

Some formulas and pattern

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Abstract

Some formulas and pattern.

Definition of zeta function

Let

$$\zeta_A^B(x) := \sum_{n=A}^B \frac{1}{n^x} \quad (1)$$

where $A, B \in \mathbb{Z}$ and $x \in \mathbb{C}$.

Consider further $a_n := 1, 2, 4, 11, 31, 83, 227, \dots$ ([OEIS-A002387](#)) which is based on $\zeta_1^\infty(1)$.

A zeta function based construction of e , γ and π

Based on the zeta function ζ and the sequence a_n , the constants γ , e and $\tau := 2\pi$ can be constructed as follows:

$$e = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_{n+0}}$$

$$\gamma = \lim_{n \rightarrow \infty} \zeta_1^{\lfloor (\frac{a_{n+1}}{a_{n+0}})^n \rfloor}(1) - n$$

$$\tau = \min(\{ x \in \mathbb{R}^+ : \lim_{n \rightarrow \infty} \frac{\zeta_{a_{n+0}}^{a_{n+1}}(1+ix)}{\zeta_{a_{n+1}}^{a_{n+2}}(1+ix)} \neq (\frac{a_{n+1}}{a_{n+0}})^{ix} \})$$

Note that even though numerical evidence indicates that the sequence a_n coincides with $[e^{n-\gamma}]$, this relationship is unproven and is not exploited in the above construction of e , γ and τ .

Euler's identity

The previous construction of π is based on the formula:

$$\lim_{n \rightarrow \infty} \frac{\zeta_{a_{n+0}}^{a_{n+1}}(x)}{\zeta_{a_{n+1}}^{a_{n+2}}(x)} = e^{x-1},$$

which holds for all $x \in \mathbb{C}$ except those x of the form $2ki\pi + 1$ with $k \in \mathbb{Z} \setminus \{0\}$.

Euler's identity $e^{i\pi} = -1$ appears as special case of above formula for $x = 1i\pi + 1$.

x	$\cos(x) + i \cdot \sin(x)$		x	$\lim \frac{\zeta_{a_{n+0}}^{a_{n+1}}(x)}{\zeta_{a_{n+1}}^{a_{n+2}}(x)}$
5π	$e^{5i\pi}$		$5i\pi + 1$	$e^{5i\pi}$
4π	$e^{4i\pi}$		$4i\pi + 1$	\sim
3π	$e^{3i\pi}$		$3i\pi + 1$	$e^{3i\pi}$
2π	$e^{2i\pi}$		$2i\pi + 1$	\sim
1π	$e^{i\pi}$		$1i\pi + 1$	$e^{i\pi}$
0π	1		$0i\pi + 1$	1
-1π	$e^{-i\pi}$		$-1i\pi + 1$	$e^{-i\pi}$
-1π	$e^{-2i\pi}$		$-2i\pi + 1$	\sim
-3π	$e^{-3i\pi}$		$-3i\pi + 1$	$e^{-3i\pi}$
-4π	$e^{-4i\pi}$		$-4i\pi + 1$	\sim
-5π	$e^{-5i\pi}$		$-5i\pi + 1$	$e^{-5i\pi}$

The wave symbol (\sim) indicates that the limes does not exist

An alternative zeta function (in esp. harmonic) based construction of π is given by:

$$\pi = \lim_{n \rightarrow \infty} n \cdot \Im\left(1 - \left(\frac{1}{\gamma - H_n}\right)^{\frac{1}{n}}\right)$$

where H_n is the n -th harmonic number. This construction arises from the indeterminate forms 0^0 and $1/0$ (see Section 4).

It can be concluded, that γ , e and π can be constructed exclusively from the harmonic series, considering the sequence a_n . In contrast to that, it is unknown if $a_n \stackrel{?}{=} [e^{n-\gamma}]$ can be constructed from those constants. This question might be of interest in an evolutionary context.

1 Some notes on the fraction of harmonic numbers

Definitions

$$A(n) := \sum_{i=1}^n \frac{1}{i} - \log(n)$$

$$B(n) := \sum_{i=1}^n \frac{1}{i} - \lfloor \sum_{i=1}^n \frac{1}{i} \rfloor$$

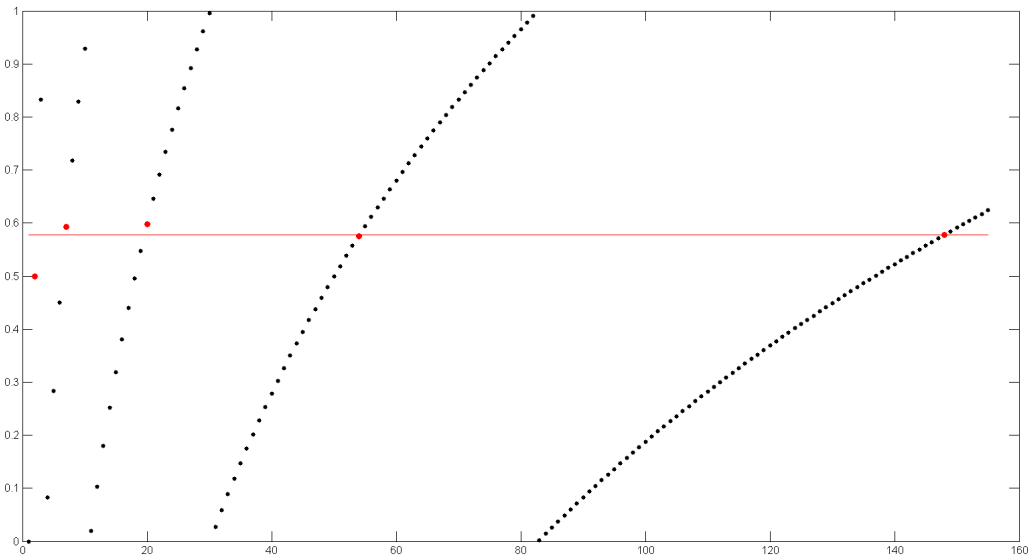
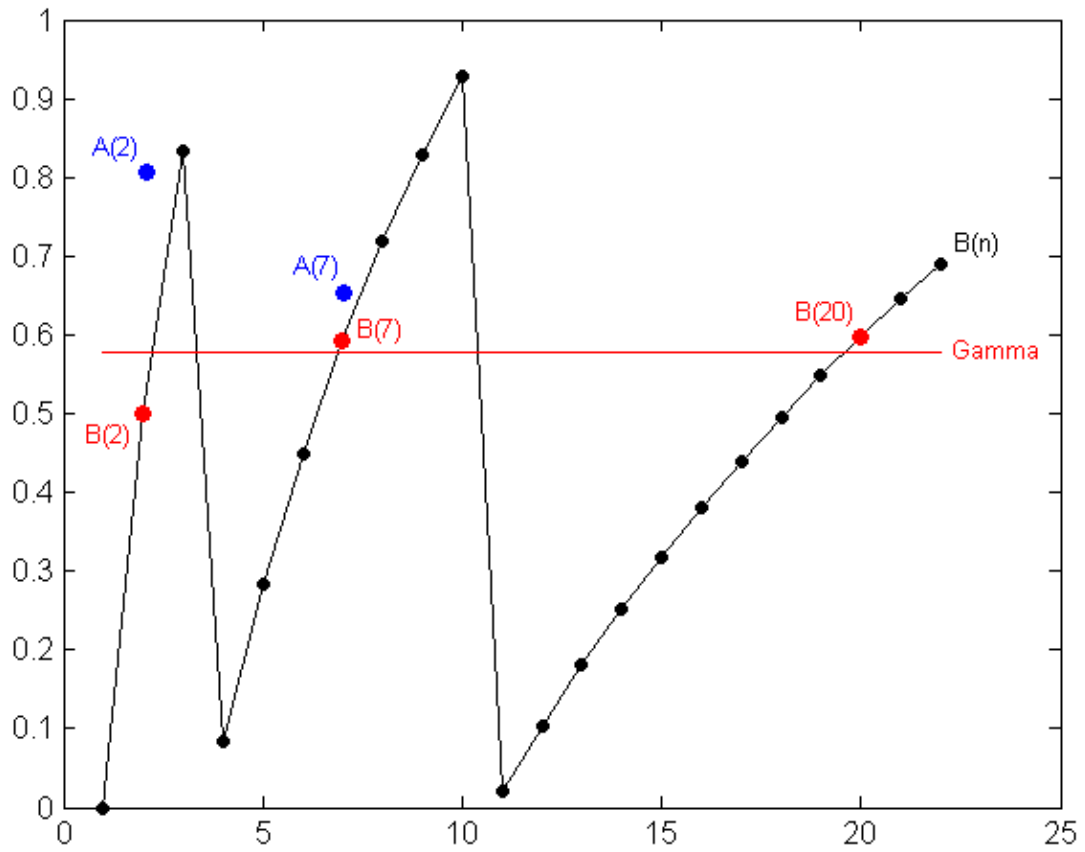
$$C(n) := \log(n) - \lfloor \log(n) \rfloor$$

$$\gamma := \lim_{n \rightarrow \infty} A(n)$$

In every period of oscillation of B there are n , so that $B(n)$ is closer to γ than $A(n)$

$$\exists n \in \mathbb{N}: B(n) - \gamma < A(n) - \gamma$$

n	$B(n)$	$B(n) - \gamma$	$<$	$A(n) - \gamma$	$A(n)$
2	0.500000000000000	-0.07721566490153	$<$	0.22963715453852	0.80685281944005
7	0.59285714285714	0.01564147795561	$<$	0.06973132890030	0.64694699380183
20	0.59773965714368	0.02052399224215	$<$	0.02479171868816	0.60200738358969
54	0.57543039374427	-0.00178527115727	$<$	0.00923068227846	0.58644634717999
148	0.57780251258124	0.00058684767971	$<$	0.00337457391559	0.58059023881713
403	0.57739240852976	0.00017674362823	$<$	0.00124018168154	0.57845584658308
1096	0.57709426741491	-0.00012139748662	$<$	0.00045613500542	0.57767179990695
2980	0.57706202025190	-0.00015364464963	$<$	0.00016777585092	0.57738344075245
8103	0.57726701163418	0.00005134673265	$<$	0.00006170427193	0.57727736917346
22026	0.57721721789959	0.00000155299805	$<$	0.00002270027317	0.57723836517470
59874	0.57722164886082	0.00000598395928	$<$	0.00000835084692	0.57722401574845
162754	0.57721387436125	-0.00000179054028	$<$	0.00000307211796	0.57721873701949
442413	0.57721590899706	0.00000024409553	$<$	0.00000113016521	0.57721679506674



The periods of oscillation of B encode $e \approx 2.71828$

$$\forall a, b \sim \text{Figure 1, 2, 3} \longrightarrow \frac{a}{b} \approx e$$

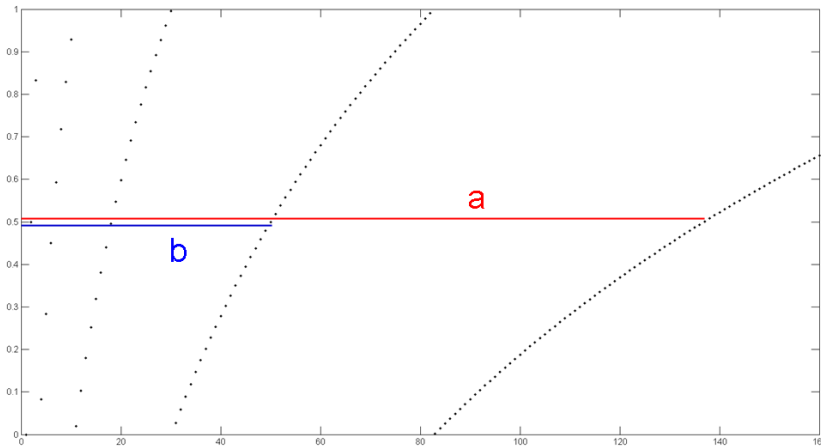


Figure 1:

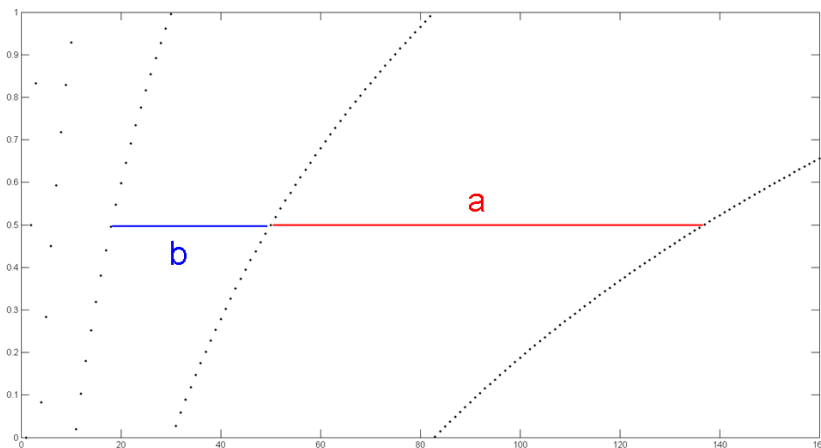


Figure 2:

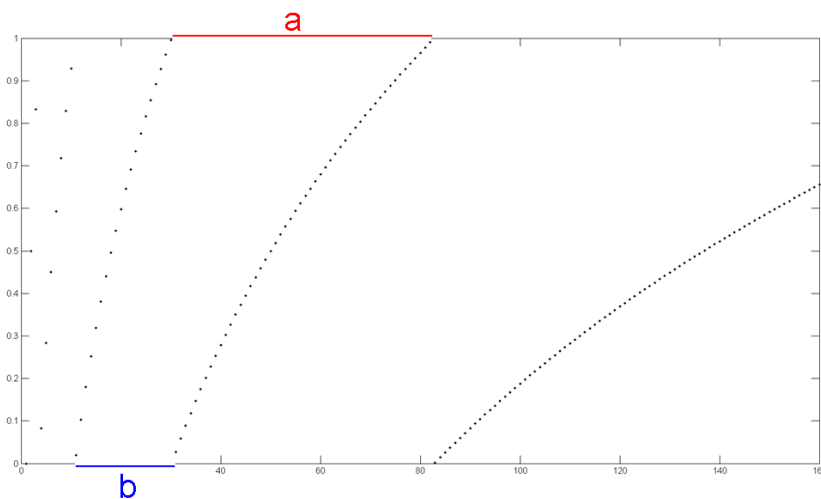


Figure 3:

Relationship between $B(n)$, $C(n)$ and γ

$$B(n) \approx D(n) := \begin{cases} C(n) + \gamma - 1 & ,\text{if } B(n) < \gamma \\ C(n) + \gamma & ,\text{if } B(n) > \gamma \end{cases}$$

n	$B(n)$	\approx	$D(n)$
1	0.000000000000000		-0.422784335098467
2	0.500000000000000	\approx	0.270362845461478
3	0.833333333333333	\approx	0.675827953569642
4	0.083333333333333	\approx	-0.036489973978577
5	0.283333333333333	\approx	0.186653577335633
6	0.450000000000000	\approx	0.368975134129588
7	0.592857142857143		1.523125813956846
8	0.717857142857143	\approx	0.656657206581369
9	0.828968253968254	\approx	0.774440242237752
10	0.928968253968254	\approx	0.879800757895579
20	0.597739657143682		1.572947938455524
100	0.187377517639621	\approx	0.182385850889625
148	0.577802512581241		1.574427938665648
1000	0.485470860550343	\approx	0.484970943883670
10000	0.787606036044348	\approx	0.787556036877716
100000	0.090146129863335	\approx	0.090141129871761
1000000	0.392726722864989	\approx	0.392726222865806
10000000	0.695311365857272	\approx	0.695311315859853

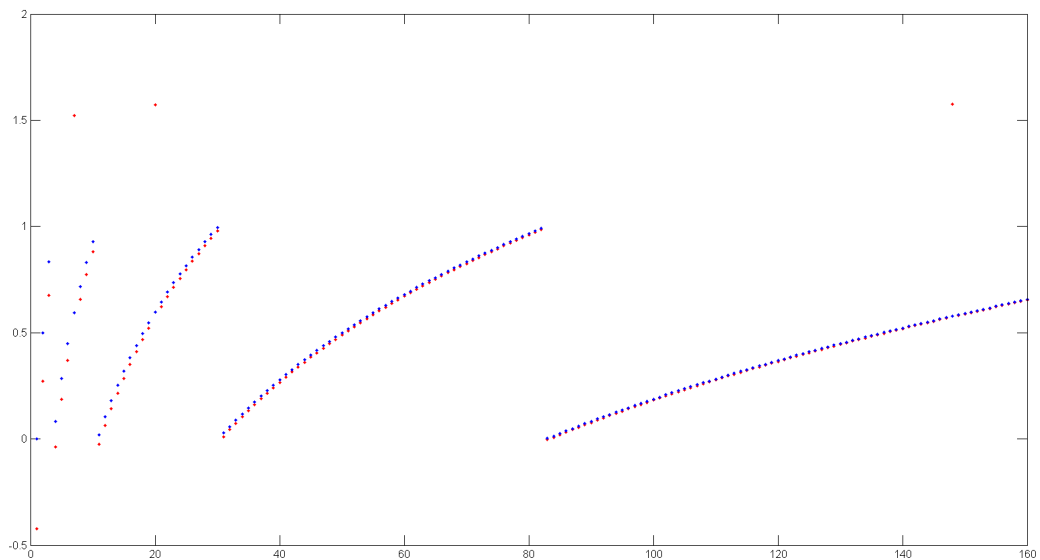
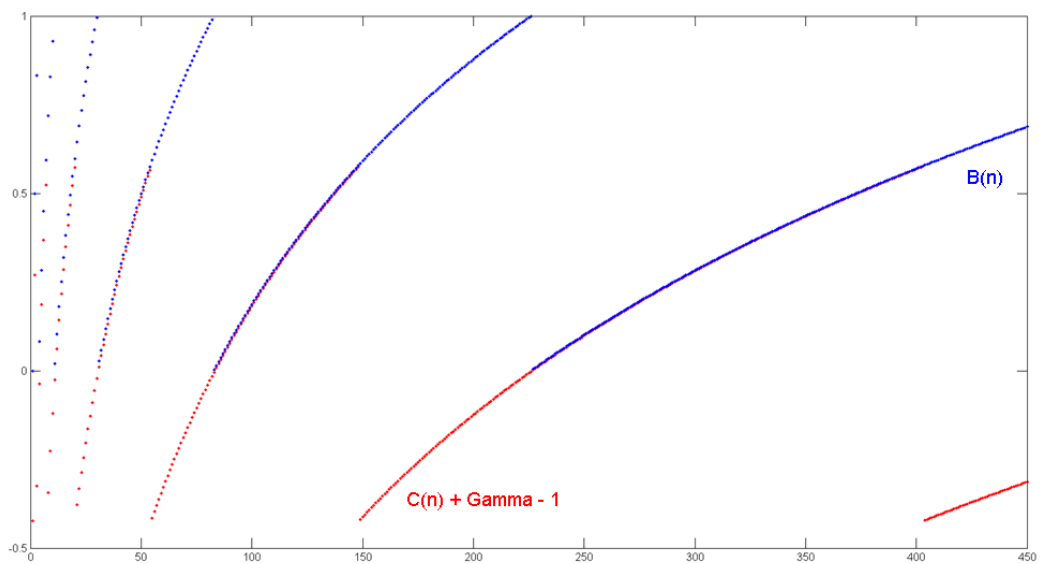
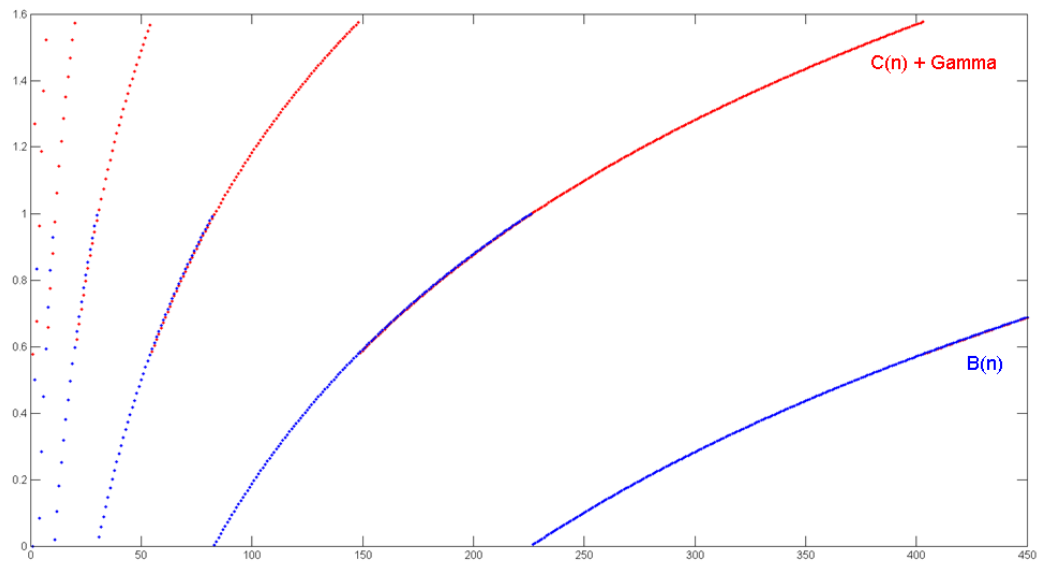
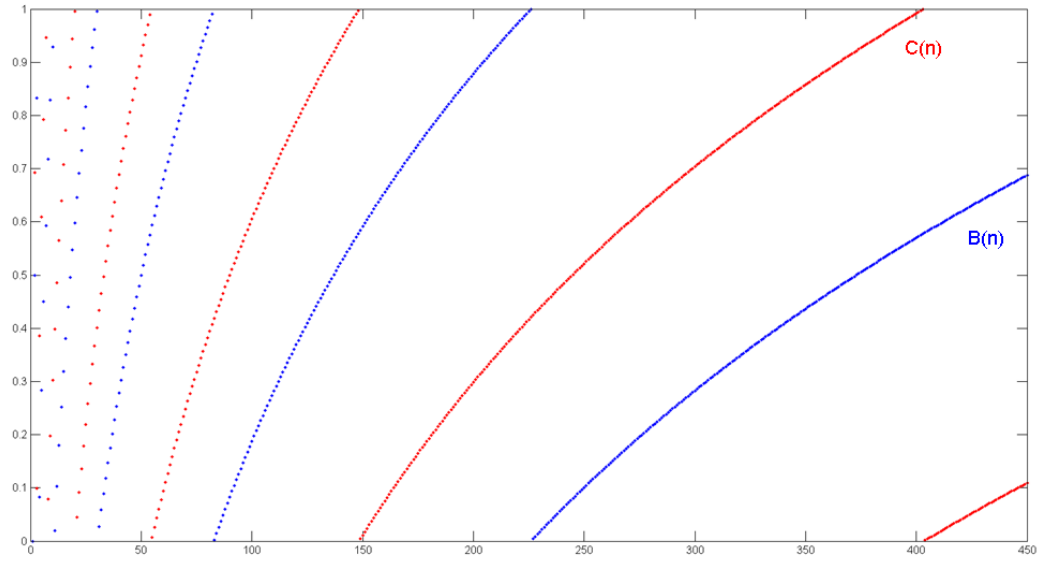


Figure 4: Plot of $B(n)$ (blue) and $F(n)$ (red)



Max/Min Relationships

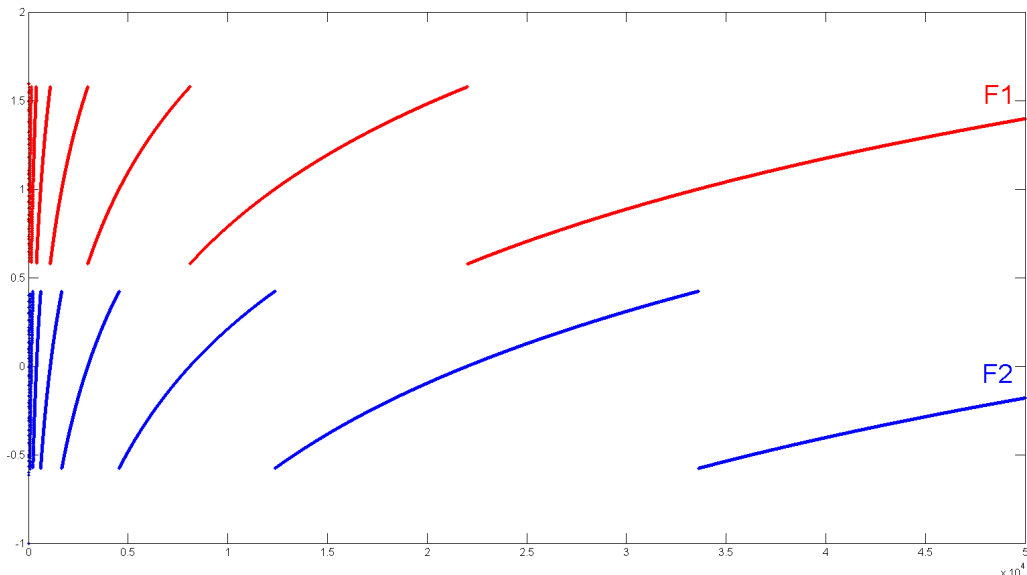
$$F_1(n) := \sum_{i=1}^n \frac{1}{i} - \lfloor \log(n) \rfloor \quad (\in \mathbb{Q})$$

$$F_2(n) := \log(n) - \left\lfloor \sum_{i=1}^n \frac{1}{i} \right\rfloor \quad (\in \mathbb{R})$$

Formula for γ :

$$\lim_{n \rightarrow \infty} \min\{ F_1 \} = \gamma$$

$$\lim_{n \rightarrow \infty} \max\{ F_2 \} = 1 - \gamma$$



n	$\min\{ F_1 \}$	$\max\{ F_2 \}$	$\min\{ F_1 \} + \max\{ F_2 \}$
1	1.0000000000000000	-1.0000000000000000	0.0000000000000000
2	1.0000000000000000	-0.306852819440055	0.693147180559945
3	0.8333333333333333	0.098612288668110	0.931945622001443
4	0.8333333333333333	0.098612288668110	0.931945622001443
5	0.8333333333333333	0.098612288668110	0.931945622001443
6	0.8333333333333333	0.098612288668110	0.931945622001443
7	0.8333333333333333	0.098612288668110	0.931945622001443
8	0.717857142857143	0.098612288668110	0.816469431525252
9	0.717857142857143	0.197224577336220	0.915081720193362
10	0.717857142857143	0.302585092994046	1.020442235851188
100	0.593612211926086	0.406719247264253	1.000331459190339
1000	0.579867656054511	0.421622267806518	1.001489923861029
10000	0.577390407488084	0.422662707570003	1.000053115058087
100000	0.577238350322196	0.422770971743748	1.000009322065944

2 Some notes on the Euler-Mascheroni constant

Formula for γ

$$\begin{aligned}\gamma &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{i} - \log(n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - \log(\lfloor e^n \rfloor) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n\end{aligned}$$

Note that the closer $e^n - \lfloor e^n \rfloor$ gets to $\frac{1}{2}$, the better is the approximation by the above formula. Therefore the subsequence $n = 1, 2, 4, 5, 6, 10, 16, 21, 85, 115, \dots$ ([OEIS-A080053](#)) gives a strictly monotonic approximation of γ by the above formula.

