On Andrica’s Conjecture, Cramér’s Conjecture, gaps Between Primes and Jacobi Theta Functions IV:
A Simple Proof for Cramér’s Conjecture

Prof. Dr. Raja Rama Gandhi¹ and Edigles Guedes²
¹Resource person in Math for Oxford University Press, Professor in Math, BITS-Vizag.
²World order Number Theorist, Pernambuco, Brazil.

1. INTRODUCTION

In On the Order of Magnitude of the Difference between Consecutive Prime Numbers [1, p. 27], 1937, Harald Cramér conjectured, using a heuristic method founded on probabilistic arguments, that

\[(1.a) \lim_{n \to \infty} \sup \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1,\]


In this paper, we prove that

\[(1.b) \lim_{n \to \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} < 1,\]

result which is stronger than the Cramér’s conjecture.

2. PRELIMINARES

The Rosser’s theorem [3] states that $p_n$ is larger than $n \log n$. This can be improved by the following pair of bounds:

\[(1) \log n + \log \log n - 1 < \frac{p_n}{n} < \log n + \log \log n,\]

for $n \geq 6$.

3. LEMMA AND THEOREMS

**THEOREM 1.** For $n \in \mathbb{N}_{\geq 6}$ then

\[
\sqrt{\frac{p_{n+1} - p_n}{p_n}} < \sqrt{2 \left(2n^2 - \left[2\sqrt{n(n+1)} + 1\right]n + 2\sqrt{n(n+1)}\right)}.
\]

*Proof.* In previous paper [4, p.__], we discover that

\[(2) \sqrt{\frac{p_{n+1} - p_n}{p_n}} < \sqrt{2(\sqrt{n+1} - \sqrt{n})\sqrt{n}},\]

for $n \in \mathbb{N}_{\geq 6}$. Squaring the inequality (1), we have

\[(3)p_{n+1} + p_n - 2\sqrt{p_{n+1}p_n} < 2n\left(2n + 1 - 2\sqrt{n(n+1)}\right)
\]

\[\Rightarrow p_{n+1} + p_n - 2\sqrt{p_{n+1}p_n} < 2n\left(2n + 1 - 2\sqrt{n(n+1)}\right)
\]

\[\Rightarrow p_{n+1} + p_n < 2\sqrt{p_{n+1}p_n} + 2n\left(2n + 1 - 2\sqrt{n(n+1)}\right).
\]

Multiplying (2) by $2\sqrt{p_n}$, we find

\[(4) 2\sqrt{p_{n+1}p_n} - 2p_n < 2\sqrt{2(\sqrt{n+1} - \sqrt{n})}\sqrt{n}\sqrt{p_n}
\]

\[\Rightarrow 2\sqrt{p_{n+1}p_n} < 2p_n + 2\sqrt{2(\sqrt{n+1} - \sqrt{n})}\sqrt{n}\sqrt{p_n}.
\]

From (3) and (4), we obtain

\[(5) p_{n+1} + p_n < 2p_n + 2\sqrt{2(\sqrt{n+1} - \sqrt{n})}\sqrt{n}\sqrt{p_n} + 2n\left(2n + 1 - 2\sqrt{n(n+1)}\right)
\]

\[\Rightarrow p_{n+1} - p_n < 2\sqrt{2(\sqrt{n+1} - \sqrt{n})}\sqrt{n}\sqrt{p_n} + 2n\left(2n + 1 - 2\sqrt{n(n+1)}\right).
\]

Dividing both members of (5) by $\sqrt{p_n}$, we encounter
On the other hand, we have that

\[
\frac{p_{n+1} - p_n}{\sqrt{p_n}} < 2\sqrt{2} \left( \sqrt{(n+1) - \sqrt{n}} \right) \sqrt{n} + \frac{2n}{\sqrt{p_n}} \left( 2n + 1 - 2\sqrt{n(n+1)} \right).
\]

THEOREM 2. For \( n \in \mathbb{N}_{\geq 0} \), then

\[
p_{n+1} - p_n < 4(n + 1 - \sqrt{n})\sqrt{\log n}.
\]

Proof. Multiplying (2) by \( \sqrt{p_{n+1}}p_n \) and \( \sqrt{p_n} \) respectively, we obtain

(6) \( p_{n+1} - \sqrt{p_{n+1}p_n} < \sqrt{2}(n + 1 - \sqrt{n})\sqrt{n}\sqrt{p_{n+1}}, \)

and

(7) \( \sqrt{p_{n+1}p_n} - p_n < \sqrt{2}(n + 1 - \sqrt{n})\sqrt{n}\sqrt{p_n}. \)

