Geometric analysis of Grover's search algorithm in the presence of perturbation

Wenliang Jin

College of Information Engineering, Yangzhou University, Yangzhou, Jiangsu 225127, P.R. China

E-mail: `wenliang20021@yahoo.cn; wenliangjin86@gmail.com`

Abstract. For an initial uniform superposition over all possible computational basis states, we explore the performance of Grover's search algorithm geometrically when imposing a perturbation on the Walsh-Hadamard transformation contained in the Grover iteration. We give the geometric picture to visualize the quantum search process in the three-dimensional space and show that Grover's search algorithm can work well with an appropriately chosen perturbation. Thereby we corroborate Grover's conclusion that if such perturbation is small, then this will not create much of an impact on the implementation of this algorithm. We also prove that Grover's path cannot achieve a geodesic in the presence of a perturbation of the Fubini-Study metric.

PACS numbers: 02.70.-c, 03.67.-a, 03.65.-w, 03.67.Lx

1. Introduction

The Grover's search algorithm [1] offers a quadratic speed-up over classical counterparts for searching an unsorted database, i.e., the problem of finding a desired state in an unsorted database of size N takes $\Theta(\sqrt{N})$ operations on a quantum computer, and it was shown to be optimal [2–4]. As explained in Refs. [5, 6], the working of Grover's search algorithm can be understood geometrically. Alvarez *et al.* [7] first pointed out that the Grover search problem is related to the geodesic for Fubini-Study metric [8–12], which is a Riemannian metric on projective Hilbert-space, and concluded that Grover's search algorithm defines a geodesic path in quantum Hilbert space. Recently, an information geometric characterization of Grover's search algorithm as a geodesic in the parameter space characterizing the pure state is presented, and some possible deviations from Grover's algorithm within this quantum information geometric framework are discussed [13].

In [14], Grover showed that his original algorithm can be implemented by replacing the Walsh-Hadamard transformation on n qubits $W = H^{\otimes n}$ by (*almost*) any unitary transformation U, where H is single-qubit Hadamard gate operation and $\otimes$ stands for tensor product. In addition to the above result he explicitly presented that if there is a small perturbation δU, it seems plausible that this will not create much of an impact on the implementation of the quantum search algorithm. In order to corroborate this conclusion and to attain a clear understanding of how the quantum system evolves for this case, here, for the sake of simplicity we first assume that we transform, on applying W, the initial zero state $|0\rangle \equiv |00\cdots 0\rangle \equiv |0\rangle^{\otimes n}$ into $|\gamma_0\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle$, an initial unbiased uniform superposition of all $N = 2^n$ computational basis states. Subsequently, we introduce a perturbation δW such that the unit-length vector $|\mu\rangle = (W + \delta W)\,|0\rangle$ embedded in the inversion-about-average operation $-\,(W + \delta W)\, I_0\, (W + \delta W)^{-1} = -\Big(I - 2\,|\mu\rangle\langle\mu|\Big)$ lies outside the two-dimensional space L spanned by $|\gamma_0\rangle$ and a unique desired state, that is, $|\mu\rangle \notin L$. Here I is the identity operator, $I_0 = I - 2\,|0\rangle\langle 0|$, and the superscript $^{-1}$ refers to the inverse of an operator. In this paper, we explore in detail the performance of Grover's search algorithm by comparing the two cases of $|\mu\rangle \notin L$ and $|\mu\rangle = |\gamma_0\rangle \in L$, which may be characterized by the vanishing of the perturbation δW. We demonstrate that the former offers a matrix representation of three-dimensional improper rotation, and further obtain a geometric picture for the time evolution of quantum systems. This suggests an intuitive geometric view of the effect of the perturbation whereby the foregoing result presented by Grover in [14] can be substantiated. Also, we show that Grover's path cannot achieve a geodesic under the influence of a perturbation of the Fubini-Study metric.

2. Three-dimensional orthogonal matrix representation

Suppose that some state $|\beta\rangle$ is the item we are looking for in an unsorted database of N entries. The initial superposition of the quantum system $|\gamma_0\rangle$ may be written in the form

$$|\gamma_0\rangle = \cos\beta_0\,|\alpha\rangle + \sin\beta_0\,|\beta\rangle = \cos\left(\frac{\theta}{2}\right)|\alpha\rangle + \sin\left(\frac{\theta}{2}\right)|\beta\rangle\,, \tag{1}$$

where the normalized basis vector $|\alpha\rangle$ and the angle β_0 (or $\theta/2$) between $|\gamma_0\rangle$ and $|\alpha\rangle$ are given by

$$|\alpha\rangle = \frac{1}{\sqrt{N-1}}\sum_{x\neq\beta}|x\rangle$$

and

$$\beta_0 = \frac{\theta}{2} = \arcsin\left(\sqrt{\frac{1}{N}}\right), \tag{2}$$

respectively, where $\arcsin(\bullet)$ is defined as $-\pi/2 \le \arcsin(\bullet) \le \pi/2$. If there is a perturbation δW which leads to the unit vector $|\mu\rangle = (W+\delta W)|0\rangle \notin L$, then the Grover iteration $G = -(W+\delta W)\,I_0\,(W+\delta W)^{-1}\,I_\beta$, $I_\beta = I - 2|\beta\rangle\langle\beta|$ being the selective inversion of the amplitude of $|\beta\rangle$ reads

$$G = -\Big(I - 2|\mu\rangle\langle\mu|\Big)\Big(I - 2|\beta\rangle\langle\beta|\Big).$$