Pattern

$$\sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n - \gamma < \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - \log(\lfloor e^n \rfloor) - \gamma$$

and

$$\left| \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n - \gamma \right| < \left| \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - \log(\lfloor e^n \rfloor) - \gamma \right|$$

Definition

$$A(n) := \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - \log(\lfloor e^n \rfloor)$$

$$B(n) := \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n$$

$$T_A(n) := \frac{1}{2} \frac{n - \log(\lfloor e^n \rfloor)}{e^n - \lfloor e^n \rfloor}$$

$$T_B(n) := \frac{\frac{1}{2} + \lfloor e^n \rfloor - e^n}{\lfloor e^n \rfloor}$$

Pattern

$$A(n) - \gamma \approx T_A(n)$$

$$B(n) - \gamma \approx T_B(n)$$

n	$A(n) - \gamma$	\approx	$T_A(n)$	$B(n) - \gamma$	\approx	$T_B(n)$
1	0.229637154538522	\approx	0.213601964634381	-0.077215664901533	\approx	-0.109140914229523
2	0.069731328900296	\approx	0.069514205140797	0.015641477955610	\approx	0.015849128724193
3	0.024791718688158	\approx	0.024946691364183	0.020523992242149	\approx	0.020723153840617
4	0.009230682278460	\approx	0.009208353109854	-0.001785271157266	\approx	-0.001817593206375
5	0.003374573915593	\approx	0.003373671569258	0.000586847679708	\approx	0.000586762820428
6	0.001240181681544	\approx	0.001240035204982	0.000176743628226	\approx	0.000176691085025
7	0.000456135005418	\approx	0.000456072655816	-0.000121397486621	\approx	-0.000121494916477
8	0.000167775850920	\approx	0.000167758271537	-0.000153644649630	\approx	-0.000153686926754
9	0.000061704271926	\approx	0.000061705221596	0.000051346732651	\approx	0.000051347948243
10	0.000022700273166	\approx	0.000022700204904	0.000001552998054	\approx	0.000001552946213
11	0.000008350846916	\approx	0.000008350860275	0.000005983959284	\approx	0.000005983979727
12	0.000003072117962	\approx	0.000003072113647	-0.000001790540282	\approx	-0.000001790548951
13	0.000001130165210	\approx	0.000001130165204	0.000000244095527	\approx	0.000000244095629
14	0.000000415763583	\approx	0.000000415764407	0.000000179472383	\approx	0.000000179473229
15	0.000000152950261	\approx	0.000000152951169	0.000000039010172	\approx	0.000000039011082
16	0.000000056265555	\approx	0.000000056267590	-0.000000002309892	\approx	-0.000000002307857
17	0.000000020695265	\approx	0.000000020699688	-0.000000010502283	\approx	-0.000000010497860
18	0.000000007612997	\approx	0.000000007614984	0.000000005521458	\approx	0.000000005523449
19	0.000000002802197	\approx	0.000000002801399	-0.000000002594347	\approx	-0.000000002595144
20	0.000000001031574	\approx	0.000000001030576	0.000000000186934	\approx	0.000000000185936

Generalization

$$n = \lfloor e^{\frac{v}{w}} \rfloor$$

$$v := \lfloor \log(n+1) \cdot (n+1) \rfloor$$

$$w := n+1$$

n	v/w	$\lfloor e^{\frac{v}{w}} \rfloor$
1	$1 / 2 = 0.50000$	1
2	$3 / 3 = 1.00000$	2
3	$5 / 4 = 1.25000$	3
4	$8 / 5 = 1.60000$	4
5	$10 / 6 = 1.66667$	5
6	$13 / 7 = 1.85714$	6
7	$16 / 8 = 2.00000$	7
8	$19 / 9 = 2.11111$	8
9	$23 / 10 = 2.30000$	9
10	$26 / 11 = 2.36364$	10

Definition

$$\tilde{A}(n) := \sum_{i=1}^n \frac{1}{i} - \log(n)$$

$$\tilde{B}(n) := \sum_{i=1}^n \frac{1}{i} - \frac{v}{w}$$

$$\tilde{T}_A(n) := \frac{1}{2} \frac{\frac{v}{w} - \log(n)}{e^{\frac{v}{w}} - n}$$

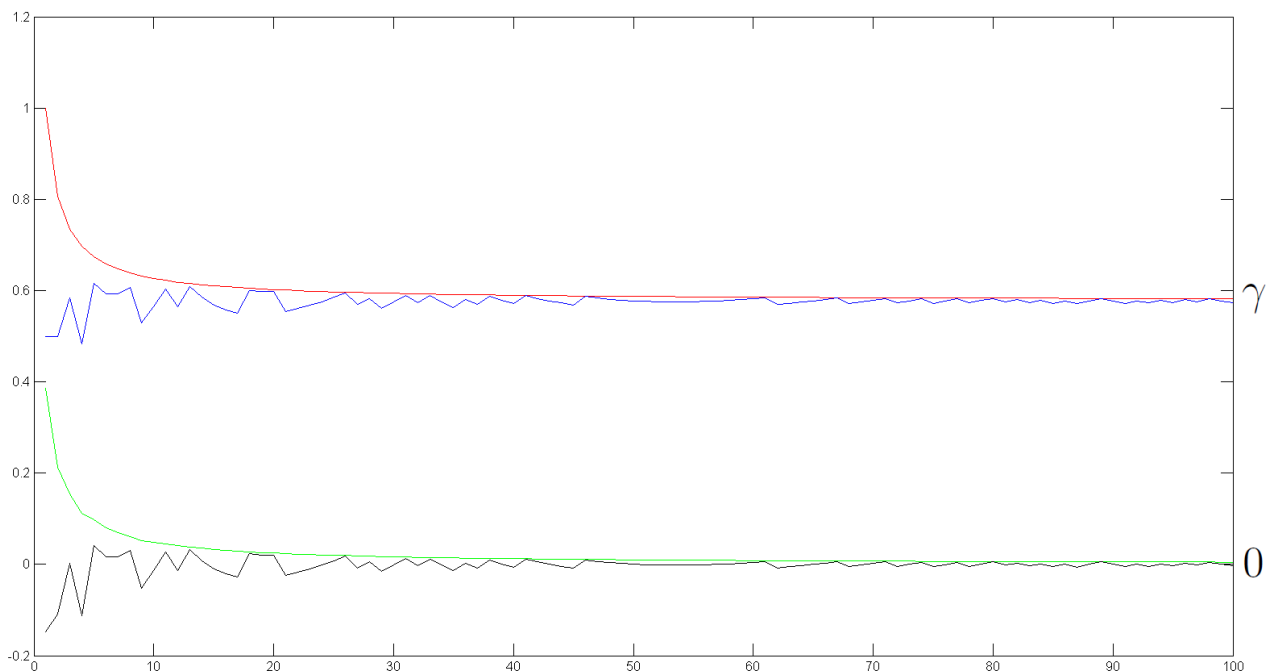
$$\tilde{T}_B(n) := \frac{\frac{1}{2} + n - e^{\frac{v}{w}}}{n}$$

Pattern (generalized)

$$\tilde{A}(n) - \gamma \approx \tilde{T}_A(n)$$

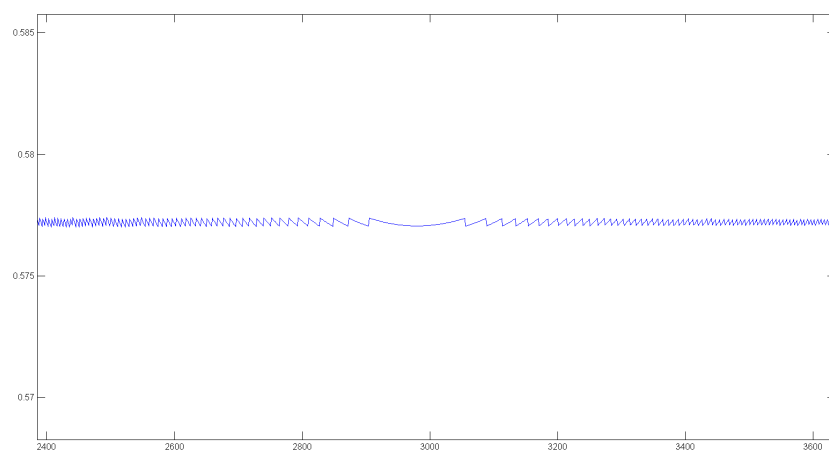
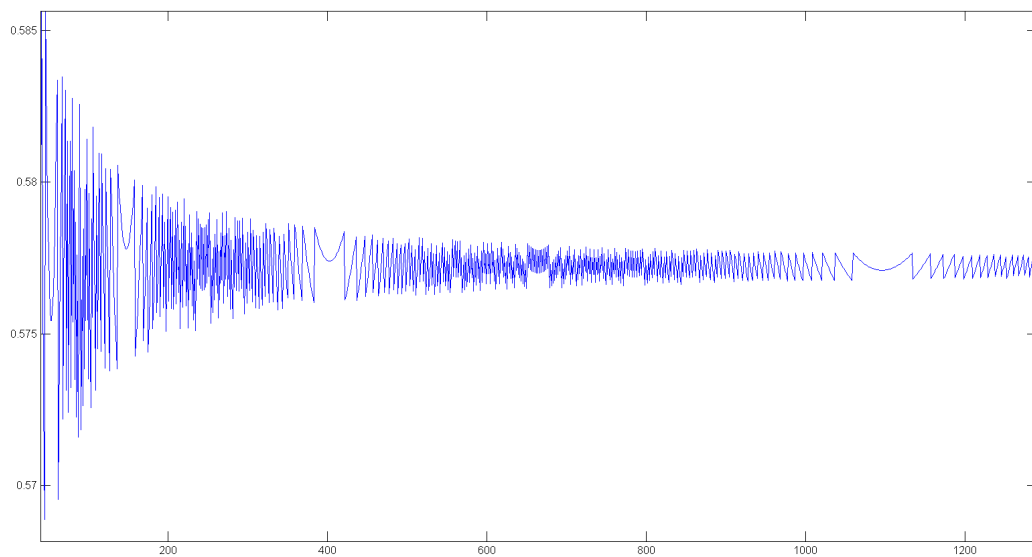
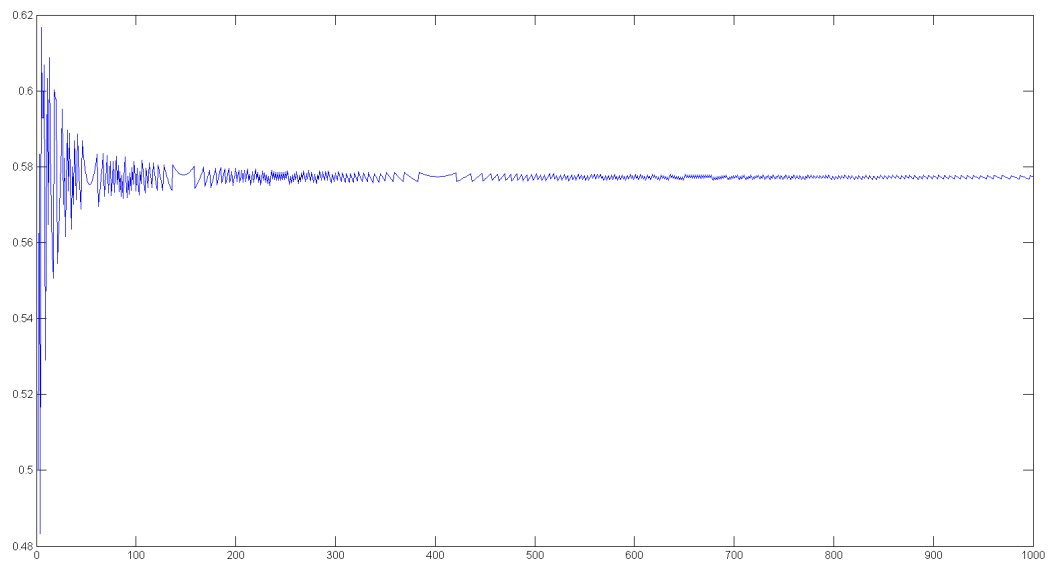
$$\tilde{B}(n) - \gamma \approx \tilde{T}_B(n)$$

