Relativity Reviewed

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Abstract

Abstract:The paper aims at focusing a greater attention on the errors that might arise due to the commonly practiced Euclidean interpretation of Curved Space time. It also brings out the fact that *classical* black holes may have world lines coming out of them. The expansion of surface area in response to infalling mass indicates towards the potential of a classical black-hole in developing outwardly directed world-lines. They are not serious "graveyards" as traditional belief deems them to be. There is also a mathematical consideration of the issue of transformations between manifolds of distinct nature.This can lead to establishing the covariance of the physical laws wrt distinct manifolds.Some interesting techniques inrelation to solving Partial Differential equations have been indicated at.

Keywords: General Relativity, Curved Spacetime,3D Space, Euclidean Background, Black Hole

1.INTRODUCTION

This letter aims at highlighting certain issues in General relativity which deserve attention. The physical curving of 3D space[in terms of its coordinate labels] due to the presence of a time-varying gravitational field, the problems due to identification of curved space in terms of its Euclidean background have been pointed out. The fact that traditional black holes do not allow world lines coming out of them has also been contradicted.

2. 3D SPACE GETS CURVED

In general relativity coordinate distances[1] and physical distances[2] are identical only for flat spacetime. But in curved spacetime especially if the curvature is strong the coordinate separations and the physical separations may become radically different. Lets consider the physical curving of 3D space in view of the above fact. We consider the x-y plane at two levels z=a and z=b in the flat spacetime context. Several pair wise points having the same value for the z-coordinate are considered on the two said planes which are parallel to each other. A gravitational change is now considered. The metric coefficients change and the physical distances of the points lying on each plane change . If points considered pair wise on each plane and also pairwise on the two separate planes their distances change with changes in the gravitational field and change may occur differently for the different pairs. The planes become undulating surfaces—space gets curved!

Let's consider a spherical planet like the earth. A dense mass approaches it in our thought experiment. The value of the metric coefficients change at each point in the concerned field changes.. Due to gravitational effects, even in our classical interpretation, the shape of the earth's surface might change due to an interaction between the changes in the spacetime curvature and inertial factors like the resistance of the earth's crust.

In our "experiment" in the first paragraph we may consider the coordinates as labels---stickers of different colors at different points on the two planes. Initially they were on a flat surface. After the gravitational change they lie on a pair of undulating surfaces. A straight line on some plane becomes a curved line – the path of a light ray bends and the straight line path of a test particle the Minkowski space[4] picks up the curved path of a planet! Would it be possible to calculate the geodesic paths [spatial-geodesics] from the above considerations with out a direct use of the geodesic Equations in tensor form, as we know them?

3. THE LIGHT CONE IN GENERAL RELATIVITY:

In relativity events are marked by coordinate values and not by physical values. Let's consider a typical GR metric[3] of the form:

$$ds^2 = g_{00}dt^2 - g_{11}dx^2 - g_{22}dy^2 - g_{33}dz^2$$

$$ds^2 = dT^2 - dL^2 \qquad \text{-------------} \ (1)$$

$$\frac{dL}{dT} = 1$$

For the null geodesic $ds^2=0$

=> $g_{00}dt^2 = g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2$

For motion along the x-direction[an infinitesimal path is being considered] we have,

$$ds^2 = g_{00}dt^2 - g_{11}dx^2 \qquad \text{-------------} \ (2)$$

For the null geodesic we have,

$$\frac{dx}{dt} = \sqrt{\frac{g_{00}}{g_{11}}} \qquad \text{--------------} \ (3)$$

The above equation represents the coordinate speed of light which may be different from "c," remembering that c=1 in the natural units.

Again for motion in the x-direction we may write:

Equation (2) may be written as:

$$ds^2 = dT^2 - dL^2$$

For the null geodesic[5],

$$\frac{dL}{dT} = 1 \qquad \text{------------} \ (4)$$

The above relation conforms to the speed of light[c=1 in the natural units]. It provides the Special relativity picture in the local context. But as we pass from cone to cone the "local variables" that is, the variables depicting the physical quantities will change themselves. The cones will get inclined with respect to each other because their axex will not remain parallel to one another anymore.

If the light cone picture in GR uses relation(3), the generators do not in general, make 45^0 with the time axis. For varying gravitational fields the surface of the cone will go on changing. As the observer advances in the positive direction of the time axis carrying the light cone, the distorted surface of the cone goes on changing. Events which are expected to be at a spacelike separation in the future are found to be at time like separation with the advancement of time and vice versa. While plotting the cone in curved. Space one has to consider physical distances corresponding to the appropriate values of the metric. Distance from the origin to the "coordinate label" x along the x-axis should not be $\int_a^b dx$. It should be represented by $\int_a^b \sqrt{g_{xx}}dx$. So we do have local light cones corresponding to equation(4) in so far as the physical variables are concerned.But events in GR are characterized by coordinate variables. The 45^0 generator for the "physical" light cone will contain different sets of coordinate points in different situations. You may consider parallel ensembles to understand the situation.