The nonorthogonal basis $\{|\alpha\rangle, |\beta\rangle, |\mu\rangle\}$ can be orthogonalized to form the orthonormal basis $\{|\alpha\rangle, |\beta\rangle, |S\rangle\}$ by the Gram-Schmidt process, where

$$|S\rangle = \frac{1}{\sqrt{1 - |\langle\alpha\mid\mu\rangle|^2 - |\langle\beta\mid\mu\rangle|^2}}\Big(|\mu\rangle - \langle\alpha\mid\mu\rangle|\alpha\rangle - \langle\beta\mid\mu\rangle|\beta\rangle\Big).$$

Since the basis vectors $|\alpha\rangle$, $|\beta\rangle$ and $|S\rangle$ are mutually orthogonal, $|S\rangle$ is a superposition of the undesired states. It follows from writing

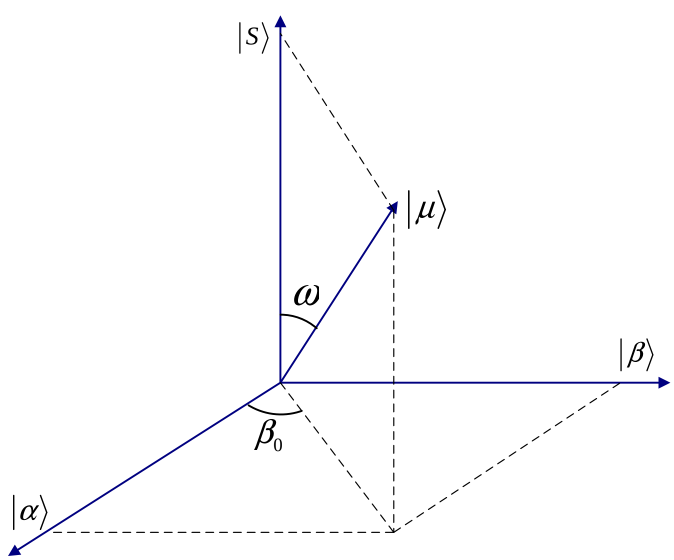

Figure 1. Illustration for the three-dimensional space spanned by the orthonormal basis $\{|\alpha\rangle, |\beta\rangle, |S\rangle\}$.

$$|\mu\rangle = \sin\omega\cos\beta_0\,|\alpha\rangle + \sin\omega\sin\beta_0\,|\beta\rangle + \cos\omega\,|S\rangle\,, \tag{3}$$

where $\omega \in (0, \pi/2]$ (see Fig. 1), that with respect to the ordered orthonormal basis $\{|\alpha\rangle, |\beta\rangle, |S\rangle\}$, the Grover iteration G can be represented by the 3×3 real orthogonal matrix

$$Q = \begin{pmatrix} -1 + 2\sin^2\omega\cos^2\beta_0 & -\sin^2\omega\sin(2\beta_0) & \sin(2\omega)\cos\beta_0 \\ \sin^2\omega\sin(2\beta_0) & 1 - 2\sin^2\omega\sin^2\beta_0 & \sin(2\omega)\sin\beta_0 \\ \sin(2\omega)\cos\beta_0 & -\sin(2\omega)\sin\beta_0 & \cos(2\omega) \end{pmatrix}. \tag{4}$$

For ease of notation, in later sections of this paper, when the perturbation δW vanishes, implying that Eq. (3) becomes $|\mu\rangle = |\gamma_0\rangle$ and thus G only acts nontrivially on the two-dimensional space spanned by $|\gamma_0\rangle$ and $|\beta\rangle$, we shall write $G_{\omega=\pi/2}$ instead of G.

3. Geometric picture of quantum search process for $|\mu\rangle \notin L$

Writing down the eigenvalue equations for Q we obtain

$$\begin{cases} Q\,|h_1\rangle = e^{i2\beta_0'}\,|h_1\rangle \\ Q\,|h_2\rangle = e^{-i2\beta_0'}\,|h_2\rangle\,, \\ Q\,|h_3\rangle = -\,|h_3\rangle \end{cases}$$

where $\beta_0' = \arcsin(\sin\omega\sin\beta_0)$, i denotes the principal square root of -1, and the normalized eigenvectors of Q corresponding to the above eigenvalues are given by

$$\begin{cases} |h_1\rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix} i\upsilon\sin\omega\cos\beta_0 \\ 1 \\ i\upsilon\cos\omega \end{pmatrix} \\ |h_2\rangle = -\dfrac{1}{\sqrt{2}}\begin{pmatrix} \upsilon\sin\omega\cos\beta_0 \\ i \\ \upsilon\cos\omega \end{pmatrix} \\ |h_3\rangle = \upsilon\begin{pmatrix} -\cos\omega \\ 0 \\ \sin\omega\cos\beta_0 \end{pmatrix} \end{cases}$$

with $\upsilon = \left(\sin^2\omega\cos^2\beta_0 + \cos^2\omega\right)^{-1/2}$, and that, for any positive integer j,

$$Q^j = \begin{pmatrix} [Q^j]_{11} & [Q^j]_{12} & [Q^j]_{13} \\ [Q^j]_{21} & [Q^j]_{22} & [Q^j]_{23} \\ [Q^j]_{31} & [Q^j]_{32} & [Q^j]_{33} \end{pmatrix}, \tag{5}$$

