On Andrica’s Conjecture, Cramér’s Conjecture, gaps Between Primes and Jacobi Theta Functions II: 
A Simple Proof of Asymptotic for Andrica’s Conjecture

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1. PRELIMINARES

In [1, p. 185] states that the prime number theorem yields:

\[(1) \pi_n \sim n \log n,\]

that is,

\[(2) \lim_{n \to \infty} \frac{n}{\pi_n \log n} = 1.\]

**LEMMA 1.** The Andrica’s conjecture is equivalent to

\[(3) \sqrt{p_{n+1}} < 1 + 2\sqrt{p_n} + p_n.\]

**Proof.** Step 1. In [2, p. __], we conclude that Andrica’s conjecture is equivalent to

\[(4) \frac{\theta_2}{\theta_3} < \frac{1}{\sqrt{p_n}} \Rightarrow \theta_2 < 1 + \frac{1}{\sqrt[p_n]}\]

where \(k := \frac{p_n}{\sqrt{p_{n+1}}}\) is a \(k\) modulus. In [3, p. 83], we encounter

\[(5) \frac{\theta_2}{\theta_3} = k^{1/2}.\]

Substituting (5) in (4) and considering \(k = \frac{p_n}{\sqrt{p_{n+1}}}\), we have

\[(6) \frac{\theta_2}{\theta_3} < \frac{1}{\sqrt[p_n]} \Rightarrow \sqrt{\frac{p_{n+1}}{p_n}} < 1 + \frac{1}{\sqrt[p_n]}\]

ergo, squaring both sides of the equation (6),

\[
\frac{p_{n+1} \cdot p_n}{p_n} < 1 + \frac{2}{\sqrt[p_n]} + \frac{1}{p_n} \Leftrightarrow p_{n+1} < 1 + 2\sqrt{p_n} + p_n.\]

2. THEOREM

**THEOREM 1** (Asymptotic for Andrica’s Conjecture). Let \(n \in \mathbb{N}\) and \(n\) sufficiently large, then

\[
\sqrt{p_{n+1}} - \sqrt{p_n} \leq 1.
\]

**Proof.** Henceforth, we will use the *reductio ad absurdum* to prove the Lemma 1. We assume that

\[(7) p_{n+1} \geq 1 + 2\sqrt{p_n} + p_n.\]

For \(n\) sufficiently large, we set (1) in (7), as follows

\[(8) \quad (n + 1) \log(n + 1) \geq 1 + 2\sqrt{n \log n} + n \log n.\]

Dividing (8) by \(\log(n + 1)\), we encounter

\[(9) n + 1 \geq \frac{1}{\log(n + 1)} + 2\frac{\sqrt{n \log n}}{\log(n + 1)} + n \frac{\log n}{\log(n + 1)}.\]

On the other hand, it is easy to see that, as \(n\) is sufficiently large, then
Substituting (10) in (9), we find
\[
\frac{\log n}{\log(n+1)} \to 1, \quad \frac{1}{\log(n+1)} \to 0, \quad \frac{\sqrt{n \log n}}{\log(n+1)} \to \infty,
\]
namely,
\[
\frac{\log n}{\log(n+1)} = 1, \quad \frac{1}{\log(n+1)} = 0, \quad \frac{\sqrt{n \log n}}{\log(n+1)} = \infty,
\]
which is false. Therefore, for \( n \) sufficiently large, \( \sqrt{\pi n} < 1 + 2\sqrt{n} + \pi n \). In face of Lemma 1, the asymptotic for Andrica's conjecture is proved.

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**REFERENCES**


[2] Prof. Dr. Raja Rama Gandhi and Guedes, Edigles, *On Andrica’s Conjecture, gaps Between Primes and Jacobi Theta Functions*,