# Some formulas and pattern

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## Abstract

Some formulas and pattern.

## Definition of zeta function

Let

$$\zeta_A^B(x) := \sum_{n=A}^{B} \frac{1}{n^x} \tag{1}$$

where $A, B \in \mathbb{Z}$ and $x \in \mathbb{C}$.

Consider further $a_n := 1, 2, 4, 11, 31, 83, 227, ...$ (OEIS-A002387) which is based on $\zeta_1^\infty(1)$.

## A zeta function based construction of $e$, $\gamma$ and $\pi$

Based on the zeta function $\zeta$ and the sequence $a_n$,
the constants $\gamma$, $e$ and $\tau := 2\pi$ can be constructed as follows:

$$e = \lim_{n\to\infty} \frac{a_{n+1}}{a_{n+0}}$$

$$\gamma = \lim_{n\to\infty} \zeta_1^{\lfloor (\frac{a_{n+1}}{a_{n+0}})^n \rfloor}(1) - n$$

$$\tau = \min(\ \{\ x \in \mathbb{R}^+ : \ \lim_{n\to\infty} \frac{\zeta_{a_{n+0}}^{a_{n+1}}(1+ix)}{\zeta_{a_{n+1}}^{a_{n+2}}(1+ix)} \ \neq \ (\frac{a_{n+1}}{a_{n+0}})^{ix} \ \}\ )$$

Note that even though numerical evidence indicates that the sequence $a_n$ coincides with $[e^{n-\gamma}]$, this relationship is unproven and is not exploited in the above construction of $e$, $\gamma$ and $\pi$.

In other words: Based on $a_n$ it is possible to construct $e$ and $\gamma$, but it is an open question, if it is possible to construct $a_n$ out of $e$ and $\gamma$. This question might be of interest in an evolutionary context.

# Euler's identity

The previous construction of $\pi$ is based on the formula:

$$\lim_{n \to \infty} \quad \frac{\zeta_{a_{n+0}}^{a_{n+1}}(x)}{\zeta_{a_{n+1}}^{a_{n+2}}(x)} \; = \; e^{x-1},$$

which holds for all $x \in \mathbb{C}$ <u>except</u> those $x$ of the form $2ki\pi + 1$ with $k \in \mathbb{Z}\backslash\{0\}$.

Euler's identity $e^{i\pi} = -1$ appears as special case of above formula for $x = 1i\pi + 1$.

| $x$ | $cos(x) + i \cdot sin(x)$ | | $x$ | $\lim \frac{\zeta_{a_{n+0}}^{a_{n+1}}(x)}{\zeta_{a_{n+1}}^{a_{n+2}}(x)}$ |
|---|---|---|---|---|
| $5\pi$ | $e^{5i\pi}$ | | $5i\pi + 1$ | $e^{5i\pi}$ |
| $4\pi$ | $e^{4i\pi}$ | | $4i\pi + 1$ | $\sim$ |
| $3\pi$ | $e^{3i\pi}$ | | $3i\pi + 1$ | $e^{3i\pi}$ |
| $2\pi$ | $e^{2i\pi}$ | | $2i\pi + 1$ | $\sim$ |
| $1\pi$ | $e^{i\pi}$ | | $1i\pi + 1$ | $e^{i\pi}$ |
| $0\pi$ | $1$ | | $0i\pi + 1$ | $1$ |
| $-1\pi$ | $e^{-i\pi}$ | | $-1i\pi + 1$ | $e^{-i\pi}$ |
| $-1\pi$ | $e^{-2i\pi}$ | | $-2i\pi + 1$ | $\sim$ |
| $-3\pi$ | $e^{-3i\pi}$ | | $-3i\pi + 1$ | $e^{-3i\pi}$ |
| $-4\pi$ | $e^{-4i\pi}$ | | $-4i\pi + 1$ | $\sim$ |
| $-5\pi$ | $e^{-5i\pi}$ | | $-5i\pi + 1$ | $e^{-5i\pi}$ |