whose entries are

$$\begin{aligned}
\left[Q^j\right]_{11} &= \cos(j2\beta_0')\sin^2\omega\cos^2\beta_0/\cos^2\beta_0' + (-1)^j\cos^2\omega/\cos^2\beta_0', \\
\left[Q^j\right]_{12} &= -\sin(j2\beta_0')\sin\omega\cos\beta_0/\cos\beta_0', \\
\left[Q^j\right]_{13} &= \left(\cos(j2\beta_0') + (-1)^{j+1}\right)\sin(2\omega)\cos\beta_0/2\cos^2\beta_0', \\
\left[Q^j\right]_{21} &= \sin(j2\beta_0')\sin\omega\cos\beta_0/\cos\beta_0', \\
\left[Q^j\right]_{22} &= \cos(j2\beta_0')\,, \\
\left[Q^j\right]_{23} &= \sin(j2\beta_0')\cos\omega/\cos\beta_0',
\end{aligned}$$

$$\left[Q^j\right]_{31} = \left(\cos\left(j2\beta_0'\right) + (-1)^{j+1}\right)\sin\left(2\omega\right)\cos\beta_0/2\cos^2\beta_0',$$
$$\left[Q^j\right]_{32} = -\sin\left(j2\beta_0'\right)\cos\omega/\cos\beta_0',$$
$$\left[Q^j\right]_{33} = \cos\left(j2\beta_0'\right)\cos^2\omega/\cos^2\beta_0' + (-1)^j\sin^2\omega\cos^2\beta_0/\cos^2\beta_0'.$$

In the special case when $\omega = \pi/2$, Eq. (5) simplifies to

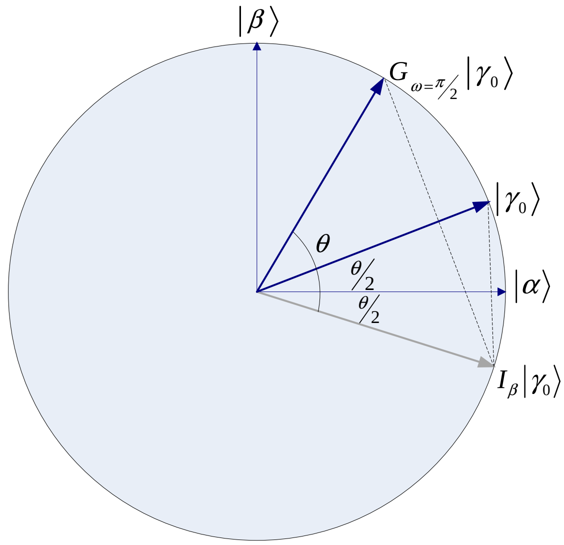

Figure 2. Geometric visualization of a single Grover iteration for $\omega = \pi/2$ (see p. 253 in [6]).

$$\left(Q_{\omega=\pi/2}\right)^j = \begin{pmatrix} \cos\left(j2\beta_0\right) & -\sin\left(j2\beta_0\right) & 0 \\ \sin\left(j2\beta_0\right) & \cos\left(j2\beta_0\right) & 0 \\ 0 & 0 & e^{-ij\pi} \end{pmatrix} \equiv \begin{pmatrix} \cos\left(j2\beta_0\right) & -\sin\left(j2\beta_0\right) \\ \sin\left(j2\beta_0\right) & \cos\left(j2\beta_0\right) \end{pmatrix}. \quad (6)$$

Then, the expression

$$\left(G_{\omega=\pi/2}\right)^j|\gamma_0\rangle = \cos\left(\frac{2j+1}{2}\theta\right)|\alpha\rangle + \sin\left(\frac{2j+1}{2}\theta\right)|\beta\rangle$$

previously given in Eq. (6.12) of Ref. [6] can be retrieved from Eqs. (2) and (6). Grover's search algorithm is shown schematically in Fig. 2 for this case.

We next consider the situation when $\omega \neq \pi/2$ and analyze in this case how the quantum system evolves in an intuitive geometric way. It can be seen that the real orthogonal matrix Q in Eq. (4) provides a matrix representation of an improper rotation, since the determinant $\det(Q) = -1$. Q may be written in the form $Q = R_0P_0$, where $R_0 = -I$ stands for an inversion of the coordinate axes through the origin, and the proper rotation matrix $P_0 = -Q \in SO\left(3\right)$ corresponds to a counterclockwise rotation through an angle

$$\varphi_\omega = \pi + 2\beta_0' \quad (7)$$

which lies in the range $(\pi, 2\pi)$ about the rotation axis $|h_3\rangle$ since φ_ω and $|h_3\rangle$ satisfy the equations $1 + 2\cos\varphi_\omega =tr(P_0)$ and $P_0|h_3\rangle = |h_3\rangle$ respectively, where tr(P_0) is the trace of P_0. Let us consider a clockwise rotation by an angle

$$\varpi = \arctan\left(\cot\omega/\cos\beta_0\right) \quad (8)$$

around $|\beta\rangle$-axis displayed in Fig. 3(a), which is denoted by T_c, where $\arctan(\bullet)$ is defined as $-\pi/2 \leq \arctan(\bullet) \leq \pi/2$. The components of the initial superposition $|\gamma_0\rangle$ previously defined by Eq. (1) in the coordinate system $I'_{\left(\left|h_3^{\perp}\right\rangle, |\beta\rangle, |h_3\rangle\right)}$ will be given by

$$|\gamma_0'\rangle = T_c\,|\gamma_0\rangle = \begin{pmatrix} \cos\varpi & 0 & \sin\varpi \\ 0 & 1 & 0 \\ -\sin\varpi & 0 & \cos\varpi \end{pmatrix} \begin{pmatrix} \cos\beta_0 \\ \sin\beta_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varpi\cos\beta_0 \\ \sin\beta_0 \\ -\sin\varpi\cos\beta_0 \end{pmatrix}.$$

