There are infinitely many primes of the form $\alpha\beta^n + \chi$, $\alpha n^\beta + \chi$ and $\alpha\beta_1^n m^{\beta_2} + \chi$

Zhen Liu

Abstract Using the method for equation reconstruction of prime sequence, this paper gives the proof that there are infinitely many primes of the form $\alpha\beta^n + \chi$, $\alpha n^\beta + \chi$ and $\alpha\beta_1^n m^{\beta_2} + \chi$.

1 Introduction

Because it can not be divisible except 1 and itself, primes are difficult to be described by appropriate expressions. This property makes prime sequence be difficult to be described such as arithmetic progression, geometric progression with the determined term formula. However, this property can make prime number establish some diophantine equations. And prime numbers can be decided by whether there is positive whole number solutions of these diophantine equations. Therefore, the expressions for solutions of these diophantine equations and its transform are used to describe the divisible property of prime number, and forming an equivalent sequence for the property. Thus, this will be easy to find the key node and the law implied to solve the problem. To this end, the theorem for equation reconstruction of prime sequence is presented and proved. Using the method, this paper gives the proof that there are infinitely many primes of the form $\alpha\beta^n + \chi$, $\alpha n^\beta + \chi$ and $\alpha\beta_1^n m^{\beta_2} + \chi$. It could be hope to provide an idea and methods to solve similar problems.

Zhen Liu

Doctor, Associate Professor of School of Marine Sciences, Sun Yat-sen University, China

Adds: Sun Yat-sen University, No.135, Xingang Xi Rd, Haizhu District, Guangzhou, Guangdong, 510275, China

E-mail: liuzh8@mail.sysu.edu.cn

In this paper, all parameters are positive whole number except where stated.

2 Proof of the theorem for equation reconstruction of prime sequence

Lemma: The prime sequence could be equivalent to the sequence with the determined general term formula through equation reconstruction of prime number for the divisible property.

Proof.

Any prime number c could be expressed as $3a \pm 1$ (a is an positive even), $4a \pm 1$ or $6a \pm 1$.

Proof is carried out the following in the case of $3a \pm 1$ first.

If $3a \pm 1$ is a prime number, it certainly can not be written $3a \pm 1 = (3x_1 \pm 1)(3x_2 \pm 1)$, otherwise, and vice versa.

Case 1: $3a+1$

$$3a+1 = (3x_1+1)(3x_2+1) = 9x_1x_2 + 3(x_1+x_2)+1$$

or

$$3a+1 = (3x_1'-1)(3x_2'-1) = 9x_1'x_2' - 3(x_1'+x_2')+1$$

Where, let $-x_1' = x_1$, $-x_2' = x_2$.

Then there is $a = 3x_1x_2 + (x_1+x_2)$.

It is easy to see that whether $3a+1$ is a prime number depends entirely on the a.

Namely $3a+1$ is a prime number that is equivalent x_1 and x_2 are both positive whole number in $a = 3x_1x_2 + (x_1+x_2)$.

Let $x_1 + x_2 = -q$, $x_1x_2 = p$

According to Vieta's formulas, equation (1) is established.

$$x^2 + qx + p = 0 \tag{1}$$

Then there is $x_{1,2} = \dfrac{-q \pm \sqrt{q^2 - 4p}}{2}$

Therefore, if x_1 and x_2 of equation (1) roots are not both positive whole number, $3a+1$ must be a prime number. Otherwise, it will be a composite number.

There is

$$a = 3p - q$$

Obviously, if $3a+1$ is a prime number, q and $\sqrt{q^2-4p}$ are not both even numbers.

Therefore, in the divisible property of prime number, c_i in prime sequence $\{c_n\}$ is equivalent to $a_i = 3p_i - q_i$ in sequence $\{a_n\}$, namely prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n\}$.

Here q_i and $\sqrt{q_i^2-4p_i}$ are not both even numbers.

In order to facilitate the expression, let $q = 2s$, $p = 2r$.

Here s and r are real numbers.

$\therefore x_{1,2} = s \pm \sqrt{s^2-2r}$

Let $\sqrt{s^2-2r} = t$

There is

$$a = 12st - 12t^2 - 2s$$

Therefore, $a_i = 3p_i - q_i$ in sequence $\{a_n\}$ (q_i and $\sqrt{q_i^2-4p_i}$ are not both even numbers) is equivalent to $a'_i = 12s_it_i - 12t_i^2 - 2s_i$ in sequence $\{a'_n\}$ (s_i and t_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a'_n\}$.