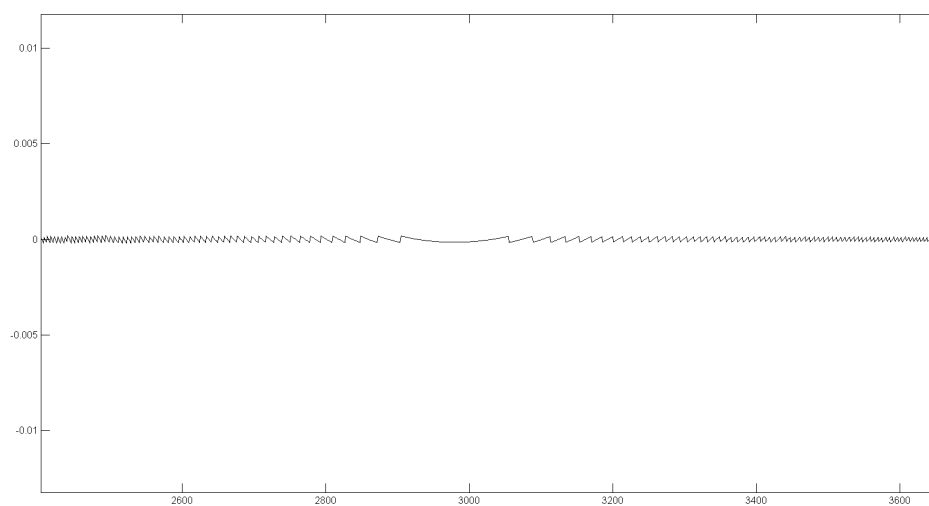
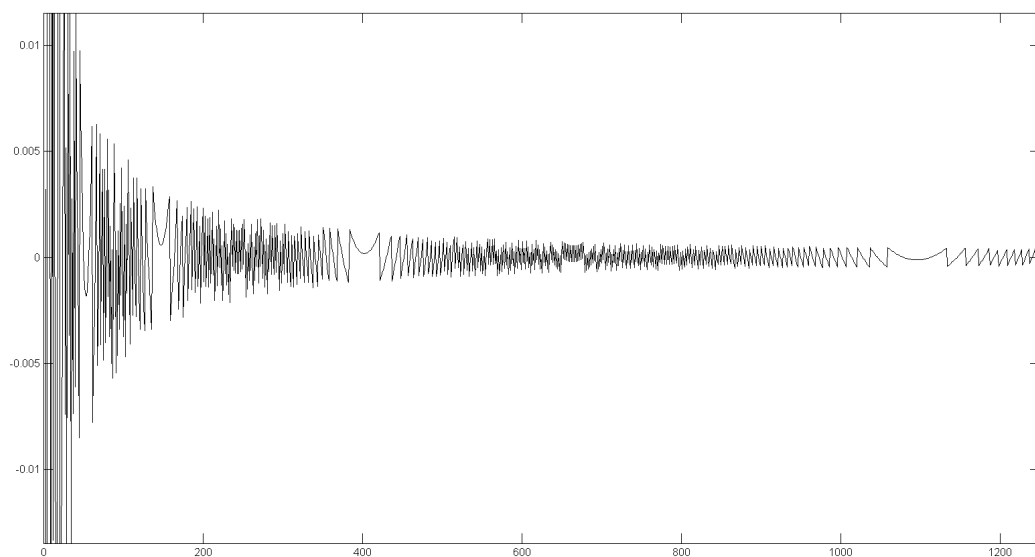
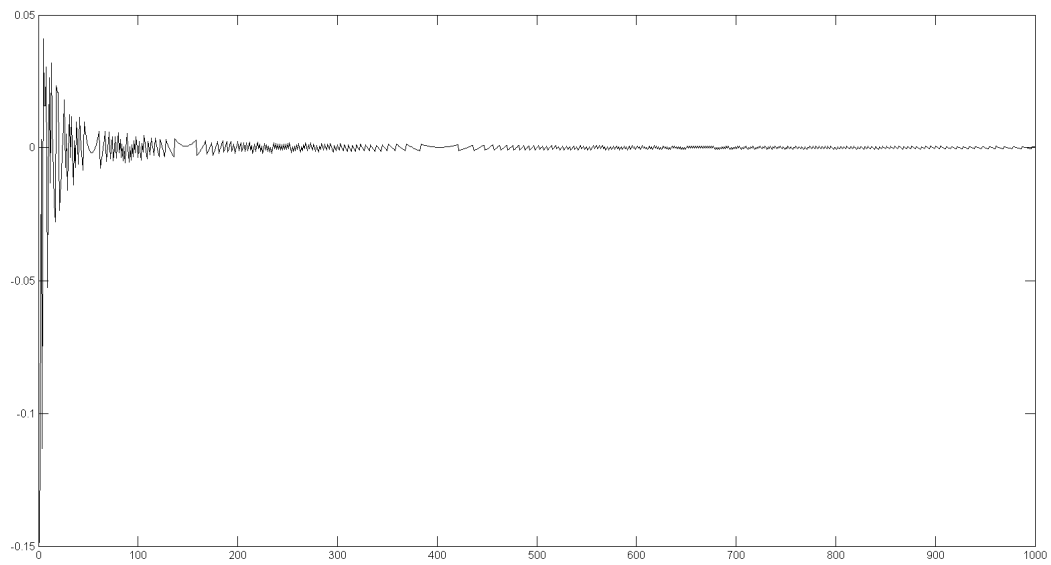
n	$\tilde{A}(n) - \gamma$	\approx	$\tilde{T}_A(n)$	$\tilde{B}(n) - \gamma$	\approx	$\tilde{T}_B(n)$	$\frac{v}{w}$
1	0.42278434	\approx	0.38537352	-0.07721566	\approx	-0.14872127	0.5000
2	0.22963715	\approx	0.21360196	-0.07721566	\approx	-0.10914091	1.0000
3	0.15750538	\approx	0.15436921	0.00611767	\approx	0.00321901	1.2500
4	0.11982331	\approx	0.11211877	-0.09388233	\approx	-0.11325811	1.6000
5	0.09667976	\approx	0.09716585	0.03945100	\approx	0.04110199	1.6667
6	0.08102487	\approx	0.08063871	0.01564148	\approx	0.01576509	1.8571
7	0.06973133	\approx	0.06951421	0.01564148	\approx	0.01584913	2.0000
8	0.06119994	\approx	0.06151555	0.02953037	\approx	0.03032361	2.1111
9	0.05452801	\approx	0.05274958	-0.04824741	\approx	-0.05268694	2.3000
10	0.04916750	\approx	0.04848925	-0.01188377	\approx	-0.01295341	2.3636
11	0.04476641	\approx	0.04502926	0.02599501	\approx	0.02650586	2.4167
12	0.04108836	\approx	0.04056090	-0.01246653	\approx	-0.01334823	2.5385
13	0.03796873	\approx	0.03833707	0.03148952	\approx	0.03196129	2.5714
14	0.03528933	\approx	0.03522353	0.00767999	\approx	0.00772028	2.6667
15	0.03296313	\approx	0.03263906	-0.00898667	\approx	-0.00950879	2.7500
16	0.03092461	\approx	0.03046081	-0.02001608	\approx	-0.02101048	2.8235
17	0.02912351	\approx	0.02860060	-0.02655203	\approx	-0.02784283	2.8889
18	0.02752066	\approx	0.02771720	0.02315557	\approx	0.02340315	2.8947
19	0.02608501	\approx	0.02624269	0.02052399	\approx	0.02073928	2.9500
20	0.02479172	\approx	0.02494669	0.02052399	\approx	0.02072315	3.0000



$\tilde{A}(n) = \text{red}$, $\tilde{B}(n) = \text{blue}$, $\tilde{T}_A(n) = \text{green}$, $\tilde{T}_B(n) = \text{black}$ ($n = 1, \dots, 100$)

Note: $\tilde{T}_A(n)$ and $\tilde{T}_B(n)$ appears to be some kind of copy of $\tilde{A}(n)$ and $\tilde{B}(n)$

Plots of $\tilde{B}(n)$ (Note: $\tilde{B}(n)$ converges to γ)

Plots of $\tilde{T}_B(n)$ (Note: $\tilde{T}_B(n)$ converges to zero)

$\tilde{T}_A(n)$ and $\tilde{T}_B(n)$ can be used as correction terms for $\tilde{A}(n)$ and $\tilde{B}(n)$

n	$ \tilde{A}(n) - \gamma $	$ \tilde{A}(n) - \tilde{T}_A(n) - \gamma $	$ \tilde{B}(n) - \gamma $	$ \tilde{B}(n) - \tilde{T}_B(n) - \gamma $	$\frac{v}{w}$
1	4.22784335e-001	3.74108145e-002	7.72156649e-002	7.15056058e-002	0.5000
2	2.29637155e-001	1.60351899e-002	7.72156649e-002	3.19252493e-002	1.0000
3	1.57505380e-001	3.13616835e-003	6.11766843e-003	2.89865425e-003	1.2500
4	1.19823307e-001	7.70454125e-003	9.38823316e-002	1.93757745e-002	1.6000
5	9.66797560e-002	4.86097554e-004	3.94510018e-002	1.65098814e-003	1.6667
6	8.10248659e-002	3.86155069e-004	1.56414780e-002	1.23616682e-004	1.8571
7	6.97313289e-002	2.17123759e-004	1.56414780e-002	2.07650769e-004	2.0000
8	6.11999363e-002	3.15613351e-004	2.95303668e-002	7.93246836e-004	2.1111
9	5.45280117e-002	1.77843584e-003	4.82474109e-002	4.43952849e-003	2.3000
10	4.91674961e-002	6.78248563e-004	1.18837746e-002	1.06963652e-003	2.3636
11	4.47664072e-002	2.62850398e-004	2.59950133e-002	5.10848073e-004	2.4167
12	4.10883635e-002	5.27465407e-004	1.24665252e-002	8.81706728e-004	2.5385
13	3.79687328e-002	3.68339974e-004	3.14895188e-002	4.71770177e-004	2.5714
14	3.52893320e-002	6.58015589e-005	7.67999499e-003	4.02839237e-005	2.6667
15	3.29631272e-002	3.24069057e-004	8.98667167e-003	5.22120607e-004	2.7500
16	3.09246061e-002	4.63796961e-004	2.00160834e-002	9.94397926e-004	2.8235
17	2.91235137e-002	5.22909915e-004	2.65520311e-002	1.29080051e-003	2.8889
18	2.75206554e-002	1.96540316e-004	2.31555712e-002	2.47581522e-004	2.8947
19	2.60850131e-002	1.57672889e-004	2.05239922e-002	2.15285219e-004	2.9500
20	2.47917187e-002	1.54972676e-004	2.05239922e-002	1.99161598e-004	3.0000
21	2.36206021e-002	3.59031254e-004	2.27660510e-002	9.03769035e-004	3.0909
22	2.25551320e-002	2.72560480e-004	1.68371973e-002	6.14026037e-004	3.1304
23	2.15816303e-002	1.79570471e-004	9.59082048e-003	3.33448745e-004	3.1667
24	2.06886825e-002	8.31189405e-005	1.25748715e-003	9.79377315e-005	3.2000
25	1.98666880e-002	1.46137118e-005	7.97328208e-003	6.23042354e-005	3.2308
26	1.91075133e-002	1.12078093e-004	1.79447921e-002	1.22579716e-004	3.2593
27	1.84042223e-002	1.21653371e-004	7.18748308e-003	2.15982971e-004	3.3214
28	1.77508639e-002	6.18995297e-006	5.12778786e-003	2.62716515e-005	3.3448
29	1.71423027e-002	1.81319188e-004	1.55618673e-002	4.41582596e-004	3.4000
30	1.65740844e-002	5.82719273e-005	1.58337269e-003	7.32665906e-005	3.4194
31	1.60423260e-002	5.83937978e-005	1.25295305e-002	8.05291112e-005	3.4375
32	1.55436277e-002	6.74691464e-005	3.56895431e-003	1.02442321e-004	3.4848
33	1.50749994e-002	5.00733127e-005	1.15825608e-002	7.04101107e-005	3.5000
34	1.46338009e-002	4.88837369e-005	1.86281731e-003	6.47391048e-005	3.5429
35	1.42176926e-002	1.30940800e-004	1.37675792e-002	3.27244657e-004	3.5833
36	1.38245934e-002	1.24768478e-005	2.74893730e-003	2.73330197e-006	3.5946
37	1.34526463e-002	7.82536329e-005	7.20838845e-003	1.54049523e-004	3.6316
38	1.31001887e-002	3.50908745e-005	9.66070737e-003	5.17842662e-005	3.6410
39	1.27657279e-002	1.83980842e-005	1.32737403e-003	1.08831938e-005	3.6750
40	1.24479199e-002	6.28009318e-005	5.98969914e-003	1.18942284e-004	3.7073

A Further Generalization

For all $k \in \mathbb{Z}$, let the set \mathbb{X}_k be defined as

$$\mathbb{X}_k := \{n \in \mathbb{Z} \mid n > -k\}$$

Consider the following family of functions

$$\Gamma := \{\hat{A}_k, \hat{B}_k : k \in \mathbb{Z}\}$$

mapping from \mathbb{X}_k to \mathbb{R} and \mathbb{Q} respectively

$$\hat{A}_k : \mathbb{X}_k \rightarrow \mathbb{R}$$

$$\hat{B}_k : \mathbb{X}_k \rightarrow \mathbb{Q}$$

where \hat{A}_k and \hat{B}_k are defined as

$$\hat{A}_k(n) := \begin{cases} -\sum_{i=1}^{-n} \frac{1}{i} - \log(n+k) & , \text{if } n < 0 \\ -\log(n+k) & , \text{if } n = 0 \\ \sum_{i=1}^n \frac{1}{i} - \log(n+k) & , \text{if } n > 0 \end{cases}$$

$$\hat{B}_k(n) := \begin{cases} -\sum_{i=1}^{-n} \frac{1}{i} - \frac{\lfloor (n+k) \log(n+k) \rfloor}{n+k} & , \text{if } n < 0 \\ -\frac{\lfloor (n+k) \log(n+k) \rfloor}{n+k} & , \text{if } n = 0 \\ \sum_{i=1}^n \frac{1}{i} - \frac{\lfloor (n+k) \log(n+k) \rfloor}{n+k} & , \text{if } n > 0 \end{cases}$$

Then

$$\tilde{A}(n) = \sum_{i=1}^n \frac{1}{i} - \log(n)$$

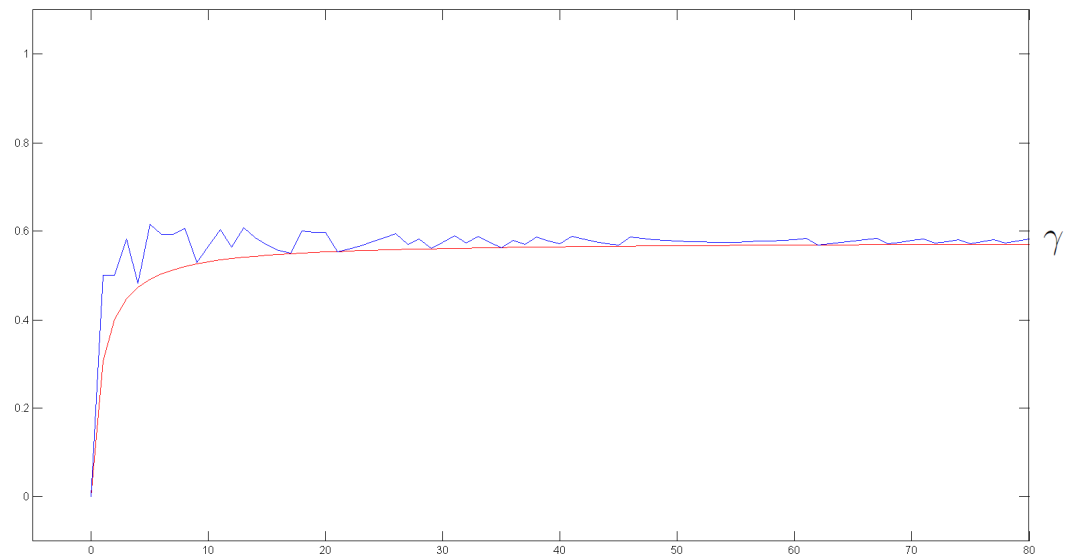
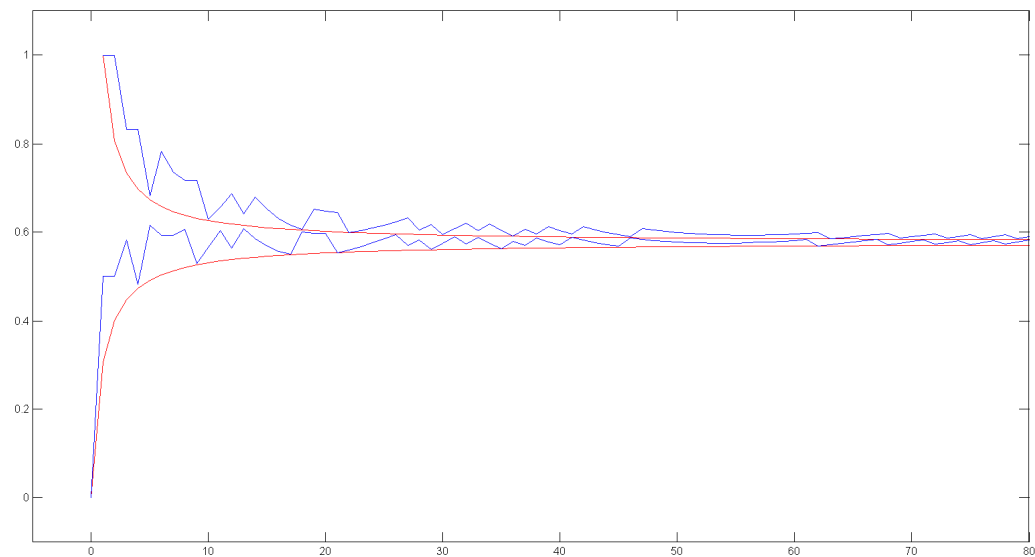
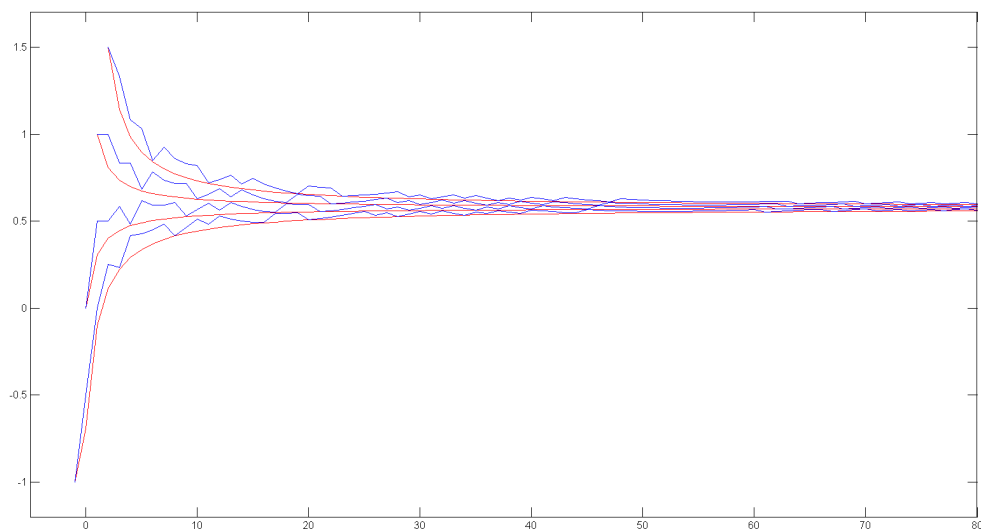
and