In this new coordinate system, the effect of Grover's algorithm after j-times repetition leads to the resulting state

$$\begin{aligned} G^j\,|\gamma_0'\rangle &= (-1)^j \begin{pmatrix} \cos(j\varphi_\omega) & -\sin(j\varphi_\omega) & 0 \\ \sin(j\varphi_\omega) & \cos(j\varphi_\omega) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varpi\cos\beta_0 \\ \sin\beta_0 \\ -\sin\varpi\cos\beta_0 \end{pmatrix} \\ &= (-1)^j \begin{pmatrix} \sqrt{1-\sin^2\varpi\cos^2\beta_0}\,\cos(j\varphi_\omega+\varphi_1) \\ \sqrt{1-\sin^2\varpi\cos^2\beta_0}\,\sin(j\varphi_\omega+\varphi_1) \\ -\sin\varpi\cos\beta_0 \end{pmatrix}, \end{aligned}$$

where

$$\varphi_1 = \arctan(\tan\beta_0/\cos\varpi). \tag{9}$$

In Fig. 3(b) it is easy to see that if we apply P_0 to $|\gamma_0'\rangle$ and $R_0P_0\,|\gamma_0'\rangle$ respectively, then these two state vectors are both rotated counterclockwise through an angle φ_ω with respect to the same rotation axis $|h_3\rangle$, which results in two identical circular cross sections that are both parallel to the $|\beta\rangle$-axis, implying that this leads to the time evolution of quantum states constrained to move on the two circles labeled C_1 and C_2 with same radius

$$r_\omega = \sqrt{1-d_\omega^2} \tag{10}$$

and centers at O_1 and O_2, where the absolute value of the projection of $|\gamma_0'\rangle$ onto $|h_3\rangle$ is given by $d_\omega = |\langle h_3 \mid \gamma_0'\rangle| = \sin\varpi\cos\beta_0$. In such a geometric picture, it is straightforward to show that Grover's search algorithm deteriorates as ω decreases thanks to Eq. (10) and the relation $|\langle\beta \mid G^j\,|\gamma_0'\rangle| \leq r_\omega$. Rotating back to the original coordinate system $I_{(|\alpha\rangle,|\beta\rangle,|S\rangle)}$, we find

$$\begin{aligned} &\left|\gamma_{(j,\omega)}\right\rangle = G^j\,|\gamma_0\rangle = T_c^{-1}G^j\,|\gamma_0'\rangle \\ &= (-1)^j \begin{pmatrix} \sqrt{1-\sin^2\varpi\cos^2\beta_0}\,\cos(j\varphi_\omega+\varphi_1)\cos\varpi + \sin^2\varpi\cos\beta_0 \\ \sqrt{1-\sin^2\varpi\cos^2\beta_0}\,\sin(j\varphi_\omega+\varphi_1) \\ \sqrt{1-\sin^2\varpi\cos^2\beta_0}\,\cos(j\varphi_\omega+\varphi_1)\sin\varpi - \sin\varpi\cos\varpi\cos\beta_0 \end{pmatrix}. \end{aligned} \tag{11}$$

Thus, for any given $\omega \in (0, \pi/2]$, we can get the success probability

$$P(j,\omega) = \left|\langle\beta \mid \gamma_{(j,\omega)}\rangle\right|^2 = \left(1-\sin^2\varpi\cos^2\beta_0\right)\sin^2(j\varphi_\omega+\varphi_1) = r_\omega^2\sin^2(j\varphi_\omega+\varphi_1)$$

after j iterations of G.

As shown in Figs. 4–6, we can see that for relatively large ω, if N is small (or relatively small), then we can still find the desired state $|\beta\rangle$ with high probability of success after repeating the Grover iteration $j = J_{\omega=\pi/2} = \left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor$ times, where $\lfloor z_0 \rfloor$