This opens up another interesting issue. Trajectories of planetary orbits, the bending of light rays etc are expressed as relationships between coordinate values and not as equations relating physical values. The equations in terms of coordinate values conform to the "Euclidean Background" and not the correct picture relevant to curved space! The standard excuse would be to put the blame on the weak curvature[6] of the space around us. But it is this weak field that maintains planets and satellites in their orbits, preventing them from moving along a straight line path as expected in flat spacetime. The weak field is not a negligible agenda as one might be tempted to think of.

4. THE BLACK HOLE PROBLEM

Now let us consider the ideas of the foregoing paragraph in relation to the black hole. One may consider the standard light cone picture[7] in relation to explanation of the singularity. The local light cones correspond to equation (3) and not to equation (4). They depict the "Euclidean Perspective" corresponding to strongly curved space. Is it necessary to account for the "deviations" if conclusions based on the Euclidean picture" are considered to be inaccurate?

4.1 IS THE BLACK HOLE REALLY A GRAVEYARD?

Just think of some mass that has fallen in to a black-hole—you may consider the Schwarzschild Black hole for the sake of simplicity.. The surface area and consequently its volume increases. This idea is consistent with the Bekenstein-Hawking formula which connects entropy of the Black Hole to its surface area[$S = \frac{k_B c^3}{G\hbar} \times \frac{A}{4}$].. The expansion of the surface area and the subsequent increase in volume of the black-hole due to infallen mass is suggestive of world lines directed outwards from the black hole, quite contrary to "traditional" belief! The points on the surface, indeed have to move outwards for such an event to occur.

Again if some charged mass enters into a Schwarzscild's Black hole and gets distributed without disturbing the spherical nature of the metric we have the Reissner Nordstrom metric[8] describing the black-hole now, instead of the Schwarzschild metric.Absence of rotation has been assumed for the sake of simplicity. The black-hole may not remain a black hole any more---it might revert to an uncollapsed star.. Any natural or artificial process that can change the metric may have serious effects on the properties of the *classical* Black-hole.

The basic problem with the classical black hole treatment, **in so far as the world-line picture is concerned**, is that it is considered in alienation from the rest of the universe—a sort of a graveyard right from the outset. Only a "test particle/s" which by definition should not disturb the world line picture "isolated black hole" is/are considered. What happens if something other than a test particle—something which can disturb/disrupt the world line picture of an isolated black hole in a serious manner---moves towards a black hole and falls into it? Is it necessary to modify the world-line picture of the classical black-hole **right from the outset to incorporate its future interaction with the external environment**? The conventional **world-line picture** of the black hole picture results from a total denial of any interaction with the external environment in its mathematical formulation which assumes spherical symmetry for the metric, and such a picture is considered good enough for any type of interaction with the external environment in the future. The mathematics considered in developing the world-line picture of the classical black-hole considers it in isolation from its environment .The same world-line picture ,strictly speaking, should not be used to understand or interpret any type of external interaction in the future where the interacting particles, object or objects do not satisfy the criteria of test particles or an everlasting spherical symmetry. Again changes in the mass [and/or charge] of the black hole in the future leading to a change in the metric or its parameters should taken into account.. Such changes may modify the very nature of the black hole and its properties and convert the classical black hole into a much more interesting object having a greater scope "future transformations". It would also be important to consider the change of the original metric due to the presence of other massive objects in the vicinity of the black-hole that have not merged into it. A resultant metric or an effective metric should provide a better account of physical phenomena.

The black hole metrics are spherically symmetric in nature---an everlasting spherical symmetry is assumed very much against practical considerations in the natural situations. The same metrics should not be considered when other bodies are present in its vicinity. A resultant metric capable of future evolution should provide a better picture of the world lines.

It is important to take note of the fact that the Bekenstein Hawking formula points towards the correctness of the theoretical speculation of properties that cannot be "observed" by for example electromagnetic waves. The Black hole itself is an object that has fallen into a region delimited by the event horizon. Nevertheless the Beckenstein Hawking formula connects its area with entropy[and consequently the change in surface area with the corresponding change in entropy]. This formula accepts and appreciates the existence of properties like surface area and entropy for a body which has collapsed into the event horizon.

It would be also interesting to mention the No Hair conjecture[9] at this juncture:
The **no-hair theorem** postulates that all black hole solutions of the Einstein-Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterized by only three *externally* observable classical parameters: mass, electric charge, and angular momentum. All other information (for which "hair" is a metaphor)

about the matter which formed a black hole or is falling into it, "disappears" behind the black-hole event horizon and is therefore permanently inaccessible to external observers.
The Black Hole itself is an object that has fallen into a region delimited by the event horizon. The externally observable parameters ,mass, charge and angular momentum may be observed only when the body was in an uncollpsed state. In the Black hole state they are no more observable
“All other information (for which "hair" is a metaphor) about the matter which formed a black hole or is falling into it, "disappears" behind the black-hole event horizon and is therefore permanently inaccessible to external observers.”

It seems from the above quotation that mass , charge and angular momentum are within theoretical speculation even after the Black –hole formation, that is, in the collapsed state. In fact the Bekenstein Hawking Formula

5. FROM CURVED SPACETIME TO FLAT SPACETIME

Our next endeavor is to investigate the possibility of going from curved space-time to flat space-time with the preservation of the line element.