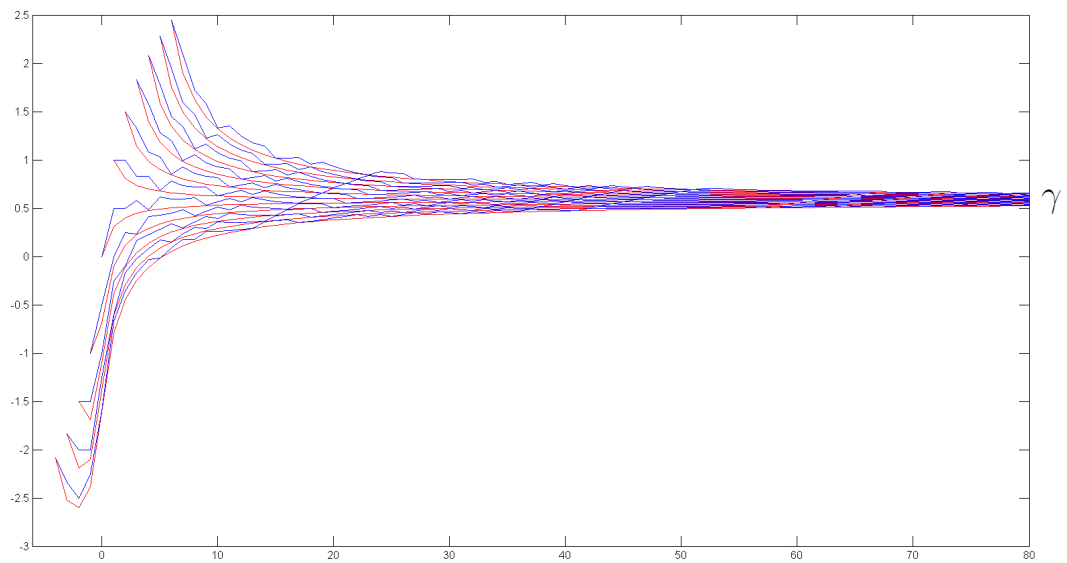
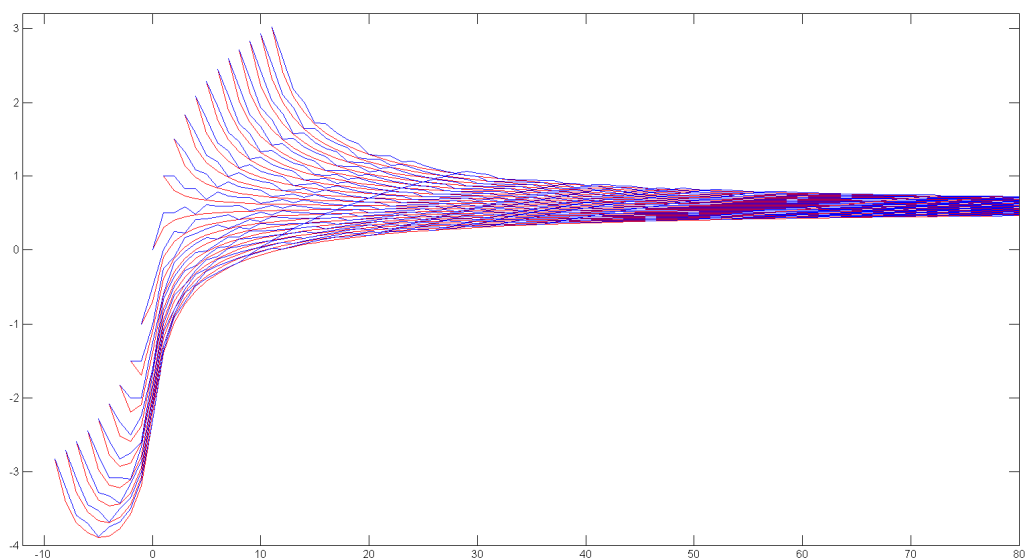
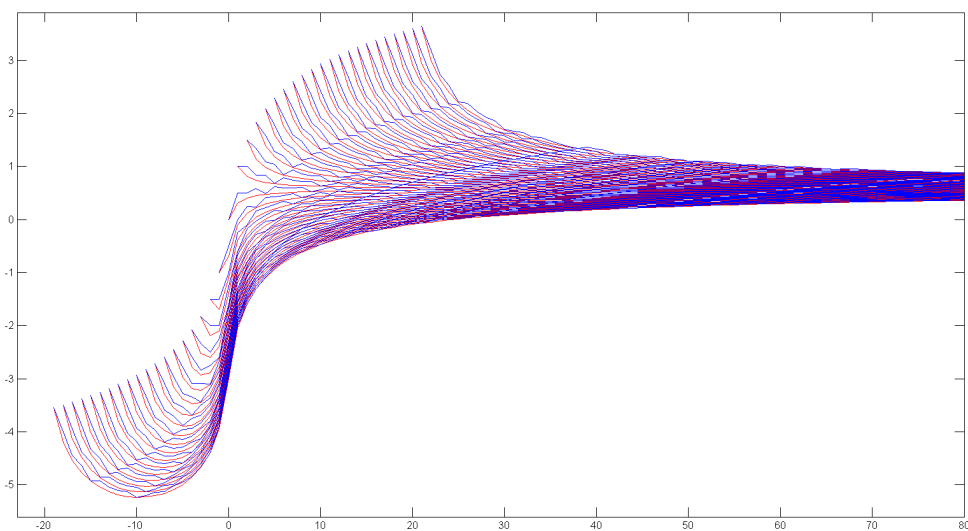
$$\tilde{B}(n) = \sum_{i=1}^n \frac{1}{i} - \frac{v}{w}$$

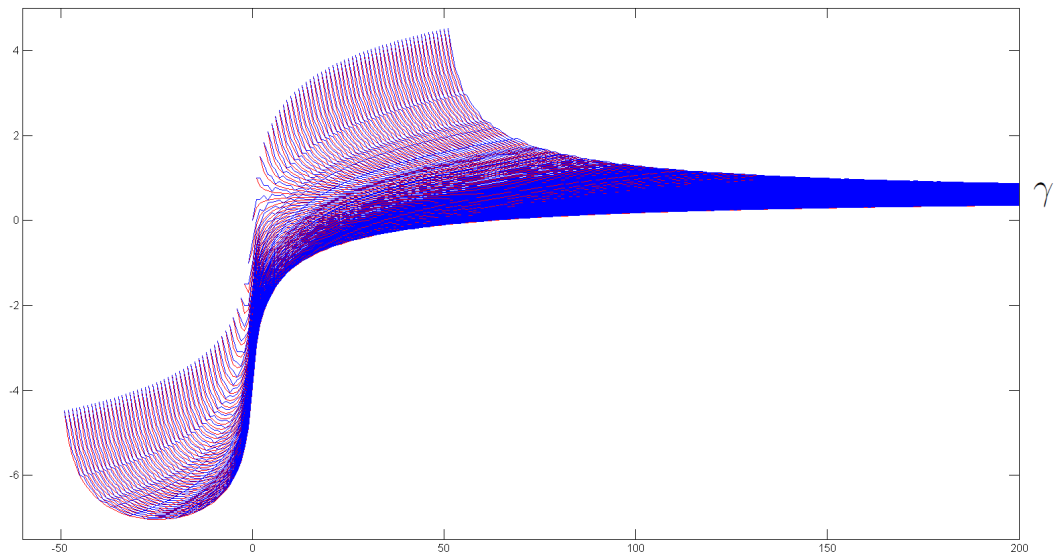
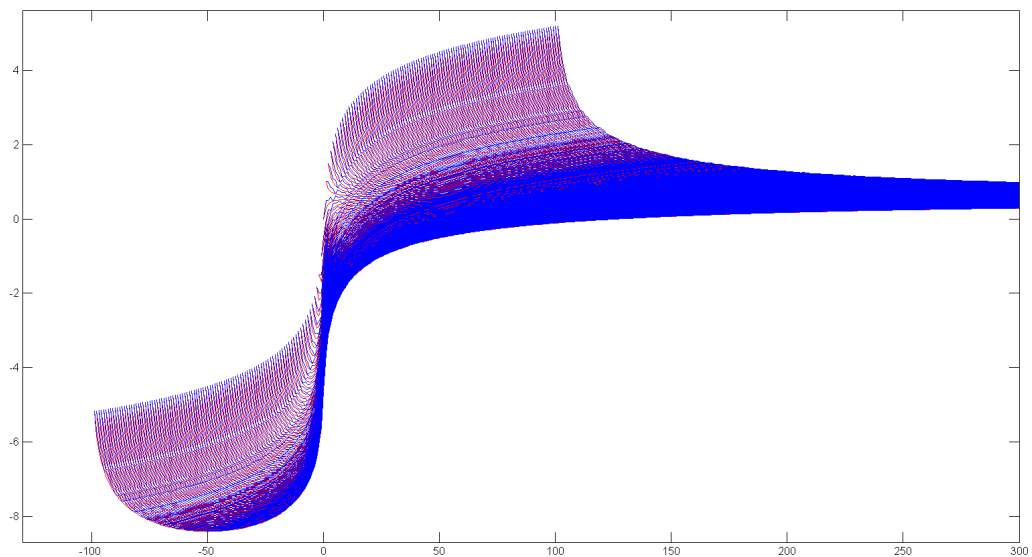
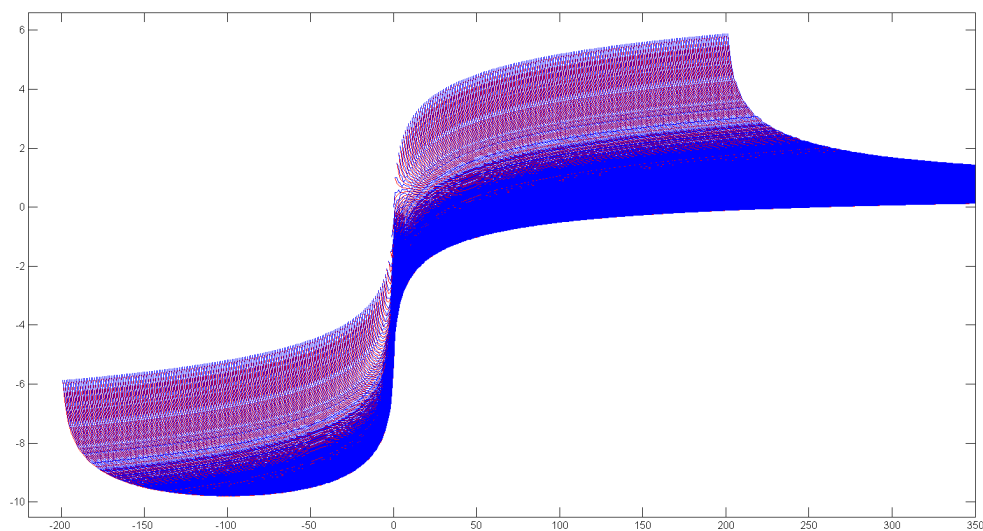
are members of Γ , because