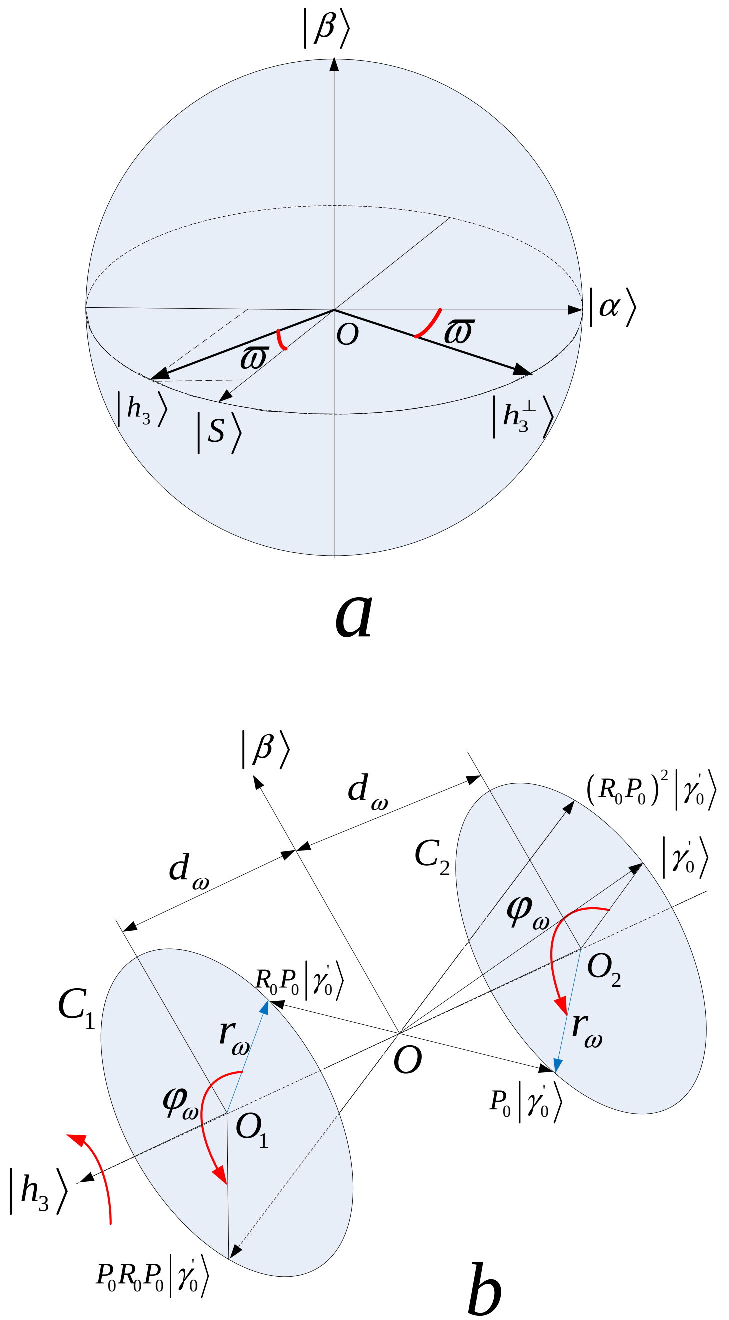

Figure 3. The Bloch sphere and the rotation axis $|h_3\rangle$ that lies in the $|\alpha\rangle$-$|S\rangle$ plane (**a**). Schematic illustration of the time evolution of quantum states for $\omega \neq \pi/2$ (**b**).

stands for the largest integer which is smaller than z_0. This follows from the fact that a relatively large ω produces a small ϖ shown in Eq. (8), leading to a large r_ω defined in Eq. (10), and, additionally, in accordance with Eqs. (7) and (9), φ_ω and φ_1 lie in a vicinity of $\pi + 2\beta_0$ and β_0, respectively.

We now suppose that N is sufficiently large. Employing Eqs. (10) and (8) and neglecting the higher-order terms of β_0, we can work out the maximum success

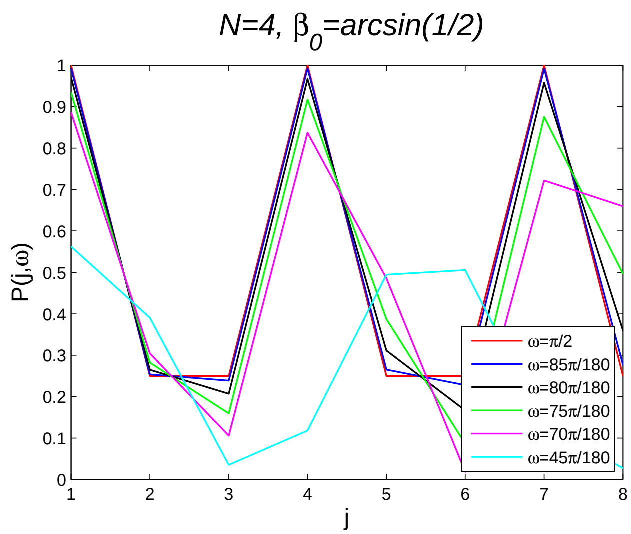

Figure 4. Success probability $P(j,\omega)$ as a function of the time step j for $N=4$ and a single ω for the cases of $\omega=\pi/2, 85\pi/180, 80\pi/180, 75\pi/180, 70\pi/180$ and $45\pi/180$.

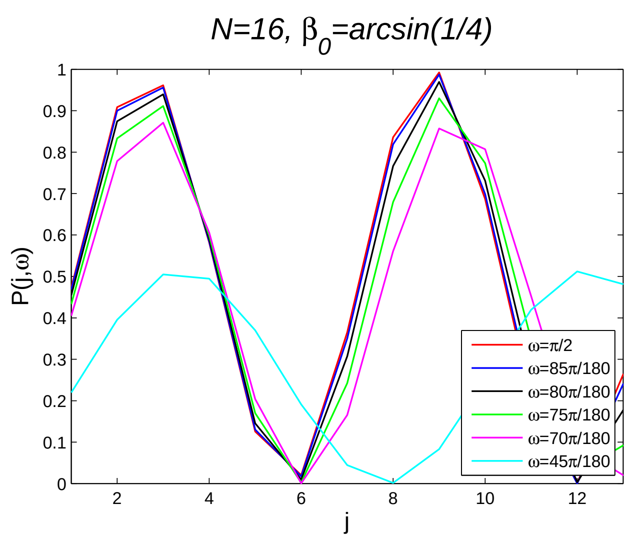

Figure 5. Success probability $P(j,\omega)$ as a function of the time step j for $N=16$ and a single ω for the cases of $\omega=\pi/2, 85\pi/180, 80\pi/180, 75\pi/180, 70\pi/180$ and $45\pi/180$.