The wave symbol ($\sim$) indicates that the limes does <u>not</u> exist

An alternative zeta function based construction of $\pi$ is given by:

$$\pi = \lim_{n \to \infty} \quad n \cdot imag(1 - (\frac{1}{\zeta_1^{\lfloor (\frac{a_{n+1}}{a_{n+0}})^n \rfloor}(1) - n - H})^{\frac{1}{n}})$$

# 1 Some notes on the fraction of harmonic numbers

Definitions

$$A(n) := \sum_{i=1}^{n} \frac{1}{i} - log(n)$$

$$B(n) := \sum_{i=1}^{n} \frac{1}{i} - \lfloor \sum_{i=1}^{n} \frac{1}{i} \rfloor$$

$$C(n) := log(n) - \lfloor log(n) \rfloor$$

$$\gamma := \lim_{n \to \infty} A(n)$$

In every period of oscillation of $B$ there are $n$, so that $B(n)$ is closer to $\gamma$ than $A(n)$

$$\exists\, n\, \in\, \mathbb{N}: \quad B(n) - \gamma \,<\, A(n) - \gamma$$

| $n$ | $B(n)$ | $B(n) - \gamma$ | $<$ | $A(n) - \gamma$ | $A(n)$ |
|---|---|---|---|---|---|
| 2 | 0.50000000000000 | -0.07721566490153 | $<$ | 0.22963715453852 | 0.80685281944005 |
| 7 | 0.59285714285714 | 0.01564147795561 | $<$ | 0.06973132890030 | 0.64694699380183 |
| 20 | 0.59773965714368 | 0.02052399224215 | $<$ | 0.02479171868816 | 0.60200738358969 |