5.1 THE TWO METRICS

We consider the following metrics[10]:

$$ds^2 = d\theta^2 + Sin^2\theta\, d\varphi^2 \text{ -------------------- } (5)$$

And

$$ds'^2 = dx^2 + dy^2 \text{ --------------------- } (6)$$

The first metric relates to a spherical surface while the second one relates to a flat surface

Is it possible to pass from metric (5) to metric (6), preserving the value of the line element, ie, by maintaining the relation $ds^2 = ds'^2$.

We use the transformations:

$$\theta = \theta(x, y)$$
$$\varphi = \varphi(x, y)$$

We have the following differentials[11]:

$$d\theta = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy$$

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$$

Using the above differentials in (5) we have:

$$ds^2 = \left(\left(\frac{\partial\theta}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial\varphi}{\partial x}\right)^2\right)dx^2 + \left(\left(\frac{\partial\theta}{\partial y}\right)^2 + \sin^2\theta\left(\frac{\partial\varphi}{\partial y}\right)^2\right)dy^2 + \left(\left(\frac{\partial\theta}{\partial x}\right)\left(\frac{\partial\theta}{\partial y}\right) + \left(\frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\varphi}{\partial y}\right)\right)dxdy$$

-------------------- (7)

We may pass from relation (5) to relation (6) without any change of the line element if the following PDEs[12] are satisfied.

SET A:

$$\left(\left(\frac{\partial\theta}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial\varphi}{\partial x}\right)^2\right) = 1 \qquad \text{------------------A1}$$

$$\left(\left(\frac{\partial\theta}{\partial y}\right)^2 + \sin^2\theta\left(\frac{\partial\varphi}{\partial y}\right)^2\right) = 1 \qquad \text{------------------- A2}$$

$$\left(\left(\frac{\partial\theta}{\partial x}\right)\left(\frac{\partial\theta}{\partial y}\right) + \left(\frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\varphi}{\partial y}\right)\right) = 0 \qquad \text{---------------- A3}$$

We look for a solution where the following relations [SET B] hold true:

$$\frac{\partial\theta}{\partial x} = \psi \qquad \text{------------------ B1}$$

$$\frac{\partial\theta}{\partial y} = \chi \qquad \text{----------------- B2}$$

$$\frac{\partial\varphi}{\partial x} = \frac{1}{\sin\theta}\chi \qquad \text{------------------- B3}$$

$$\frac{\partial\varphi}{\partial y} = -\frac{1}{\sin\theta}\psi \qquad \text{------------------- B4}$$

$$\psi^2 + \chi^2 = 1 \qquad \text{------------- B5}$$

If the above relations[SET B] hold then SET A gets automatically satisfied.

Exact Differential[13] Conditions[SET C]:

$$\frac{\partial\psi}{\partial y}=\frac{\partial\chi}{\partial x} \qquad \text{---------------} \text{ (C1)}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\sin\theta}\chi\right)=-\frac{\partial}{\partial x}\left(\frac{1}{\sin\theta}\psi\right) \qquad \text{-----------} \text{ (C2)}$$

From (C2) we have,

$$-\frac{\cos\theta}{\sin^2\theta}\frac{\partial\theta}{\partial y}\chi+\frac{1}{\sin\theta}\frac{\partial\chi}{\partial y}=\frac{\cos\theta}{\sin^2\theta}\frac{\partial\theta}{\partial x}\psi-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial x}$$

Using B1 and B2 in the above step we have,

$$\frac{\cos\theta}{\sin^2\theta}(\psi^2+\chi^2)=\frac{1}{\sin\theta}(\frac{\partial\psi}{\partial x}+\frac{\partial\chi}{\partial y})$$

Or,

$$\cot\theta=\frac{\partial\psi}{\partial x}+\frac{\partial\chi}{\partial y}$$

[Since, $\psi^2+\chi^2=1$, according to B5]

From the last step we have,

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\cot\theta \qquad \text{-------------} \text{ (D)}$$

[B1 and B2 have been used in deriving relation D from the previous step]

6. SOLVING THE PDE

Let's solve the PDE expressed by (D) , subject to the constraint PDE: $\psi^2+\chi^2=1$,that is, subject to:

$$\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2=1$$

We write equation D as:

$$\frac{\partial^2(\theta-f(\theta)+f(\theta))}{\partial x^2}+\frac{\partial^2(\theta-f(\theta)+f(\theta))}{\partial y^2}=\cot\theta \qquad \text{-------------------} \text{ (E)}$$

We break up (E) into two parts:

$$\frac{\partial^2(\theta - f(\theta))}{\partial x^2} + \frac{\partial^2(\theta - f(\theta))}{\partial y^2} = 0 \qquad \text{---------------------- F1}$$

And

$$\frac{\partial^2 f(\theta)}{\partial x^2} + \frac{\partial^2 f(\theta)}{\partial y^2} = \cot\theta \qquad \text{--------------------- F2}$$

If F1 and F2 produce identical values of θ in terms of x and y our job is done.