probability

$$P(j=J_\omega,\omega)=r_\omega^2 \doteq \cos^2\varpi \doteq \sin^2\omega$$

for any fixed $\omega\in(0,\pi/2]$, where its corresponding number J_ω of iterations satisfies the relation $2J_\omega\beta_0'+\varphi_1=\pi/2$. When $\beta_0\ll\omega$, it readily follows from this relation that

$$J_\omega \doteq \left\lfloor \frac{J_{\omega=\pi/2}}{\sin\omega} \right\rfloor$$

with the aid of Eq. (9), together with $\cos\varpi\doteq\sin\omega$ and

$$\beta_0'=\arcsin(\sin\omega\sin\beta_0)\doteq\sin\omega\sin\beta_0\doteq\beta_0\sin\omega.$$

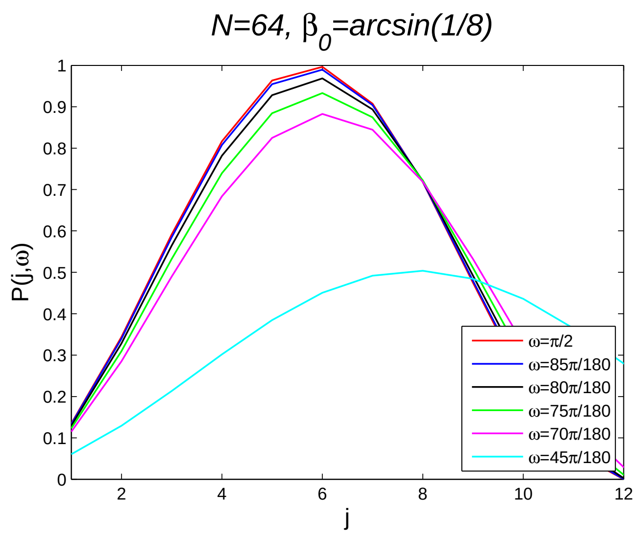

Figure 6. Success probability $P\left(j,\omega\right)$ as a function of the time step j for $N=64$ and a single ω for the cases of $\omega=\pi/2, 85\pi/180, 80\pi/180, 75\pi/180, 70\pi/180$ and $45\pi/180$.

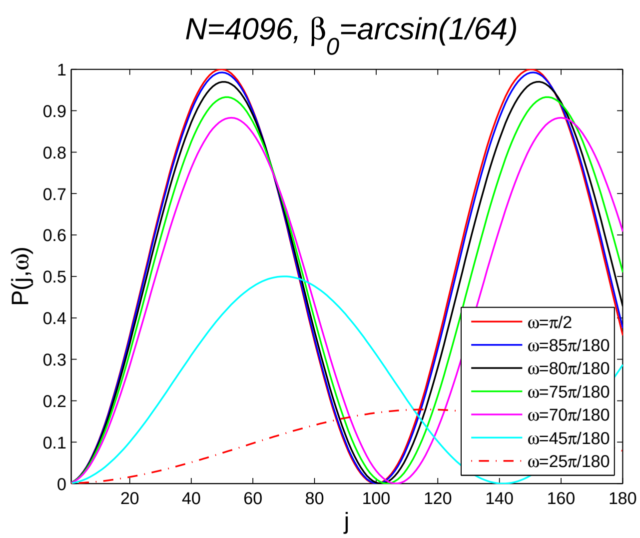

Figure 7. Success probability $P\left(j,\omega\right)$ as a function of the time step j for $N=4096$ and a single ω for the cases of $\omega=\pi/2, 85\pi/180, 80\pi/180, 75\pi/180, 70\pi/180, 45\pi/180$ and $25\pi/180$.

Note that it makes no sense to talk about J_ω for the case when $\omega\sim\beta_0$, because in this situation $P\left(j=J_\omega,\omega\right)\sim 1/N\rightarrow 0$. The validity of the formulae $P\left(j=J_\omega,\omega\right)$ and J_ω in the case of N being a large number is depicted in Fig. 7.

4. On the problem of Grover's path not being a geodesic in the presence of a Fubini-Study metric perturbation

For notational convenience we let $|\beta\rangle = |0\rangle$. From the discussion in Section 2, we have known that $|S\rangle$ is a superposition of the undesired states, so it has the form

$|S\rangle = \sum_{x=1}^{N-1} g_x |x\rangle$, where the real numbers $g_1, \cdots, g_{N-1}$ are not all zero and satisfy

$$\sum_{x=1}^{N-1} g_x^2 = 1 \tag{12}$$

and

$$\sum_{x=1}^{N-1} g_x = 0. \tag{13}$$

As is clear from Eq. (11), for all positive integers j', the state vectors $\left|\gamma_{(j=2j'-1,\omega\neq\pi/2)}\right\rangle$ and $\left|\gamma_{(j=2j',\omega\neq\pi/2)}\right\rangle$ lie on the circles C_1 and C_2, respectively. Hence, for any given $\omega \neq \pi/2$, if we replace $j\varphi_\omega + \varphi_1 \to \vartheta$ for even numbers j and introduce the quantum phases $\phi_x(\vartheta, \omega)$, then we can approximate one of the discrete paths consisting of $\left|\gamma_{(j=2j',\omega)}\right\rangle$ and $|\gamma_0\rangle$ by a path

$$\left|\gamma_{(\vartheta,\omega)}\right\rangle = \sum_{x=0}^{N-1} \sqrt{p_x(\vartheta,\omega)} e^{i\phi_x(\vartheta,\omega)} |x\rangle \tag{14}$$

depending on a continuous parameter ϑ, where

$$p_x(\vartheta,\omega) = |c_x(\vartheta,\omega)|^2 = (c_x(\vartheta,\omega))^2, \tag{15}$$