Summing (6) with (7), member by member, we have

(8) \( p_{n+1} - p_n < \sqrt{2}(n + 1 - \sqrt{n})\sqrt{n}(\sqrt{p_{n+1}} + \sqrt{p_n}). \)

From (1) and (8), we find

(9) \( p_{n+1} - p_n < \sqrt{2}(n + 1 - \sqrt{n})\sqrt{n}(\sqrt{p_{n+1}} + \sqrt{p_n}) < \sqrt{2}(n + 1 - \sqrt{n})(\sqrt{n(n+1)\log(n+1)} + (n+1)\log\log(n+1) + \sqrt{n\log n + n\log\log n}) \)

\(< \sqrt{2}(n + 1 - \sqrt{n})\sqrt{n}(\sqrt{2(n+1)\log(n+1) + 2n\log n}) \)

\(= 2(n + 1 - \sqrt{n})\sqrt{n}(\sqrt{n\log(n+1) + \sqrt{n\log n}}). \)

Since \( \sqrt{(n+1)\log(n+1)} \geq \sqrt{n\log n} \), we find

\( p_{n+1} - p_n < 2(n + 1 - \sqrt{n})\sqrt{n}(2\sqrt{n\log n}) \)

\(= 4n(\sqrt{n+1} - \sqrt{n})\sqrt{\log n}. \)

THEOREM 3 (Stronger Cramér’s conjecture).

\( \lim_{n \to \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} < 1. \)

Proof. In Theorem 2, see [6], we have

(10) \( \left( \frac{\theta^2 - \theta_1^2}{\theta_2^2} \right)(n\log n + n\log\log n - n) < g(p_n) < \left( \frac{\theta_2^2 - \theta_1^2}{\theta_2^2} \right)(n\log n + n\log\log n), \)

where \( k = p_n \) to be a \( k \) modulus. In other words,

(11) \( \left( \frac{\theta^2 - \theta_1^2}{\theta_2^2} \right)(n\log n + n\log\log n - n) < p_{n+1} - p_n < \left( \frac{\theta_2^2 - \theta_1^2}{\theta_2^2} \right)(n\log n + n\log\log n). \)
Dividing (11) by \( \left( \frac{x^2-y^2}{y^2} \right) (n \log n + n \log \log n) \), we encounter
\[
\frac{n \log n + n \log \log n - n}{n \log n + n \log \log n} < \left( \frac{\frac{y^2}{y^2}}{\frac{x^2}{x^2}} \right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1.
\]

ergo,
\[
\frac{\log n + \log \log n - 1}{\log n + \log \log n} < \left( \frac{\frac{y^2}{y^2}}{\frac{x^2}{x^2}} \right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1.
\]

But, by the Rosser’s theorem, we find
\[
\frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n - 1 - n \log n - n \log \log n + n}{n \log n + n \log \log n} < \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1.
\]

wherefore,
\[
\frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{n \log n + n \log \log n} < \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1.
\]

Dividing (15) by (13), we find
\[
\frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{n \log n + n \log \log n} \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1.
\]

From (12) and (16), we have
\[
\frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1.
\]

therefore,
\[
\frac{\log n + \log \log n - 1}{\log n + \log \log n} < \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1.
\]

consequently,
\[
\frac{p_{n+1} - p_n}{n \log n + n \log \log n} < \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{n \log n + n \log \log n}.
\]

On the other hand, applying the Rosser’s theorem, we obtain
\[
\frac{p_{n+1} - p_n}{n \log n + n \log \log n} < \frac{1}{n \log n + n \log \log n}.
\]

Dividing (19) by (20), we encounter
\[
\frac{p_{n+1} - p_n}{(\log p_n)^2} < \frac{1}{n \log n + n \log \log n}.
\]

Applying the limit as \( n \to \infty \) in both members of inequality above, we obtain
\[
\lim_{n \to \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} < \lim_{n \to \infty} \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{n \log n + n \log \log n} = 1.
\]

ACKNOWLEDGMENTS

I thank Prof. Dr. K. Raja Rama Gandhi and your society for their encouragement and support during the development of this paper.
REFERENCES

[4] Prof. Dr. Raja Rama Gandhi and Edigles Guedes, *On Andrica’s Conjecture, gaps Between Primes and Jacobi Theta Functions III: A Simple Proof for Andrica’s Conjecture*