Now,

$$\frac{\partial f(\theta)}{\partial x} = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\Rightarrow \frac{\partial^2 f(\theta)}{\partial x^2} = \frac{\partial f}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 f}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2$$

Again,

$$\frac{\partial f(\theta)}{\partial y} = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\Rightarrow \frac{\partial^2 f(\theta)}{\partial y^2} = \frac{\partial f}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 f}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2$$

Thus we have,

$$\frac{\partial^2 f(\theta)}{\partial x^2} + \frac{\partial^2 f(\theta)}{\partial y^2} \equiv \frac{\partial f}{\partial \theta}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + \frac{\partial^2 f}{\partial \theta^2}\left(\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right)$$

PDE F2 reduces to:

$$\frac{\partial f}{\partial \theta}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + \frac{\partial^2 f}{\partial \theta^2}\left(\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right) = \cot\theta$$

Or,

$$\frac{d^2 f(\theta)}{d\theta^2} + \cot\theta\,\frac{df(\theta)}{d\theta} = \cot\theta \qquad \text{---------------- G}$$

The above equation is an ODE and not a PDE. We may take advantage of the situation
Solving (G) we get the functional form of f . One may think in terms of a series solution.
F1 is simply laplace's equation:
We write:

$\theta - f(\theta) = L(x, y)$ -------------- H
Where, L(x,y) represents the solution to Laplace's equation in two dimensions. It may be written as an infinite series involving a huge number of constants.
In (H) we use the functional form of f as obtained from G.

Indeed by double differentiating H, we have,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - \left(\frac{\partial^2 f(\theta)}{\partial x^2} + \frac{\partial^2 f(\theta)}{\partial y^2} \right) = \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2}$$

Or,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - \left(\frac{\partial f}{\partial\theta}\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} \right) + \frac{\partial^2 f}{\partial\theta^2}\left(\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 \right) \right) = 0 \quad \text{----- H1}$$

[Since: $\frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} = 0$]
We write H1 as:

$$\frac{\partial^2 f}{\partial\theta^2}\left(\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 \right) + \frac{\partial f}{\partial\theta}\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} \right) = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} \qquad \text{------- H2}$$

Let's compare this equation ie,[H2] with ODE (G):

$$\frac{d^2 f(\theta)}{d\theta^2} + \cot\theta \frac{df(\theta)}{d\theta} = \cot\theta$$

$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 1$ and $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \cot\theta$ are the simplest options for G and H2 to hold simultaneously. If one is satisfied the other automatically gets satisfied. The pde: $\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 1$ goes into the formulation of the PDE : $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \cot\theta$. But we cannot deduce the first PDE exclusively by using the second one.This is indicative of two classes of solution for the second PDE:

1. Solutions that satisfy both the PDE's. **This is brought out by the fact that PDE $\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 1$ goes into the construction/formulation of the PDE $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \cot\theta$**
2. Solutions that satisfy the second one but not the first one. Two examples of this case:

$$\int_\alpha^\theta \frac{dt}{\sqrt{\ln|K \sin t|}} = Ax + By + C$$

[Limits of integration from α(constant) to θ, a variable.]
Where,
$A^2 + B^2 = 2$
A,B and K are constants

We may consider a solution in the form $\theta = \theta(x)$. The function u does not change in the y-direction. That is , $\frac{\partial\theta}{\partial y} = 0$. Now we have ,

$$\frac{\partial^2\theta}{\partial x^2} = \cot\theta$$

Writing, $p = \frac{d\theta}{dx}$ we have $\frac{dp}{dx} = \cot\theta$ or, $\frac{dp}{d\theta}\frac{d\theta}{dx} = \cot\theta$ or, $pdp = \cot\theta\, d\theta$

Or, $p^2 = 2\ln|\sin\theta| + C$ From here we may proceed to solve the main PDE but the constraint PDE will not be satisfied.

Incidentally we are interested in the first category of solutions. The solution H has two types of constants: constants pertaining to the function $f(\theta)$ and those pertaining to L(x,y), the solution to Laplace's equation in two-dimensions. We may relate two sets of constants so that one of the PDE's gets satisfied—the constraint PDE or the original PDE . The other should automatically get satisfied.

6.1 SOLVING THE ODE:

$$\frac{d^2 f(\theta)}{d\theta^2} + \cot\theta\,\frac{df(\theta)}{d\theta} = \cot\theta \qquad \text{------------------- (I.1)}$$

We write:

$$\frac{d^2 f}{d\theta^2} + \cot\theta\,\frac{df}{d\theta} = \cot\theta - \sum_0^\infty \phi_i(\theta) + \sum_0^\infty \phi_i(\theta)$$

Now we consider two separate equations hoping to get the same solution for $f(\theta)$ from both the equations for proper choice of $\phi_i(\theta)$:

$$\frac{\partial^2 f}{\partial\theta^2} = \sum_0^\infty \phi_i(\theta) \qquad \text{---------------- (I.2)}$$

$$\cot\theta\,\frac{\partial f}{\partial\theta} = \cot\theta - \sum_0^\infty \phi_i(\theta) \qquad \text{------------- (I.3)}$$

Relation (3) may be written as:

$$\frac{\partial f}{\partial\theta} = 1 - \tan\theta \sum_0^\infty \phi_i(\theta) \qquad \text{-------------- (I.4)}$$