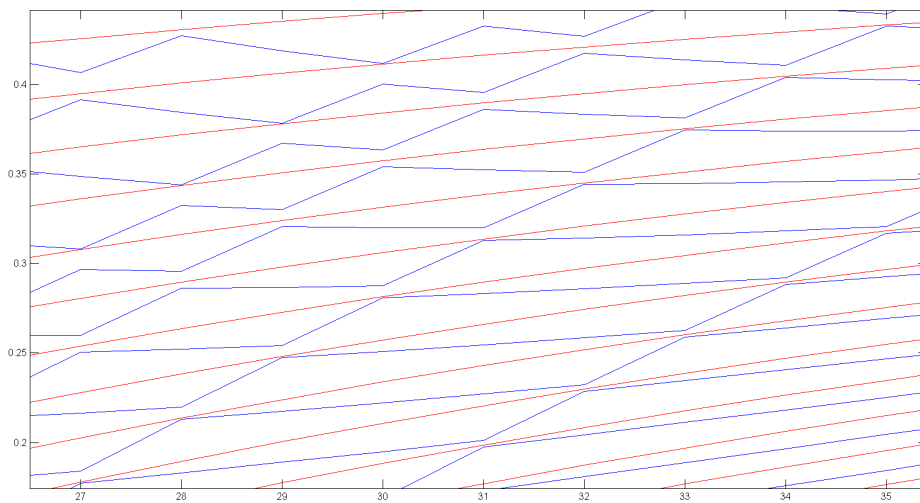
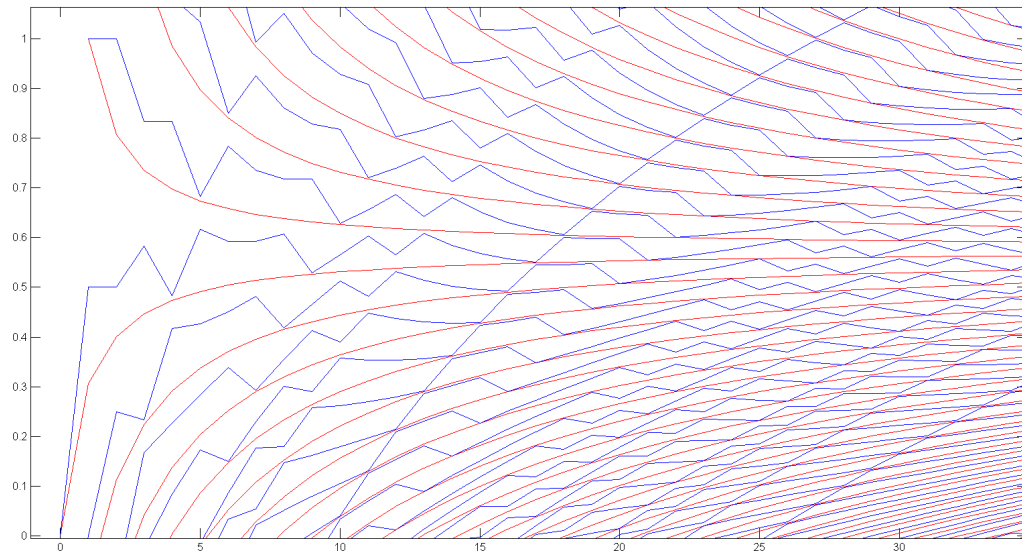
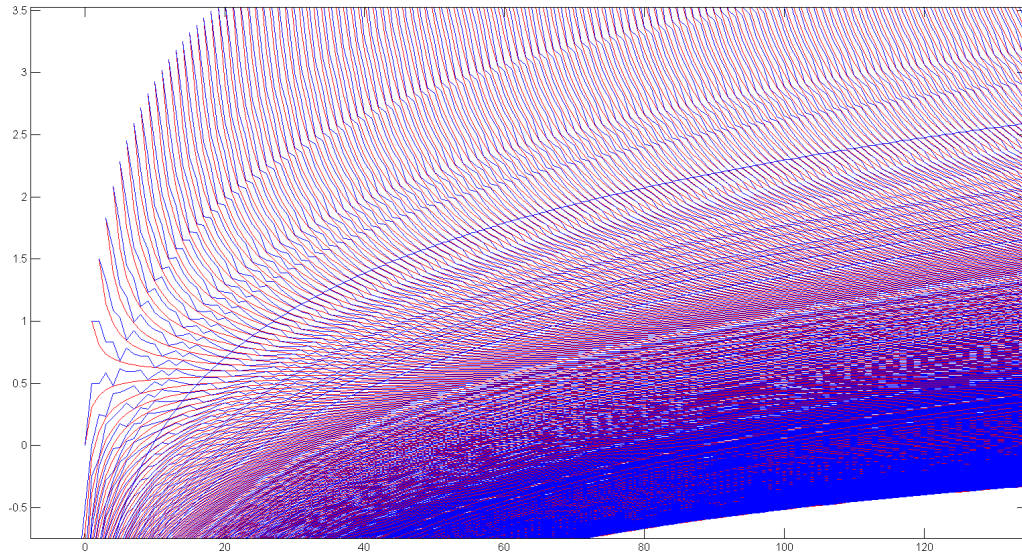
$$\hat{A}_0(n) = \tilde{A}(n) \quad (\mathbb{X}_0 = 1, 2, 3, \dots)$$

$$\hat{B}_1(n) = \tilde{B}(n) \quad (\mathbb{X}_1 = 0, 1, 2, 3, \dots)$$

Plot of $\hat{A}_1(n)$ (red) and $\hat{B}_1(n)$ (blue)Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{0, 1\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-1, 0, 1, 2\}$ 

Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-5, -4, \dots, 4, 5\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-10, \dots, 10\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-20, \dots, 20\}$ 

Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-50, \dots, 50\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-100, \dots, 100\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-200, \dots, 200\}$ 

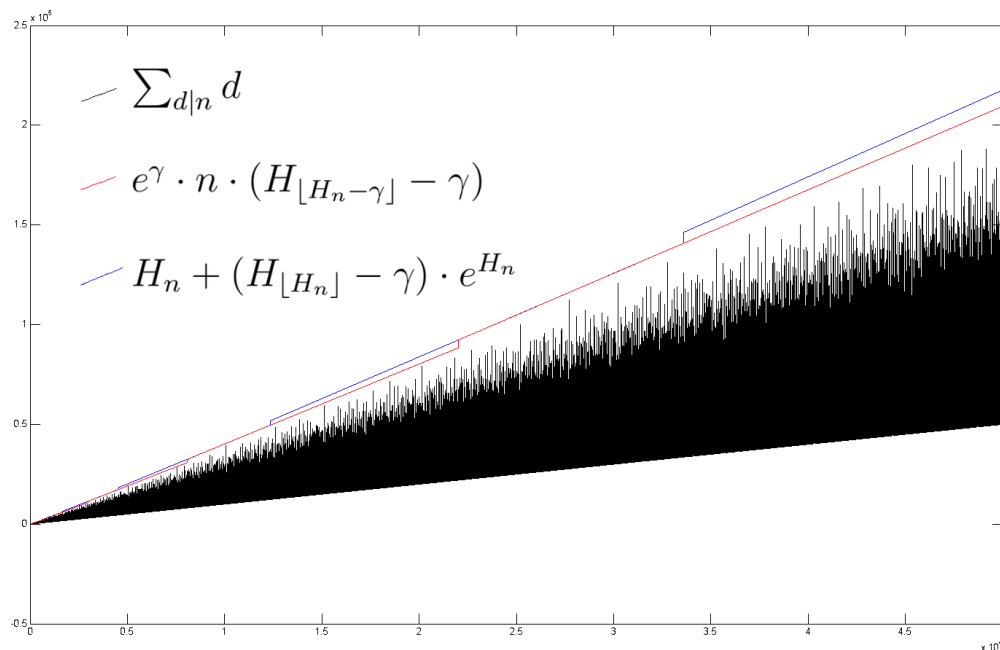
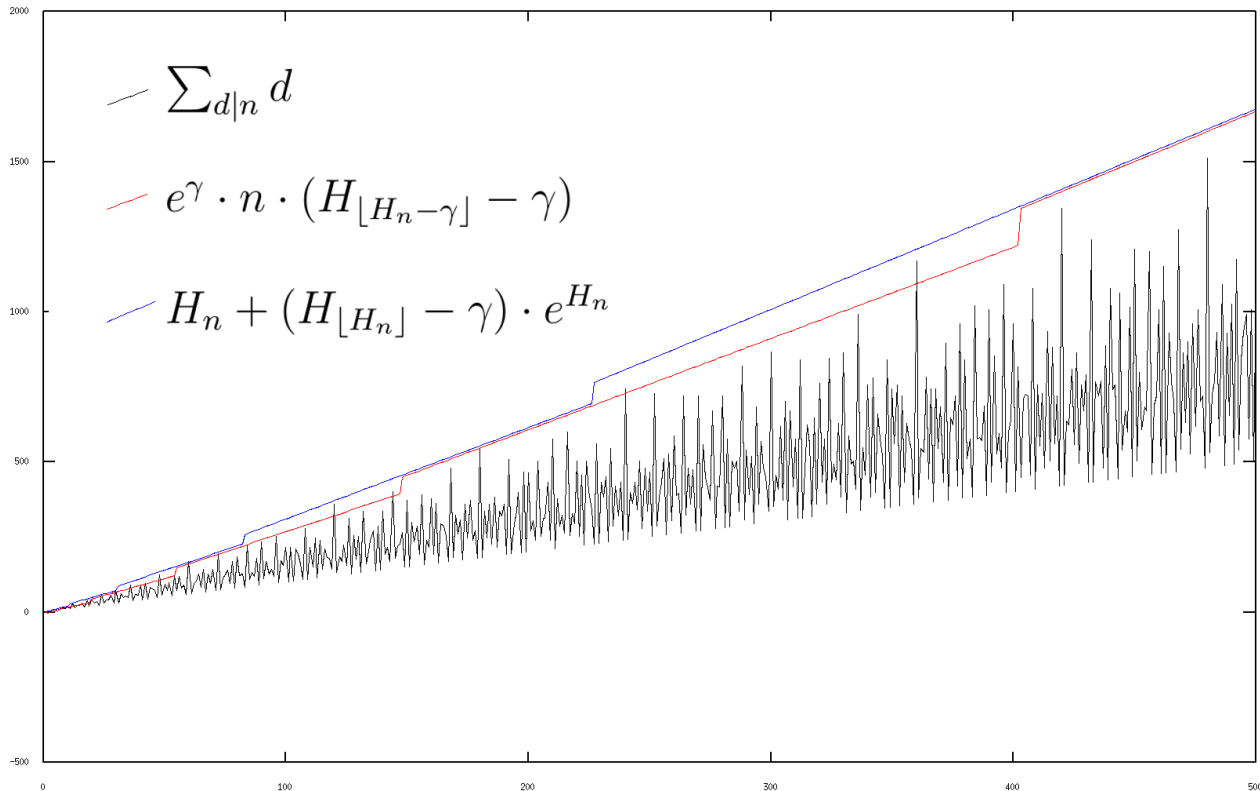


Pattern: $\hat{A}_k(n) < \hat{B}_k(n) < \hat{A}_{k-1}(n) \quad \forall k \in \mathbb{Z}, n \in \mathbb{X}_{k-1}$
(In words: The red and blue lines never touch or cross each other)

3 Pattern related to divisor function

Substituting $\log(x)$ by $H_{\lfloor x \rfloor} - \gamma$ in Lagarias criterion for the RH and Groenwall's theorem, gives the following pattern:

$$e^\gamma \cdot n \cdot (H_{\lfloor H_n - \gamma \rfloor} - \gamma) < H_n + (H_{\lfloor H_n \rfloor} - \gamma) \cdot e^{H_n}$$



Note that the "steps" of $H_n + (H_{\lfloor H_n \rfloor} - \gamma) \cdot e^{H_n}$ correspond with $a_n \stackrel{?}{=} \lfloor e^{n-\gamma} \rfloor$ (see Page 1).

4 A Note on the indeterminate forms 0^0 and $\frac{1}{0}$

While 0^0 is "normally" one, there is the following exception:

$$a_n := \frac{1}{n} \quad (a_n \rightarrow 0^+)$$

$$b_n := \frac{1}{\gamma - H_n} \quad (b_n \rightarrow 0^-)$$

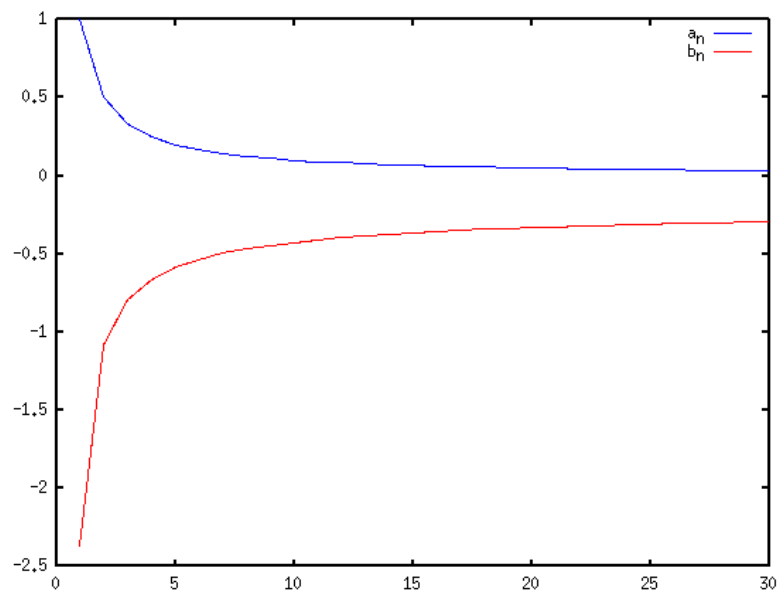
then

$$\lim_{n \rightarrow \infty} a_n^{b_n} \rightarrow 0^0 = e$$

$$\lim_{n \rightarrow \infty} b_n^{a_n} \rightarrow 0^0 = 1 + 0 \cdot \pi i$$

and

$$\lim_{n \rightarrow \infty} \frac{b_n^{a_n}}{a_n} \rightarrow \frac{1}{0} = \infty + \pi i$$



Matlab/Octave source code + Output:

```
function main
gamma = 0.5772156649;
n = 100000;
a = 1/n;
b = 1/(gamma-H(n));
printf('\n %10.9f' , a^b);
printf('\n %10.9f + %10.9f * i', b^a, imag(b^a));
printf('\n %10.9f + %10.9f * i', b^a/a, imag(b^a)/a);
% Harmonic Number
function [h] = H(n)
    h = 0;
    for i = 1 : n
        h = h + 1 / i;
    end
end
```

Output:

```
>> 2.718280648
>> 0.999975565 + 0.000031415 * i
>> 99997.556509714 + 3.141515890 * i
```