It is obvious that

$$t = \frac{s \pm \sqrt{s^2 + \frac{2s-a}{3}}}{2}$$

Let $s^2 + \dfrac{2s-a}{3} = e^2$

Then there is

$$a = 3s^2 + 2s - 3e^2$$

Therefore, $a_i' = 12s_i t_i - 12t_i^2 - 2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) is equivalent to $a_i'' = 3s_i^2 + 2s_i - 3e_i^2$ in sequence $\{a_n''\}$ (s_i and e_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n''\}$.

It is obvious that

$$s = \frac{-1 \pm \sqrt{9e^2 + 3a + 1}}{3}$$

Let $9e^2 + 3a + 1 = (3h+1)^2$

Then there is

$$3a + 1 = (3h+1)^2 - (3e)^2$$

$$a = 3h^2 + 2h - 3e^2$$

Therefore, $a_i'' = 3s_i^2 + 2s_i - 3e_i^2$ in sequence $\{a_n''\}$ (s_i and e_i are not both positive whole number solutions) is equivalent to $3a_n''' + 1 = (3h_i + 1)^2 - (3e_i)^2$ in sequence $\{a_n''\}$ (e_i and h_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'''\}$.

Case 2: $3a - 1$

$$3a - 1 = (3x_1' + 1)(3x_2' - 1) = 9x_1'x_2' + 3(x_2' - x_1') - 1$$

Where, let $-x_1' = x_1$, $x_2' = x_2$.

Then there is $a = -3x_1x_2 + (x_1 + x_2)$.

Namely $3a - 1$ is a prime number that is equivalent x_1 and x_2 are both positive whole number in $a=-3x_1x_2+(x_1+x_2)$.

Let $x_1+x_2=-q$, $x_1x_2=p$. Here p is negative whole number.

According to Vieta's formulas, equation (2) is established.

$$x^2+qx+p=0 \tag{2}$$

Then there is $x_{1,2}=\dfrac{-q\pm\sqrt{q^2-4p}}{2}$.

Therefore, if x_1 and x_2 of equation (2) roots are not both positive whole number, $3a-1$ must be a prime number. Otherwise, it will be a composite number.

There is

$$a=-3p-q$$

Obviously, if $3a-1$ is a prime number, q and $\sqrt{q^2-4p}$ are not both even numbers.

Therefore, in the divisible property of prime number, c_i in prime sequence $\{c_n\}$ is equivalent to $a_i=-3p_i-q_i$ in sequence $\{a_n\}$, namely prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n\}$.

Using the same argument as in the case 1, we can easily get

$$a=12t^2-12st-2s$$

Therefore, $a_i=-3p_i-q_i$ in sequence $\{a_n\}$ (q_i and $\sqrt{q_i^2-4p_i}$ are not both even numbers) is equivalent to $a_i'=12t_i^2-12s_it_i-2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'\}$.

It is obvious that

$$t=\frac{s\pm\sqrt{s^2+\dfrac{2s+a}{3}}}{2}$$

Let $s^2 + \frac{2s+a}{3} = e^2$

Then there is

$$a = 3e^2 - 3s^2 - 2s$$

Therefore, $a_i' = 12t_i^2 - 12s_i t_i - 2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) is equivalent to $a_i'' = 3e_i^2 - 3s_i^2 - 2s$ in sequence $\{a_n''\}$ (s_i and e_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n''\}$.

It is obvious that

$$s = \frac{-1 \pm \sqrt{9e^2 - 3a + 1}}{3}$$

Let $9e^2 - 3a + 1 = (3h+1)^2$

Then there is

$$3a - 1 = (3e)^2 - (3h+1)^2$$

$$a = 3e^2 - 3h^2 - 2h$$

Therefore, $a_i'' = 3e_i^2 - 3s_i^2 - 2s$ in sequence $\{a_n''\}$ (s_i and e_i are not both positive whole number solutions) is equivalent to $3a_n''' - 1 = (3e_i)^2 - (3h_i + 1)^2$ in sequence $\{a_n''\}$ (e_i and h_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'''\}$.

The prime sequence that prime number c could be expressed as $4a \pm 1$ or $6a \pm 1$, have equivalent methods that are similar to the case of $3a \pm 1$. It can be proved in the same way as shown in the case of $3a \pm 1$ before. Of course, some new equivalent sequences are reconstructed through establishing other forms equations.

This completes the proof.

According to above proof, in the divisible property of prime number, the prime sequence $\{c_n\}$ without term formula is analyzed by using the sequence $\{a_n\}$、$\{a'_n\}$、$\{a''_n\}$、$\{a'''_n\}$ with term formula. This will be easy to find the key node and the law implied to solve the problem.