From relation (4) we have,

$f = \theta - \int \phi_1 \tan\theta\, d\theta - \int \phi_2 \tan\theta\, d\theta - \int \phi_3 \tan\theta\, d\theta - \int \phi_4 \tan\theta\, d\theta - \ldots\ldots\ldots\ldots$ infinite no of terms ----------------- (I.5)

From(2) we have,

$f = \int\left(\int \phi_1 d\theta\right)d\theta + \int\left(\int \phi_2 d\theta\right)d\theta + \int\left(\int \phi_3 d\theta\right)d\theta + \int\left(\int \phi_4 d\theta\right)d\theta + \ldots\ldots$ infinite no of terms ----------------- (I.6)

The following relations are imposed:

$\theta - \int \phi_1 \tan\theta\, d\theta - \int \phi_2 \tan\theta\, d\theta - \int \phi_3 \tan\theta\, d\theta = \int\left(\int \phi_1 d\theta\right)d\theta$ --------- (I.7)

$-\int \phi_4 \tan\theta\, d\theta = \int\left(\int \phi_2 d\theta\right)d\theta$

$-\int \phi_5 \tan\theta\, d\theta = \int\left(\int \phi_3 d\theta\right)d\theta$

$-\int \phi_6 \tan\theta\, d\theta = \int\left(\int \phi_4 d\theta\right)d\theta$

In general we have,

$-\int \phi_i \tan\theta\, d\theta = \int\left(\int \phi_{i-2} d\theta\right)d\theta$

For i=4,5,6…………

=> $-\phi_i \tan\theta = \int \phi_{i-2} d\theta$ ------------------ (I.8)

Relation (7) may be rewritten as:

$\theta - \int (\phi_1 + \phi_2 + \phi_3) \tan\theta\, d\theta = \int\left(\int \phi_1 d\theta\right)d\theta$

Or,

$1 - (\phi_1 + \phi_2 + \phi_3) \tan\theta = \int \phi_1 d\theta$ ------------------ (I.9)

One has to obtain $\phi_{1,}\phi_2$ and ϕ_3 satisfying (9). The other $\phi_i s$ have to be determined from (I.8).

The issue of convergence for the series on the right hand side of I.2 and I.3 on some given interval is important. The right hand side of the above relation may always converge on some continuous interval of θ

6.2 Antiderivatives of "Unfriendly Functions"

The expression $\int \varphi(\theta)d\theta$ may not have an exact solution in the closed form. *But we can always express the integrand as a polynomial of the nth degree on some specified interval by using standard interpolation techniques and hence obtain an approximate analytical expression of the antiderivative on the said interval by term by term integration of the integrand ,expressed in the form of a polynomial.*

EXAMPLES
Suppose I write,

$\phi_1 = \cot\theta$, equation (9) gives us,

$$\phi_1 + \phi_2 = -\cot\theta \int \cot\theta\, d\theta$$
Or,

$$\phi_1 + \phi_2 = -\cot\theta \ln|\sin\theta| + C$$

Let's take,

$$\phi_1 = -0.5\cot\theta \ln|\sin\theta|$$
And ,

$$\phi_2 = -0.5\cot\theta \ln|\sin\theta| + C$$
Or, we may write
$$\phi_1 = -0.5\cot\theta \ln|\sin\theta| \times \sin^2\theta$$
And
$$\phi_2 = -0.5\cot\theta \ln|\sin\theta| \times \cos^2\theta + C$$
Here we have made use of the relation:

$$\sin^2\theta + \cos^2\theta = 1$$

Using relation (I.8) we can calculate the other phi s. At each stage of integration an extra constant is generated for the LHS of H. on the RHS of H we again have another set of constants!

Let's consider another initial choice of $\phi_1 = \dfrac{\sin^2\theta}{\tan\theta} = \sin\theta\cos\theta$

$$1 - (\phi_1 + \phi_2 + \phi_3)\tan\theta = \int \phi_1 d\theta$$
$$1 - \sin^2\theta - (\phi_2 + \phi_3)\tan\theta = \int \sin\theta\cos\theta\, d\theta$$
$$(\phi_2 + \phi_3)\tan\theta = \cos^2\theta - 0.25\cos 2\theta + C$$
$$(\phi_2 + \phi_3) = \cot\theta(\cos^2\theta - 0.25\cos 2\theta + C)$$

=> $(\phi_2 + \phi_3)\tan\theta = \cos^2\theta - \int \sin\theta\cos\theta\, d\theta$

$$(\phi_2 + \phi_3)\tan\theta = \cos^2\theta - 0.25\cos 2\theta + C$$

Or,
$$(\phi_2 + \phi_3) = \cot\theta\,(\cos^2\theta - 0.25\cos 2\theta + C)$$
We may select ϕ_2 and ϕ_3 satisfying the RHS of the above relation and proceed with relation(I.8). At each stage of integration an extra constant gets added.[Here also we can make use of the relation: $\sin^2\theta + \cos^2\theta = 1$