5 Harmonic approximation of the prime counting function

It can be observed, that the following approximation of $\pi(x)$ and $Li(x) = \int_2^x 1/\log(t) dt$ holds for small x (lower than ~ 2000 , H_x is the x -th harmonic number):

$$\pi(x) \approx I_e(x) := \frac{x}{H_x - e\gamma}$$

$$Li(x) \approx I_\pi(x) := \frac{x}{H_x - \pi\gamma}$$

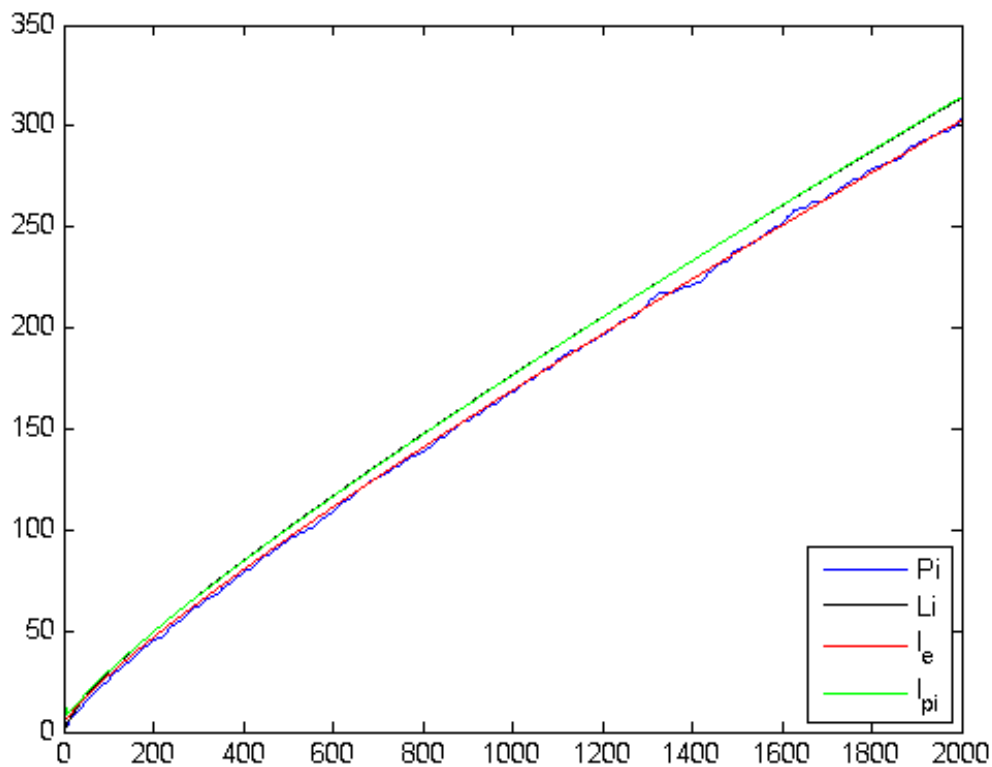


Figure 5: Observe how the red line fits the blue line and how the green line fits the black line.

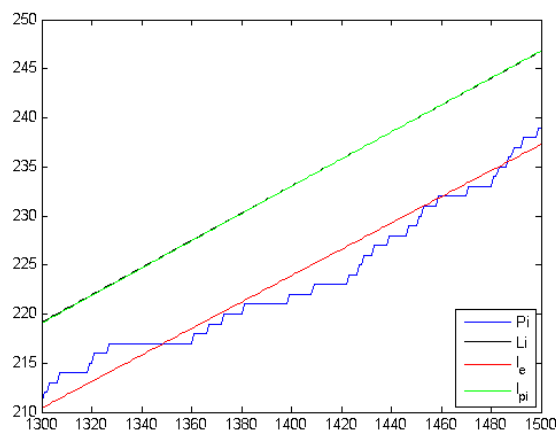


Figure 6: Close up for $x = 1300, \dots, 1500$

However, for larger x this relationship doesn't hold any more. It appears that $\pi(x)$ *moves up*, while $Li(x)$ *moves down* in respect to $I_e(x)$ and $I_\pi(x)$ as their reference asymptotes.

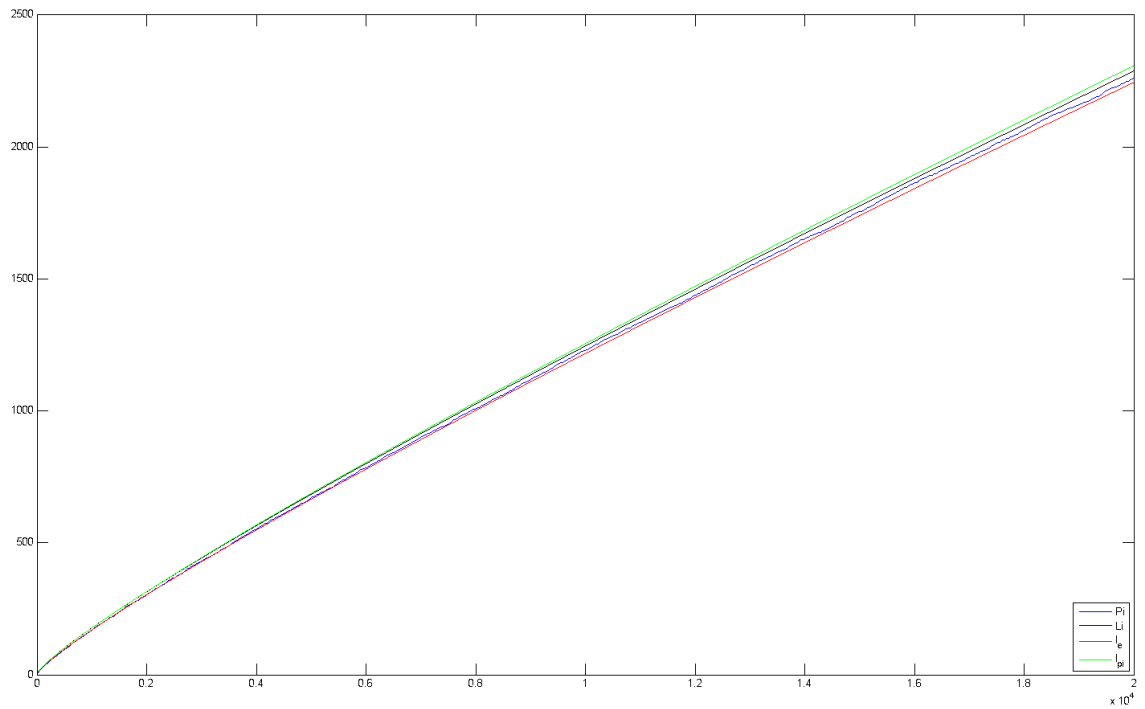


Figure 7: Observe how blue and black move to the inner region defined by the red and green

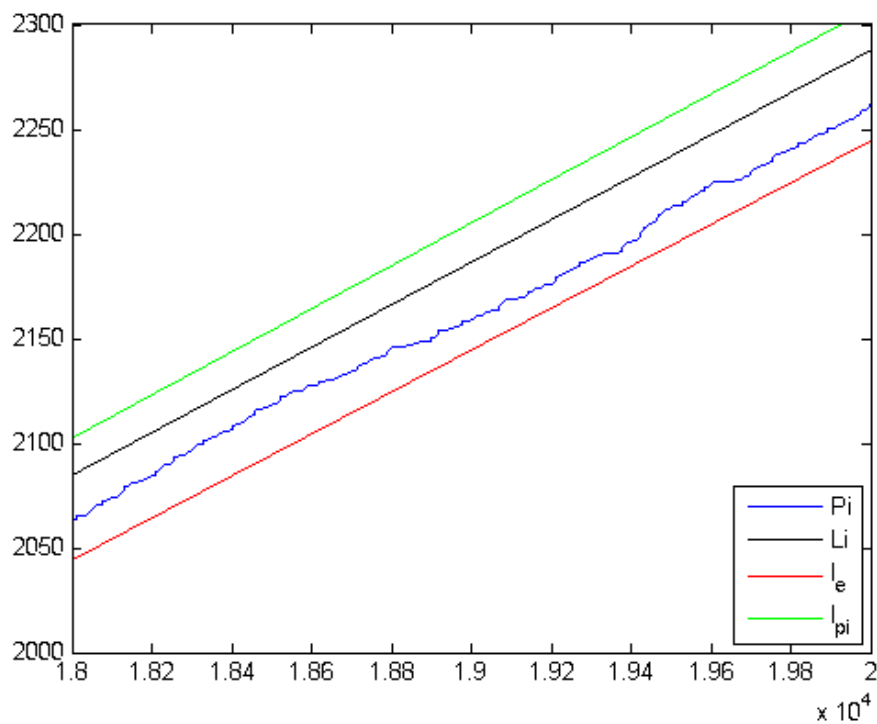


Figure 8: Close up for $x = 18000, \dots, 20000$

Considering the idea that $I_e(x)$ and $I_\pi(x)$ act as reference asymptotes for $\pi(x)$, a general approximation formula $A_{\pi(x)}$ for $\pi(x)$ is constructed as follows:

$$A_{\pi(x)} := w_1 \cdot I_e(x) + w_2 \cdot I_\pi(x) - e$$

where w_1 and w_2 are weights for a convex combination of $I_e(x)$ and $I_\pi(x)$. Their values are defined as follows:

$$w_1 := \frac{1}{H_x} + \gamma$$

$$w_2 := 1 - w_1$$

$A_{\pi(x)}$ has also an offset of e , as this value has shown to be numerically effective. The formula $A_{\pi(x)}$ is a heuristic attempt to create an approximation of $\pi(x)$ using $I_e(x)$ and $I_\pi(x)$ as reference asymptotes, whereas $A_{\pi(x)}$ moves to the inner region defined by $I_e(x)$ and $I_\pi(x)$. The beneath plots illustrate, how $A_{\pi(x)}$ moves to the inner region of $I_e(x)$ and $I_\pi(x)$, similar to $\pi(x)$.

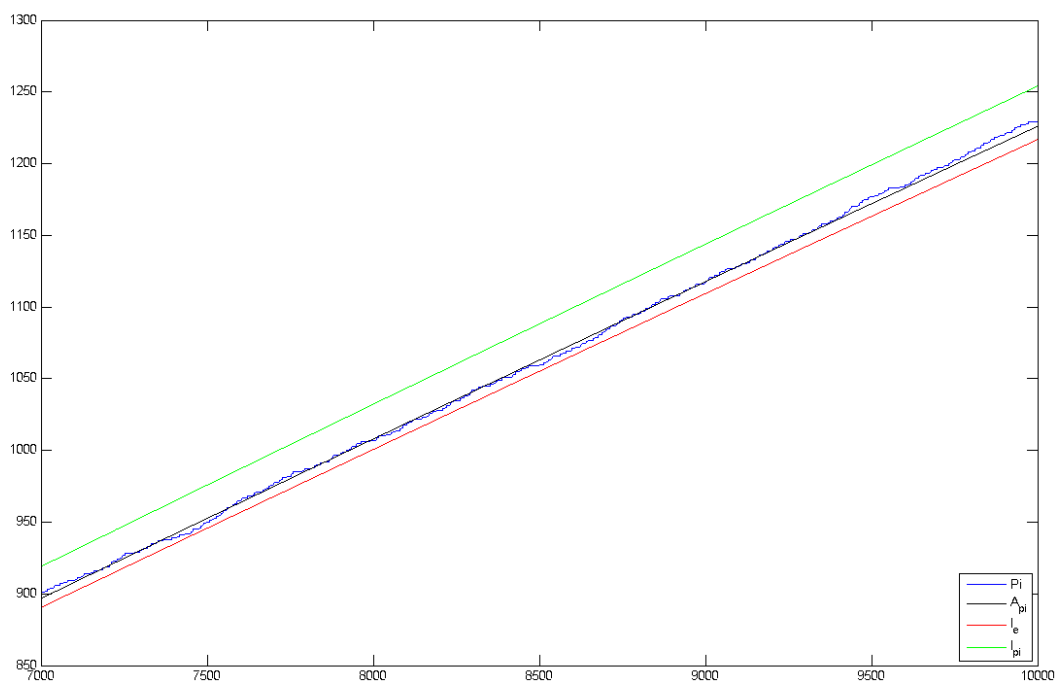


Figure 9: Plot for $x = 7000, \dots, 10000$. Observe how $A_{\pi(x)}$ (black) fits $\pi(x)$ (blue)

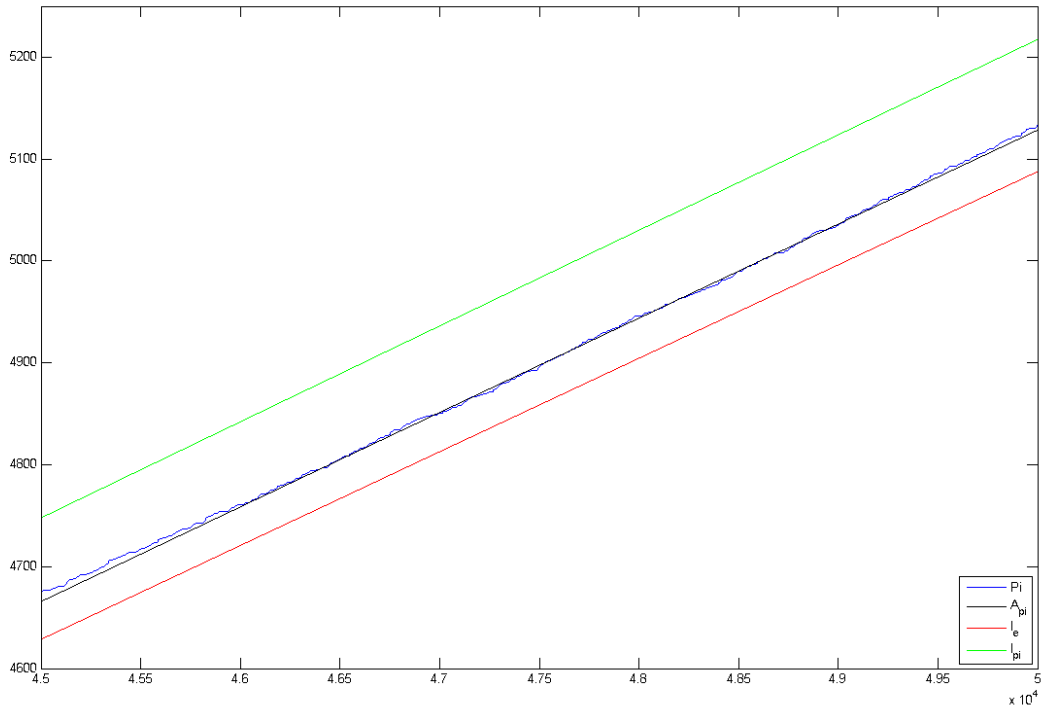


Figure 10: Plot for $x = 45000, \dots, 50000$. Observe how $A_{\pi(x)}$ (black) fits $\pi(x)$ (blue)

It can be shown, that the average distance $|A_{\pi(x)} - \pi(x)|$ is about the same as $|Li(x) - \frac{1}{2}Li(\sqrt{x}) - \pi(x)|$ for $n=4, \dots, 100000$. For lower values than 100000, the average distance by $A_{\pi(x)}$ is often better than the average distance of $|Li(x) - \frac{1}{2}Li(\sqrt{x}) - \pi(x)|$ (see plots beneath).