3 Proof of existing infinitely many primes of the form $\alpha\beta^n + \chi$, $\alpha n^\beta + \chi$ and $\alpha\beta_1^n m^{\beta_2} + \chi$.

Theorem: There are infinitely many primes of the form $\alpha\beta^n + \chi$, $\alpha n^\beta + \chi$ and $\alpha\beta_1^n m^{\beta_2} + \chi$, where α, β and χ are constants, and α and β are both positive whole numbers, χ is an integer, n and m are both any positive whole number.

Proof.

It proves the Theorem with the reduction to absurdity follows.

If the Theorem is not true, it becomes: there is an even number a_0 large enough that makes all primes of the form $\alpha\beta^n + \chi$, $\alpha n^\beta + \chi$ and $\alpha\beta_1^n m^{\beta_2} + \chi$ and bigger than $3a_0 \pm 1$ be composite numbers.

According to the Lemma for equation reconstruction of prime sequence, there are

$$3a - 1 = (3x_1 + 1)(3x_2 - 1) \tag{3}$$

$$3a + 1 = (3x'_1 + 1)(3x'_2 + 1) \text{ or } 3a + 1 = (3x'_1 - 1)(3x'_2 - 1) \tag{4}$$

Where, $a = 2l$, l is positive whole number.

Therefore, when a is large enough, at least one of equation (9) and equation (10) has integer solutions.

Using the same argument as in the proof of equation reconstruction of prime sequence, we can easily get this statement fellows.

For equation (3), there are

$$\frac{a}{2} = 6t_1^2 - 6s_1t_1 - s_1$$

$$t_1 = \frac{s_1 \pm \sqrt{s_1^2 + \frac{2s_1 + a}{3}}}{2}$$

Let $s_1^2 + \frac{2s_1 + a}{3} = e_1^2$

Then, there is

$$s_1 = \frac{-1 \pm \sqrt{9e_1^2 - 6l + 1}}{3}$$

Let $9e_1^2 - 6l + 1 = (3h_1 + 1)^2$, Namely it makes s_1 be a positive whole number.

Then, there is

$$6l - 1 = 9e_1^2 - (3h_1 + 1)^2$$

$$a = 2l = 3e_1^2 - 3h_1^2 - 2h_1$$

$$3a - 1 = 9e_1^2 - (3h_1 + 1)^2 \tag{5}$$

For equation (4), there are

$$\frac{a}{2} = 6s_2t_2 - 6t_2^2 - s_2$$

$$t_2 = \frac{s_2 \pm \sqrt{s_2^2 + \frac{2s_2 - a}{3}}}{2}$$

Let $s_2^2 + \frac{2s_2 - a}{3} = e_2^2$

Then, there is

$$s_2 = \frac{-1 \pm \sqrt{9e_2^2 + 6l + 1}}{3}$$

Let $9e_2^2 + 6l + 1 = (3h_2 + 1)^2$, Namely it makes s_2 be a positive whole number.

Then, there is

$$6l + 1 = (3h_2 + 1)^2 - 9e_2^2$$

$$a = 2l = 3h_2^2 + 2h_2 - 3e_2^2$$

$$3a+1=(3h_2+1)^2-9e_2^2 \tag{6}$$

According to the equation (5) and equation (6), there is

$$3a\pm 1=\left|(3h+1)^2-9e^2\right|$$

Case 1: $3a\pm 1=\alpha\beta^n+\chi$

Then, if the Theorem is not true, it becomes: there is an even number a_0 large enough that makes the equation $\alpha\beta^n+\chi=\left|(3h+1)^2-9e^2\right|$ have integer solutions for all $\alpha\beta^n+\chi>3a_0\pm 1$.

When $\alpha\beta^n+\chi=(3h+1)^2-9e^2$, let

$$\alpha\beta^{n_0}+\chi=3a_0+1=(3h_0+1)^2-9e_0^2=A$$

Where, n_0 is a constant and is also an integer.

Then the equation $\alpha\beta^{n_0\cdot n}+\chi=[3(h_0+\Delta_h)+1]^2-9(e_0+\Delta_e)^2$ has integer solutions for any n.

There is

$$\Delta_h=\frac{1}{3}\left[-3h_0-1\pm\sqrt{(3h_0+1)^2+\left(9\Delta_e^2+18e_0\Delta_e+\alpha\beta^{n_0+n}+\chi-A\right)}\right]$$

$\therefore$ $(3h_0+1)^2+\left(9\Delta_e^2+18e_0\Delta_e+\alpha\beta^{n_0+n}+\chi-A\right)$ is a square number.