7 The PDE and its Intricacies

So we have a set of non-linear transformations between two distinct manifolds. It is important to take note of the fact that we may have a set of transformations connecting flat spacetime and curved spacetime—that simply does not make the manifolds identical. We may establish a correspondence between the points of a chair and a table—that does not make them identical!
Calculations:
PDE's
$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 1 \text{ --------- (1)}$$
$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = Cot\theta \text{ -----------(2)}$$

Let's look for a solution in the form:
$$\frac{\partial\theta}{\partial x} = Sin(v(x,y)) \text{ ---------- (3)}$$
And,

$$\frac{\partial\theta}{\partial y} = Cos(v(x,y)) \text{ --------- (4)}$$
The PDE:
$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 1$$
Is automatically satisfied.
Exact Differential Condition:
$$\frac{\partial^2\theta}{\partial y\partial x} = \frac{\partial^2\theta}{\partial x\partial y}$$
=>
$$Cos(v)\frac{\partial v}{\partial y} = -Sin(v)\frac{\partial v}{\partial x} \text{ ------- (5)}$$
By using (3) and (4),the PDE:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=Cot\theta$$

May be written as:

$$Cos(v)\frac{\partial v}{\partial x}-Sin(v)\frac{\partial v}{\partial y}=Cot\theta \quad \text{----- (6)}$$

By applying (5) on (6) we have,

$$\frac{\partial v}{\partial x}=Cot\theta\, Cos(v) \quad \text{------------------ (7)}$$

$$\frac{\partial v}{\partial y}=-Cot\theta\, Sin(v) \quad \text{------------------ (8)}$$

Again

$$\frac{\partial^2 v}{\partial y\partial x}=\frac{\partial^2 v}{\partial x\partial y}$$

If each of the above quantities are much greater than one we may write as an approximation:

$$\frac{\partial^2 v}{\partial y\partial x}\approx\frac{\partial^2 v}{\partial x\partial y}+1\; \frac{\partial^2 v}{\partial y\partial x}\approx\frac{\partial^2 v}{\partial x\partial y}+1 \quad \text{--------------- (9)}$$

Implies:

$$\frac{\partial}{\partial y}(Cot\theta\, Cos(v))\simeq-\frac{\partial}{\partial x}(Cot\theta\, Sin(v))-1$$

Or,

$$-Co\sec^2\theta\,\frac{\partial\theta}{\partial y}Cos(v)-Cot\theta\, Sin(v)\frac{\partial v}{\partial y}=-[-Co\sec^2\theta\,\frac{\partial\theta}{\partial x}Sin(v)+Cot\theta\, Cos(v)\frac{\partial v}{\partial x}]-1$$

Or,

$$-Co\sec^2\theta\, Cos^2(v)+Cot^2\theta\, Sin^2(v)=Co\sec^2\theta\, Sin^2(v)-Cot^2\theta\, Cos^2(v)-1$$

Or,

$$Co\sec^2\theta\,(Cos^2 v+Sin^2 v)-Cot^2\theta\,(Cos^2 v+Sin^2 v)=1$$

Or,

$$(Co\sec^2\theta-Cot^2\theta)(Cos^2 v+Sin^2 v)=1$$

The above result is a consistent one.

If we used $\frac{\partial^2 v}{\partial y\partial x}=\frac{\partial^2 v}{\partial x\partial y}$ instead of $\frac{\partial^2 v}{\partial y\partial x}\approx\frac{\partial^2 v}{\partial x\partial y}+1$ we would have got an inconsistent result.

Now from relations (7) and (8) we have,

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \quad \text{------------ (10)}$$

Which is Laplace's equation in 2-Dimensions.

. The function v=v(x,y) has to be chosen in such a manner that relation (9) is satisfied. The values of $\frac{\partial^2 v}{\partial y \partial x}$ and $\frac{\partial^2 v}{\partial x \partial y}$ have to be much greater than one on some large macroscopic region for all conditions to be satisfied.
Let's try out:

$$v(x,y) = \sum_{i=1}^{n} (A_i e^{p_i x} + B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y)) \text{ --------- (11)}$$

v(x,y) stated above is a solution to the 2D Laplace's Equation.

$$\frac{\partial v}{\partial x} = \sum_{i=1}^{n} p_i (A_i e^{p_i x} - B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))$$

$$\frac{\partial v}{\partial y} = \sum_{i=1}^{n} p_i (A_i e^{p_i x} + B_i e^{-p_i x})(-C_i Sin(p_i y) + D_i Cos(p_i y))$$

$$\frac{\partial^2 \theta}{\partial y \partial x} = Cos(v)\frac{\partial v}{\partial y} = \sum_{i=1}^{n} p_i \left((A_i e^{p_i x} + B_i e^{-p_i x})(-C_i Sin(p_i y) + D_i Cos(p_i y))\right)$$
$$\times Cos\left((A_i e^{p_i x} + B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right)$$

$$\frac{\partial^2 \theta}{\partial x \partial y} = Sin(v)\frac{\partial v}{\partial x} = \sum_{i=1}^{n} p_i \left((A_i e^{p_i x} - B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right)$$
$$\times Sin\left((A_i e^{p_i x} + B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right)$$

$$\frac{\partial^2 \theta}{\partial y \partial x} = Cos(v)\frac{\partial v}{\partial y} = \sum_{i=1}^{n} p_i (A_i e^{p_i x} + B_i e^{-p_i x})(-C_i Sin(p_i y) + D_i Cos(p_i y))$$
$$\times Cos\left((A_i e^{p_i x} + B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right)$$

$$\frac{\partial^2 \theta}{\partial x \partial y} = Sin(v)\frac{\partial v}{\partial x} = \sum_{i=1}^{n} p_i \left((A_i e^{p_i x} - B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right)$$
$$\times Sin\left((A_i e^{p_i x} + B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right)$$