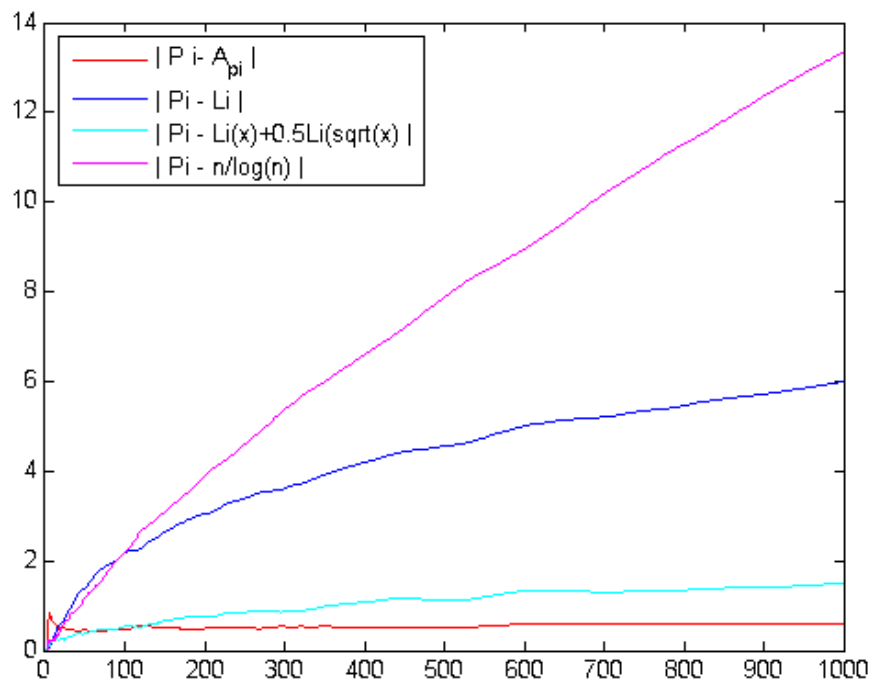
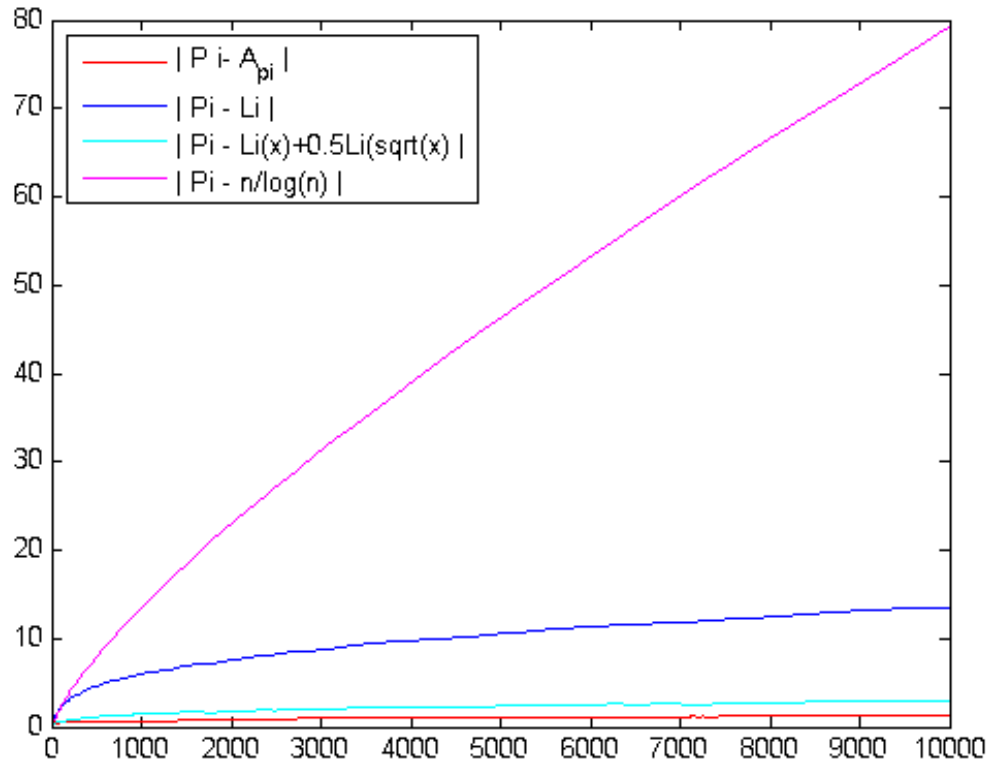
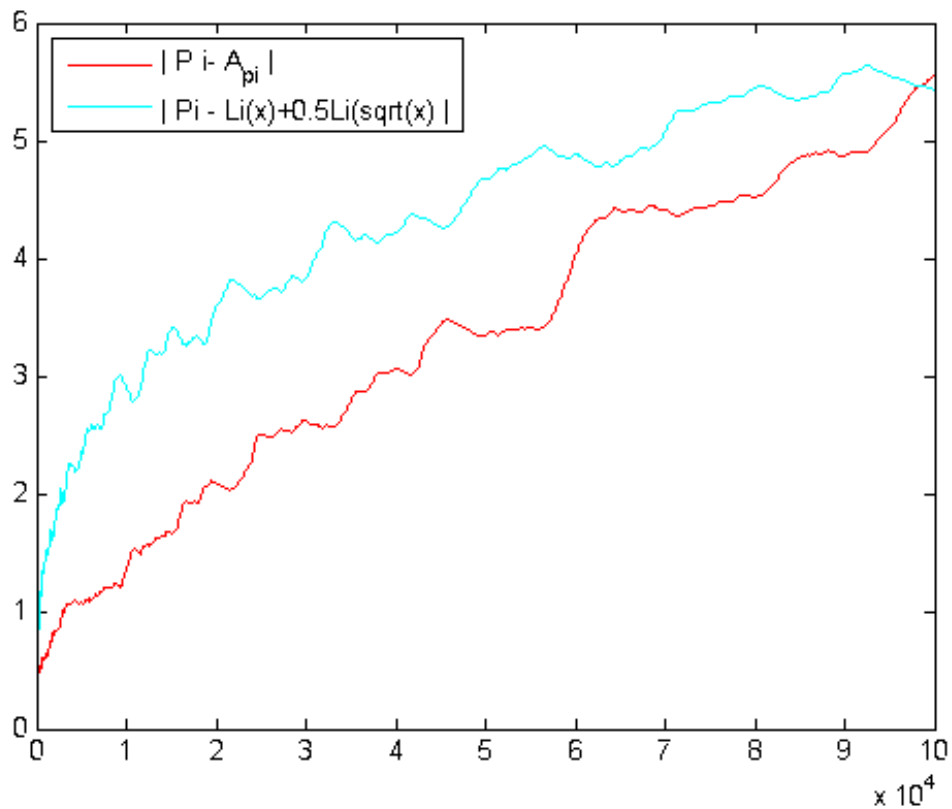


Figure 11: Comparison of average distance for $n=4, \dots, 1000$

Figure 12: Comparison of average distance for $n=4, \dots, 10000$ Figure 13: Comparison of average distance for $n=4, \dots, 100000$

6 Appendix: Source code to construct e, γ and π

Programming language: Matlab/Octave

```

function construct % This program numerically approximates 'e', 'gamma' and 'pi'
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n = 1; % Initialize index 'n' for sequence 'a(n)'
a(n) = 1; % Define first element of 'a(n)' as 1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for k = 1:50000 % Generate sequence 'a(n)' which is OEIS-A004080 for 'n>=1'

    if Zeta(1,k ,1) - floor(Zeta(1,k ,1)) < ...
        Zeta(1,k-1,1) - floor(Zeta(1,k-1,1)) % Check if 'k' is part of 'a(n)'

        n = n + 1; % Increase index 'n'
        a(n) = k; % Update sequence 'a(n)'

    end
end
a % Print sequence 'a(n)'
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
EULER = a(n) / a(n-1) % Print approximation of 'e'
GAMMA = Zeta(1,floor(EULER^n),1) - n % Print approximation of 'gamma'
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
TAU = 0; % Initialize 'tau' as zero
x = 0; % Initialize 'x' as zero
tolerance = 1; % Tolerance for equality 'exp()=lim zeta()/zeta()'
stepsize = 0.001; % Stepsize to increase 'x' from zero to 'infinity'
i = sqrt(-1); % Define 'i' as imaginary unit sqrt(-1)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
while (TAU == 0) % Loop that increases 'x' and checks equality

    x = x + stepsize; % Increase 'x' by stepsize increment

    ratio = Zeta(a(n-2), a(n-1)-1, 1+i*x) / ...
        Zeta(a(n-1), a(n-0)-1, 1+i*x); % zeta()/zeta()

    % Check if equality 'exp()=lim zeta()/zeta()' fails
    if ( abs(ratio-EULER^(i*x)) > tolerance )

        TAU = x % If equality fails, 'tau' is found and printed

    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
PI = TAU / 2 % Print 'pi' as half of 'tau'

% Zetafunction from 'A' to 'B' with input argument 'x'
function [z] = Zeta( A, B, x)
z = 0;
for n = A : B
    z = z + 1 / n^x;
end

```

```

function main

N = 100
PRIMES = primes(N);
GAMMA = 0.57721566490153286060;

for n=1:N
    a(n) = Pi(PRIMES,n);
    b(n) = Li(n);
    c(n) = Li_improved(n);

    Ie(n) = n / (H(n) - exp(1)*GAMMA);
    Ipi(n) = n / (H(n) - pi*GAMMA);

    Api(n) = A(n);
end

plot(a)
hold on
hold on
plot(Api, 'k')
plot(Ie, 'r')
hold on
plot(Ipi, 'g')
%axis([18000 20000 2000 2300]);
legend('Pi', 'A_{pi}', 'I_{e}', 'I_{pi}', 'Location', 'SouthEast');
hold off

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function y=A(n)
GAMMA = 0.57721566490153286060;
HARMONIC = H(n);
Ie = n / (HARMONIC - exp(1)*GAMMA);
Ipi = n / (HARMONIC - pi*GAMMA);
w1 = 1 / HARMONIC + GAMMA;
w2 = 1 - w1;
y = w1 * Ie + w2 * Ipi - exp(1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function y=H(n)
if(n<=100)
y = 0; for i=1:n; y = y + 1/i; end
else
GAMMA = 0.57721566490153286060;
y = log(n) + GAMMA + 1/(2*n) - 1/(12*n^2);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function y=Pi(p,n)
%y=size(primes(n),2);
y = 0;
for i=1:size(p,2)
if(p(i) <= n)
y = y + 1;

```

```
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function y=Li(n)
F = @(x)1./log(x);
y = quad(F,2,n);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function y=Li_improved(n)
F = @(x)1./log(x);
y = quad(F,2,n) - 0.5 * quad(F,2,sqrt(n));
```

```

function main

N = 100000
PRIMES = primes(N);
GAMMA = 0.57721566490153286060;

for n=4:N
    a(n) = Pi(PRIMES,n);
    b(n) = Li(n);
    c(n) = Li_improved(n);
    d(n) = n / log(n);
    e(n) = n / (H(n) - 3/2);    % Locker-Ernst 1959

    Ie(n) = n / (H(n) - exp(1)*GAMMA);
    Ipi(n) = n / (H(n) - pi*GAMMA);

    Api(n) = A(n);

    dis_Api(n) = a(n) - Api(n);
    dis_Li(n) = a(n) - b(n);
    dis_LiX(n) = a(n) - c(n);
    dis_LOG(n) = a(n) - d(n);
    dis_LE(n) = a(n) - e(n);

    aver_Api(n) = sum(abs(dis_Api))/n;
    aver_Li(n) = sum(abs(dis_Li))/n;
    aver_LiX(n) = sum(abs(dis_LiX))/n;
    aver_LOG(n) = sum(abs(dis_LOG))/n;
    aver_LE(n) = sum(abs(dis_LE))/n;

    % Print average Distance
    fprintf('\n %i: Api: %f Li: %f LiX: %f LOG: %f LE:
           %f',n,aver_Api(n),aver_Li(n),aver_LiX(n),aver_LOG(n),aver_LE(n));
end

%%%%%%%%%
figure()
plot(aver_Api,'r')
hold on
plot(aver_Li)
hold on
plot(aver_LiX,'c')
hold on
plot(aver_LOG,'m')
legend('| P i- A_{pi} |', '| Pi - Li |', '| Pi - Li(x)+0.5Li(sqrt(x) |', '| Pi -
      n/log(n) |', 'Location','NorthWest');
hold off

%%%%%%%%%
figure()
plot(a)
hold on
plot(Api,'k')
hold on

```

```

plot(Ie, 'r')
hold on
plot(Ipi, 'g')
%axis([18000 20000 2000 2300]);
legend('Pi', 'A_{pi}', 'I_{e}', 'I_{pi}', 'Location', 'SouthEast');
hold off

```

```

%%%%%%%%%%%%%%
function y=A(n)
GAMMA = 0.57721566490153286060;
HARMONIC = H(n);
Ie = n / (HARMONIC - exp(1)*GAMMA);
Ipi = n / (HARMONIC - pi*GAMMA);
w1 = 1 / HARMONIC + GAMMA;
w2 = 1 - w1;
y = w1 * Ie + w2 * Ipi - exp(1);

%%%%%%%%%%%%%%
function y=H(n)
if(n<=100)
y = 0; for i=1:n; y = y + 1/i; end
else
GAMMA = 0.57721566490153286060;
y = log(n) + GAMMA + 1/(2*n) - 1/(12*n^2);
end
%%%%%%%%%%%%%%
function y=Pi(p,n)
%y=size(primes(n),2);
y = 0;
for i=1:size(p,2)
if(p(i) <= n)
y = y + 1;
end
end
end
%%%%%%%%%%%%%%
function y=Li(n)
F = @(x)1./log(x);
y = quad(F,2,n);
%%%%%%%%%%%%%%
function y=Li_improved(n)
F = @(x)1./log(x);
y = quad(F,2,n) - 0.5 * quad(F,2,sqrt(n));

```