where $c_0(\vartheta,\omega) = r_\omega \sin\vartheta$ and

$$c_x(\vartheta,\omega) = \frac{r_\omega \cos\vartheta \cos\varpi + \sin^2\varpi \cos\beta_0}{\sqrt{N-1}} + g_x(r_\omega \cos\vartheta \sin\varpi - \sin\varpi\cos\varpi\cos\beta_0) \tag{16}$$

for $x \neq 0$ with r_ω defined as in Eq. (10). By the same argument, the formula (14) also formally holds when we consider another continuous path which is emanated from the discrete path $\left|\gamma_{(j=2j'-1,\omega)}\right\rangle$. Consequently, the N-dimensional probability distribution vector

$$\overrightarrow{p_\omega} = (p_0(\vartheta,\omega), p_1(\vartheta,\omega), \cdots, p_{N-1}(\vartheta,\omega)) \tag{17}$$

virtually defines a path characterizing Grover's algorithm on "probability space" for both of the two discrete paths as above described. Making use of Eqs. (12), (13) and (15), we can evaluate the Fisher's information function $\mathcal{F}(\vartheta,\omega)$ [15] associated with the quantum mechanical wave-vector (14) as

$$\mathcal{F}(\vartheta,\omega) = \sum_{x=0}^{N-1} \frac{\dot{p_x}^2}{p_x} = 4r_\omega^2 \tag{18}$$

for a fixed $\omega \in (0, \pi/2)$, where $\dot{p_x} = \frac{dp_x(\vartheta,\omega)}{d\vartheta}$. In the event that $\omega = \pi/2$, the above results also hold true except that now those two continuous paths coincide with each other since, when $\omega = \pi/2$, Eq. (8) becomes $\varpi = 0$, and so $\langle S \mid \gamma_{(j,\omega)}\rangle$ appearing in Eq. (11) and d_ω are both equal to zero.

For a pure state described by a normalized state vector $\left|\gamma_{(\vartheta,\omega)}\right\rangle$ (14), the Fubini-Study metric on a finite dimensional complex Hilbert space has the form (see p. 5 in [16])

$$ds_{F-S}^2 = \langle d\gamma_{(j,\omega)} \mid d\gamma_{(j,\omega)}\rangle - \langle \gamma_{(j,\omega)} \mid d\gamma_{(j,\omega)}\rangle\langle d\gamma_{(j,\omega)} \mid \gamma_{(j,\omega)}\rangle. \tag{19}$$

Then, according to Eq. (19), we can derive

$$ds_{F-S}^2 = \frac{1}{4}\sum_{x=0}^{N-1}\frac{dp_x^2}{p_x} + \left[\sum_{x=0}^{N-1} p_x d\phi_x^2 - \left(\sum_{x=0}^{N-1} p_x d\phi_x\right)^2\right]$$

or

$$\begin{aligned} ds_{F-S}^2 &= \left\{\frac{1}{4}\sum_{x=0}^{N-1}\frac{\dot{p}_x^2}{p_x} + \left[\sum_{x=0}^{N-1} p_x \dot{\phi}_x^2 - \left(\sum_{x=0}^{N-1} p_x \dot{\phi}_x\right)^2\right]\right\} d\vartheta^2 \\ &= \frac{1}{4}\left(\mathcal{F}(\vartheta,\omega) + 4\sigma_{\dot{Y}_\vartheta}^2\right) d\vartheta^2 \end{aligned} \tag{20}$$

for any fixed $\omega \in (0, \pi/2]$ under the conditions (12) and (13), where $\dot{\phi}_x = \frac{d\phi_x(\vartheta,\omega)}{d\vartheta}$, $\sigma_{\dot{Y}_\vartheta}^2$ is the variance of the random variable

$$\dot{Y}_\vartheta = \left(\dot{\phi}_0(\vartheta,\omega), \dot{\phi}_1(\vartheta,\omega), \cdots, \dot{\phi}_{N-1}(\vartheta,\omega)\right)$$

and $\sigma_{\dot{Y}_\vartheta}^2 = 0$ for all $\omega \in (0, \pi/2]$, as required by the Grover's search algorithm. As discussed above, we see, then, that for the case when $\omega \neq \pi/2$, Eq. (20) can also be regarded as a resulting perturbed metric if we introduce such a nonvanishing perturbation $\delta\left(\underline{ds_{F-S}^2}\right) = -d_{\omega\neq\pi/2}^2 d\vartheta^2$ on the (unperturbed) Fubini-Study metric

$$\underline{ds_{F-S}^2} = \frac{1}{4}\left(\mathcal{F}(\vartheta, \omega = \pi/2) + 4\sigma_{\dot{Y}_\vartheta}^2\right) d\vartheta^2,$$

i.e., this consideration allows us to formulate the decomposition of the perturbed metric ds_{F-S}^2 in the following manner

$$ds_{F-S}^2 = \underline{ds_{F-S}^2} + \delta\left(\underline{ds_{F-S}^2}\right).$$

Taking into account the normalization constraint on the parametric probabilities $p_x(\vartheta,\omega)$, $\sum_{x=0}^{N-1} p_x(\vartheta,\omega) = 1$, and the introduction of a Lagrange multiplier λ, we get the Lagrangian

$$\mathcal{L}\left(\dot{p}_x(\vartheta,\omega), p_x(\vartheta,\omega)\right) = \left[\sum_{x=0}^{N-1}\frac{\dot{p}_x^2(\vartheta,\omega)}{p_x(\vartheta,\omega)}\right]^{\frac{1}{2}} - \lambda\left(\sum_{x=0}^{N-1} p_x(\vartheta,\omega) - 1\right). \tag{21}$$