Therefore,

$$\sum_{i=1}^{n} p_i \left((A_i e^{p_i x} + B_i e^{-p_i x})(-C_i Sin(p_i y) + D_i Cos(p_i y))\right)$$
$$\times Cos\left((A_i e^{p_i x} + B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right)$$
$$= \sum_{i=1}^{n} p_i \left((A_i e^{p_i x} - B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right)$$
$$\times Sin\left((A_i e^{p_i x} + B_i e^{-p_i x})(C_i Cos(p_i y) + D_i Sin(p_i y))\right) \text{ -------------- (12)}$$

If we choose "n" ordered pairs(x_i,y_i) we have n equations of type (12) with greater than "n" unknowns p_i , A_i,B_i,C_i and D_i.Generally speaking we have multiple valued solutions for the said unknowns by solving the transcendental equations. The multiple valued nature of the function v(x,y) can allow further relaxations on the relation: $\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$

Solutions do exist for an arbitrary number of discrete points no matter how large the number is. This is compatible with the fact that PDE (1) goes into the construction of PDE (2).

7.1 On Perfect Differentials

We may consider the Taylor expansion of a function of two variables, z= f(x,y):

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \frac{1}{1!}\left(f_x'(x,y)\Delta x + f_y'(x,y)\Delta y\right) + \frac{1}{2!}\left(f_{xx}''(x,y)\Delta x^2 + f_{yy}''^{2}(x,y)\Delta y^2 + 2f_{xy}''\Delta x\Delta y\right)$$

$$+\frac{1}{3!}\left(f_{xxx}'''(x,y)\Delta x^3 + f_{yyy}'''^{3}(x,y)\Delta y^3 + 3f_{xxy}'''\Delta x^2\Delta y + 3f_{xyy}'''\Delta x\ \Delta y^2\right) + ... \text{ --------- (1)}$$

Or,

$$f(x+\Delta x, y+\Delta y) - f(x,y) = \frac{1}{1!}\left(f_x'(x,y)\Delta x + f_y'(x,y)\Delta y\right) + \frac{1}{2!}\left(f_{xx}''(x,y)\Delta x^2 + f_{yy}''^{2}(x,y)\Delta y^2 + 2f_{xy}''\Delta x\Delta y\right)$$

$$+\frac{1}{3!}\left(f_{xxx}'''(x,y)\Delta x^3 + f_{yyy}'''^{3}(x,y)\Delta y^3 + 3f_{xxy}'''\Delta x^2\Delta y + 3f_{xyy}'''\Delta x\ \Delta y^2\right) + ...\text{--------- (2)}$$

Or,

$$\Delta f = \frac{1}{1!}\left(f_x'(x,y)\Delta x + f_y'(x,y)\Delta y\right) + \frac{1}{2!}\left(f_{xx}''(x,y)\Delta x^2 + f_{yy}''^{2}(x,y)\Delta y^2 + 2f_{xy}''\Delta x\Delta y\right)$$

$$+\frac{1}{3!}\left(f_{xxx}'''(x,y)\Delta x^3 + f_{yyy}'''^{3}(x,y)\Delta y^3 + 3f_{xxy}'''\Delta x^2\Delta y + 3f_{xyy}'''\Delta x\ \Delta y^2\right) + ... \text{ -------- (3)}$$

In the limit we have,

$$df = \frac{1}{1!}\left(f_x'(x,y)dx + f_y'(x,y)dy\right) + \frac{1}{2!}\left(f_{xx}''(x,y)dx^2 + f_{yy}''^{2}(x,y)dy^2 + 2f_{xy}''dxdy\right)$$

$$+$$

$$\int_a^b \frac{1}{3!}\left(f_{xxx}'''(x,y)dx^3 + f_{yyy}'''^{3}(x,y)dy^3 + 3f_{xxy}'''dx^2dy + 3f_{xyy}'''dx\ dy^2\right) + ...(infinite\_no\_of\_terms)$$

$$+\frac{1}{3!}\left(f_{xxx}'''(x,y)dx^3 + f_{yyy}'''^{3}(x,y)dy^3 + 3f_{xxy}'''dx^2dy + 3f_{xyy}'''dx\ dy^2\right) + ... \text{ ---------- (4)}$$

$$df \approx \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \text{ ----- (5)}$$

Relation (5) is only an approximate result.
We may think of a rigorous one of the type

$$df = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \quad \text{----- (6)}$$

Where the RHS of (6) is equal to the RHS of (4)

