_w(s) \quad \text{for all } f \in Y_\eta,\ s \in G \text{ and } w \in \widehat{F}. \tag{12}$$

We say that $f \in L^{\eta_w}(H_w)$ *$\mathcal{F}$-uniformly with respect to $w \in W$* iff there are $R^* > 0$ and $\nu > 0$ with

$$\{w \in W : \eta_w(\nu\,f) \leq R^*\} \in \mathcal{F}. \tag{13}$$

Let now ϕ_w, $w \in W$, be as in (2) and $\tau \in \mathcal{T}$. We say that $(\phi_w)_w$ *satisfies property* $(*)$ iff there exist $E_3 > 0$, $\lambda' > 0$ and $\underline{F} \in \mathcal{F}$ with

$$\rho_w(\lambda'\,\phi_w(\cdot, s)) \leq E_3 \text{ for each } s \in G \text{ and } \sup_{w \in \underline{F}} \rho_w(\lambda' \phi_w(\cdot, s)) = O(\tau(s)) \tag{14}$$

with respect to the filter $\mathcal{H}_\theta$ of all neighborhoods of θ.

We state our main theorem about rates of approximation with respect to filter convergence for $T_w f - f$, where T_w, $w \in W$, are our operators defined in (4), and $f \in Lip(\tau)$ for a fixed $\tau \in \mathcal{T}$.

Theorem 0.2 *Let ρ be a quasi-convex and monotone modular on $L^0(G)$, ρ_w, η_w, $w \in W$, be monotone modulars on $L^0(H_w)$, such that the triple (ρ_w, ψ_w, η_w) is $\mathcal{F}$-properly directed with respect to a net $(c_w)_w$ in $\mathbb{R}$, where $c_w = O(\xi(w))$ with respect to $\mathcal{F}$.*

Let K_w, L_w, l_w satisfy the above assumptions. Let $\xi \in \Xi$ and $\tau \in \mathcal{T}$ be fixed.

Assume that $\mathbb{K}$ is $(\mathcal{F}, \xi)$-singular with respect to l_w and π_w, η_w is $\mathcal{F}$-subbounded, $f \in L^\rho(G) \cap Lip(\tau) \cap Y_\eta$, and $f \in L^{\eta_w}(H_w)$ $\mathcal{F}$-uniformly with respect to $w \in W$.

Suppose that the family of measures $(m_w)_w$ defined in (8) is $\mathcal{F}$-compatible with the pair (ρ, ρ_w) with respect to a net $(b_w)_w$, with $b_w = O(\xi(w))$ with respect to $\mathcal{F}$, and let $(\phi_w)_w$ satisfy property $(*)$ as in (14).

Finally, assume that there is a neighborhood U of θ with $U \in \mathcal{U}$ and

$$\int_U l_w(s)\tau(s)\,d\mu(s) = O(\xi(w)) \quad \text{with respect to } \mathcal{F}. \tag{15}$$

Then there is a constant $c > 0$ with

$$\rho(c(T_w f - f)) = O(\xi(w)) \quad \text{with respect to } \mathcal{F},$$

where

$$(T_w f)(s) = \int_{H_w} K_w(s, t, f(t))\,d\mu_w(t)$$

is as in (4).