Then let

$$(3h_0+1)^2+\left(9\Delta_e^2+18e_0\Delta_e+\alpha\beta^{n_0+n}+\chi-A\right)=k^2$$

$$(3h_0+1)^2=B$$

There is

$$\Delta_e=\frac{1}{3}\left[-3e_0\pm\sqrt{9e_0^2-\left(\alpha\beta^{n_0+n}+\chi-A+B-k^2\right)}\right]$$

$\therefore$ $9e_0^2-\alpha\beta^{n_0+n}-\chi+A-B+k^2$ is also a square number.

Then, there is $9e_0^2-\alpha\beta^{n_0+n}-\chi+A-B+k^2=\delta_{n+n_0}^2$ for arbitrary continuous n.

And there is

$$\delta_{n+n_0}^2-\delta_{n_0}^2=\alpha\beta^{n_0}\left(1-\beta^n\right)$$

It is easy to see that this is also in contradiction with the difference between square numbers $n^2 - n_0^2$.

When $\alpha\beta^n + \chi = 9e^2 - (3h+1)^2$, let

$$\alpha\beta^{n_0} + \chi = 3a_0 - 1 = 9e_0^2 - (3h_0 + 1)^2 = A$$

Where, n_0 is a constant and is also an integer.

Then the equation $\alpha\beta^{n_0 \cdot n} + \chi = 9(e_0 + \Delta_e)^2 - [3(h_0 + \Delta_h) + 1]^2$ has integer solutions for any n.

There is

$$\Delta_h = \frac{1}{3}\left[-3h_0 - 1 \pm \sqrt{(3h_0 + 1)^2 + \left(9\Delta_e^2 + 18e_0\Delta_e - \alpha\beta^{n_0+n} - \chi - A\right)}\right]$$

$\therefore (3h_0 + 1)^2 + \left(9\Delta_e^2 + 18e_0\Delta_e - \alpha\beta^{n_0+n} - \chi - A\right)$ is a square number.

Then let

$$(3h_0 + 1)^2 + \left(9\Delta_e^2 + 18e_0\Delta_e - \alpha\beta^{n_0+n} - \chi - A\right) = k^2$$

$$(3h_0 + 1)^2 = B$$

There is

$$\Delta_e = \frac{1}{3}\left[-3e_0 \pm \sqrt{9e_0^2 + \left(\alpha\beta^{n_0+n} + \chi + A - B + k^2\right)}\right]$$

$\therefore \alpha\beta^{n_0+n} + \chi + 9e_0^2 + A - B + k^2$ is also a square number.

Then, there is $\alpha\beta^{n_0+n} + \chi + 9e_0^2 + A - B + k^2 = \delta_{n+n_0}^2$ for arbitrary continuous n.

And there is

$$\delta_{n+n_0}^2 - \delta_{n_0}^2 = \alpha\beta^{n_0}\left(\beta^n - 1\right)$$

It is easy to see that this is also in contradiction with the difference between square numbers $n^2 - n_0^2$.

$\therefore$ There are infinitely many primes of the form $\alpha\beta^n + \chi$.

Case 2: $\alpha n^\beta + \chi = (3h+1)^2 - 9e^2$

Using the same method as in the proof of Case 1, we can easily get: there is

$\pm\alpha(n_0+n)^{\beta}\pm\chi+9e_0^2+A-B+k^2=\delta_{n+n_0}^2$ for arbitrary continuous n.

And there is

$$\delta_{n+n_0}^2-\delta_{n_0}^2=\pm\alpha[(n_0+n)^{\beta}-n_0^{\beta}]$$

It is easy to see that this is also in contradiction with the difference between square numbers $n^2-n_0^2$.

∴There are infinitely many primes of the form $\alpha n^{\beta}+\chi$.

Case 3: $\alpha\beta_1^n m^{\beta_2}+\chi=(3h+1)^2-9e^2$

Using the same method as in the proof of Case 1, we can easily get: there is $\pm\alpha\beta_1^n m^{\beta_2}\pm\chi+9e_0^2+A-B+k^2=\delta_{n+n_0}^2$ for arbitrary continuous n.

And there is

$$\delta_{n+n_0}^2-\delta_{n_0}^2=\pm\alpha\beta_1^{n_0}[\beta_1^n(m_0+m)^{\beta_2}-m_0^{\beta_2}]$$

It is easy to see that this is also in contradiction with the difference between square numbers $n^2-n_0^2$.s

∴There are infinitely many primes of the form $\alpha\beta_1^n m^{\beta_2}+\chi$.

Therefore, the Theorem is true.

It is now obvious that the theorem holds.

This completes the proof.

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