Accordingly, the geodesic path related to Grover's algorithm can be found by minimizing

$$\mathtt{S} = \int_{\mathtt{A}_0}^{\mathtt{A}_1}\sqrt{ds_{F-S}^2} = \frac{1}{2}\int_{\mathtt{A}_0}^{\mathtt{A}_1}\mathcal{L}\left(\dot{p}_x(\vartheta,\omega), p_x(\vartheta,\omega)\right) d\vartheta \tag{22}$$

with $\sum_{x=0}^{N-1} p_x(\vartheta,\omega) = 1$. If we define $p_x(\vartheta,\omega) = q_x^2(\vartheta,\omega)$, Eq. (21) may be rewritten as follows:

$$\mathcal{L}\left(\dot{q}_x(\vartheta,\omega), q_x(\vartheta,\omega)\right) = \left[4\sum_{x=0}^{N-1}\dot{q}_x^2(\vartheta,\omega)\right]^{\frac{1}{2}} - \lambda\left(\sum_{x=0}^{N-1} q_x^2(\vartheta,\omega) - 1\right). \tag{23}$$

The critical value of the integral (22) is furnished by the Euler-Lagrange equations

$$\frac{d}{d\vartheta}\left(\frac{\partial\mathcal{L}\left(\dot{q}_x(\vartheta,\omega), q_x(\vartheta,\omega)\right)}{\partial\dot{q}_x}\right) - \frac{\partial\mathcal{L}\left(\dot{q}_x(\vartheta,\omega), q_x(\vartheta,\omega)\right)}{\partial q_x} = 0 \tag{24}$$

for each $x \in \{0, 1, \cdots, N-1\}$ by direct differentiation of Eq. (23), where $\partial\mathcal{L}/\partial\dot{q}_x$ and $\partial\mathcal{L}/\partial q_x$ are the partial derivatives. It follows from Eq. (24) that

$$\frac{d^2 q_x(\vartheta,\omega)}{d\vartheta^2} + \lambda r_\omega q_x(\vartheta,\omega) = 0 \tag{25}$$

by using the fact that for a fixed $\omega \in (0, \pi/2]$, the formula (18) does not rely on ϑ. Likewise, in view of the restriction $\sum_{x=0}^{N-1} p_x(\vartheta,\omega) = 1$, we take $\lambda = 1/r_\omega$, and thus Eq. (25) becomes

$$\frac{d^2 q_x(\vartheta,\omega)}{d\vartheta^2} + q_x(\vartheta,\omega) = 0. \tag{26}$$

Whatever values we choose for g_x in Eq. (16), it can be shown that the Grover's path (15) is not a solution to the differential equation (26). This implies that in the presence of a perturbation of the Fubini-Study metric as considered above, the N-dimensional probability vector $\overrightarrow{p_\omega}$ defined in Eq. (17) fails to be a geodesic path.

5. Conclusions

We have investigated the behavior of Grover's search algorithm for the case when the Grover iteration G allows for a perturbation that engenders the unit vector $|\mu\rangle \notin L$. We have shown that the 3×3 real orthogonal matrix corresponding to G provides a matrix representation of an improper rotation, and further, that given any $\omega \in (0, \pi/2)$, the elements in the sequence $\{|\gamma_0\rangle, G|\gamma_0\rangle, G^2|\gamma_0\rangle, \cdots\}$ alternately lie on two circles with the same radius. Moreover, the obtained result, for the fact that Grover's search algorithm can still work well even when ω is not quite large, exhibited the robustness of this algorithm against a suitably chosen perturbation–that is, a moderate but not just small perturbation responsible for the emergence of such a not-quite large angle. The prediction that there will not be much of an impact on the implementation of the quantum search algorithm for a small perturbation [14], thus, was substantiated by the geometrically exact description of quantum search process in the three-dimensional space. Finally, we have demonstrated that the Grover's path, due to the Fubini-Study metric perturbation, cannot become a geodesic in the complex Hilbert space geometry of all computational basis states.

References

[1] Grover L K 1997 *Phys. Rev. Lett.* **79** 325
[2] Bennett C H, Bernstein E, Brassard G and Vazirani U arXiv:quant-ph/9701001v1
[3] Buhrman H, Cleve R, Wolf R de and Zalka C arXiv:cs.CC/9904019
[4] Zalka C 1999 *Phys. Rev. A* **60** 2746
[5] Jozsa R arXiv:quant-ph/9901021v1
[6] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[7] Alvarez J J and Gómez C arXiv:quant-ph/9910115v3
[8] Wootters W K 1981 *Phys. Rev. D* **23** 357
[9] Anandan J and Aharonov Y 1990 *Phys. Rev. Lett.* **65** 1697

[10] Pati A K and Joshi A 1993 *Europhys. Lett.* **21** 723
[11] Brody D C and Hughston L P arXiv:quant-ph/9906086v2
[12] Miyake A and Wadati M 2001 *Phys. Rev. A* **64** 042317
[13] Cafaro C and Mancini S 2012 *Physica A* **391** 1610
[14] Grover L K 1998 *Phys. Rev. Lett.* **80** 4329
[15] Brandt S 1970 *Statistical and computational methods in data analysis* (New York: North Holland Publishing Company)
[16] Shabat B V 1992 *Introduction to Complex Analysis, Part II: Functions of Several Variables* (Providence: American Mathematical Society)