An approximate formula:

$$\int_a^b df \simeq \int_a^b \left(f_x'(x,y)dx + f_y'(x,y)dy \right) \text{------------- (7)}$$

A better formula:

$$\int_a^b df = \int_a^b \left(f_x'(x,y)dx + f_y'(x,y)dy \right) + \int_a^b \frac{1}{2!} \left(f_{xx}''(x,y)dx^2 + f_{yy}''^2\,(x,y)dy^2 + 2f_{xy}''dxdy \right) +$$

$$+$$

$$\int_a^b \frac{1}{3!} \left(f_{xxx}'''(x,y)dx^3 + f_{yyy}'''^3\,(x,y)dy^3 + 3f_{xxy}'''dx^2dy + 3f_{xyy}'''dx\;dy^2 \right) + \text{............}(infinite\_no\_of\_terms)$$

-------------- (8)

The required condition for perfect differentials:

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial y \partial x} \quad \text{---- (9)}$$

Instead of

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{----- (10)}$$

Let's consider a simple example in the single variable function case:

$y = e^{-x^2}$

$dy = -2xe^{-x^2} dx$, if the higher order infinitesimals are excluded

Therefore ,

$dy = 0$ at x=0 in the absence of the higher order differentials!

It is important that we include the higher order infinitesimals in serious work.

8.SOME INTERESTING POINTS

For a coordinate transformation we need to specify the metric in the transformed frame apart from the transformation rules

"What's your name?" may be translated into several languages. The meaning does not change. But there could be some language or rather we may think of devising one where the meaning of the stated string is "Hello , how are you?".This language alphabet characters having the same shape as we have in English. Words of course have different meanings.

If "Harry Dickens" changes his name to "George Brown" by an affidavit the person does not change. But there could be a different person in the same town having the name George brown.

We cannot change the physical nature of an object by coordinate transformation. The only point is that the graphs would be different. A circular ring in the physical world will have graphs of different shapes for various transformations. If the ring is removed from sight and a physicist[who has not seen this particular object before] and he is asked to comment on the shape of the object in the physical world he would run into deep trouble. As a good physicist he is not supposed to have preference towards any particular type of reference frame.

The null geodesic is a straight line in Minkowaski space. By some transformation I can convert it into a curved line. No harm if we treat the new space as simply some mathematical workspace. But there could be some other manifold where light follows exactly the same curved path we have obtained by our transformation

8.1. ON THE UNIVERSALITY OF THE PHYSICAL LAWS

[A SIMPLE ILLUSTRATION]

Let's consider a 2D orthogonal [x-y] system having origin at O in the flat space ontext..A is a point on the x-axis and B is another on the -axis.ABC is a right angled triangle with

$$A'B'^2 = O'A'^2 + O'B'^2$$

We now transform to a non-orthogonal system in the same manifold[flat space]We make the angle between the axes x' and y' =theta not equal to a right angle.The axes are maintained as straight lines. We simply change the inclination between them.

A----->A'
B----->B'
O---->O'

Since ds^2 is preserved

OA=O'A'
OB=O'B'
AB=A'B'

AB is the shortest distance between A and B in the original distance. If you try to figure the situation on a piece of flat paper[after the transformation], the distance A'B' along a straight line path in the new frame will not be equal to AB

Actually the straight line AB is becoming a curved line. A'B' is a curved line. What about the straight line distance between A' and B'?
Should we allow such straight lines in the transformed situation.

We indeed have the relation:

$$A'B'^2 = O'A'^2 + O'B'^2$$

In the new frame.We have Pythagorean for a triangle for a triangle without a right angle--of course on a curved surface. The problem consists in the fact that we may A'B' as a straight line. We may consider the x'-y' surface to become a curved one after the transformation from the x-y system to remove the problem. We simply do not have straight line path between A' and B' in the new frame!

We have passed into a new manifold with the preservation of ds^2 but with the non-preservation of angles.We may do some sort of re- labeling of coordinates to make the new system orthogonal.

The basic point is that we may have two types of transformation

1We change the relative orientation of the axes leaving the system[manifold]undisturbed

2.We distort the x-y plane itself into a curved surface into an undulating one such that ds^2 remains invariant for any pair of points in the process of transformation

The significance of these Transformations:

The tensor object is defined on a particular type of manifold. Coordinate transformations pertaining to the manifold concerned should not change the properties of any physical object pertaining to it. The principle of covariance is based on the invariant form of the tensor equations wrt coordinate transformations on some particular type of manifold.

But we can see from the foregoing discussions we may pass from one manifold to another, keeping the line element invariant giving a mathematical foundation to the universality of the physical laws--that they have the same tensor form[covariant form] in all distinct manifolds.

Any failure to pass between different manifolds[by suitable transformations] would restrict the tensor object to a particular type of manifold.Such a situation would be a hindrance towards the claim of the universality of the physical laws [in covariant form].

9. CONCLUSION

The basic aim of this paper is to create an awareness in regard of the issues discussed in it—to modify the picture of the black hole and also to understand that a “Euclidean Interpretation “ of curved space time may be something quite remote from reality. Finally the passage between distinct manifolds has been discussed to formulate the universality of the physical laws irrespective of distinct manifolds.

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