| 54 | 0.57543039374427 | -0.00178527115727 | $<$ | 0.00923068227846 | 0.58644634717999 |
| 148 | 0.57780251258124 | 0.00058684767971 | $<$ | 0.00337457391559 | 0.58059023881713 |
| 403 | 0.57739240852976 | 0.00017674362823 | $<$ | 0.00124018168154 | 0.57845584658308 |
| 1096 | 0.57709426741491 | -0.00012139748662 | $<$ | 0.00045613500542 | 0.57767179990695 |
| 2980 | 0.57706202025190 | -0.00015364464963 | $<$ | 0.00016777585092 | 0.57738344075245 |
| 8103 | 0.57726701163418 | 0.00005134673265 | $<$ | 0.00006170427193 | 0.57727736917346 |
| 22026 | 0.57721721789959 | 0.00000155299805 | $<$ | 0.00002270027317 | 0.57723836517470 |
| 59874 | 0.57722164886082 | 0.00000598395928 | $<$ | 0.00000835084692 | 0.57722401574845 |
| 162754 | 0.57721387436125 | -0.00000179054028 | $<$ | 0.00000307211796 | 0.57721873701949 |
| 442413 | 0.57721590899706 | 0.00000024409553 | $<$ | 0.00000113016521 | 0.57721679506674 |
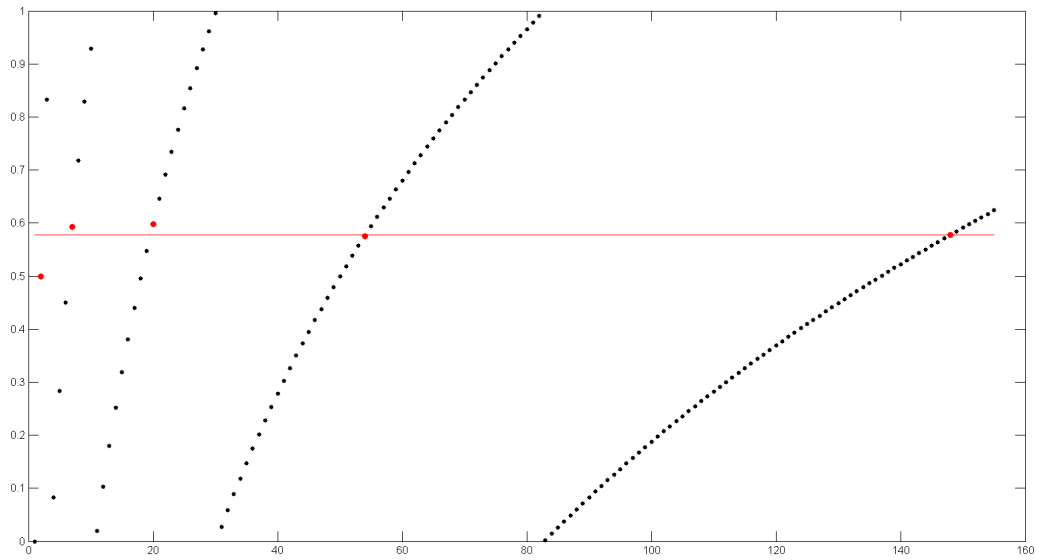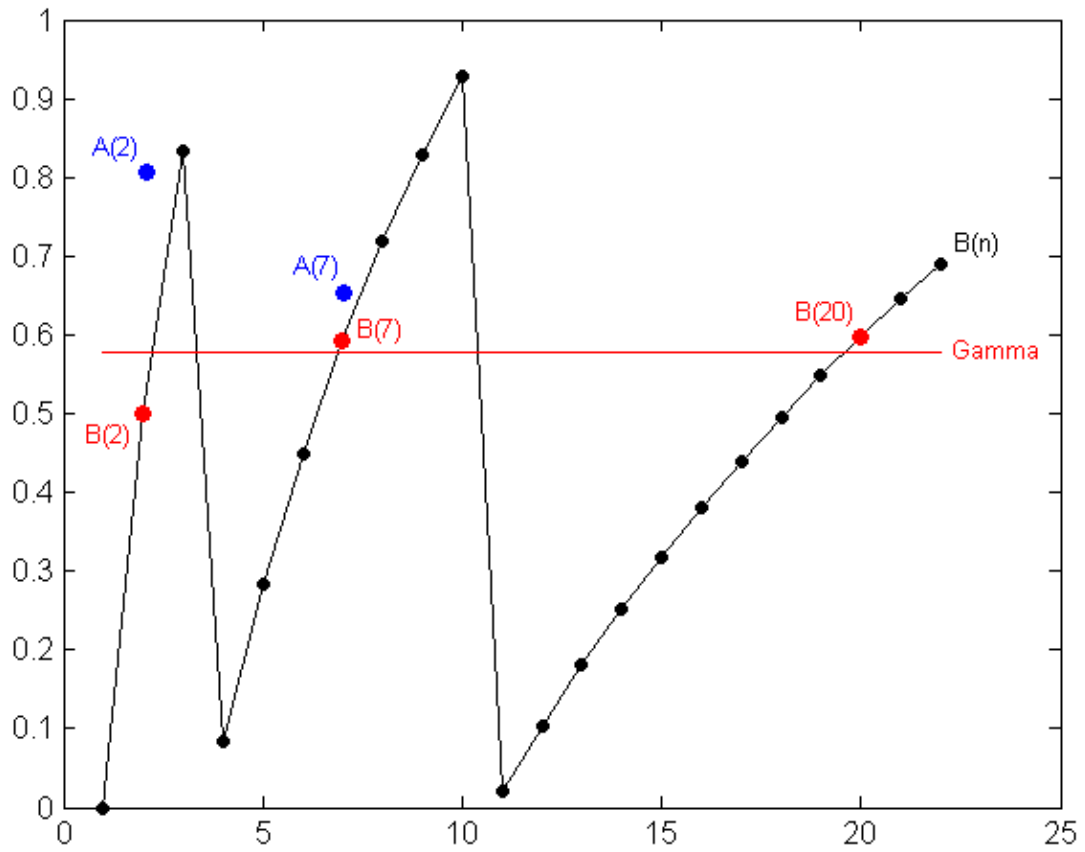
The periods of oscillation of $B$ encode $e \approx 2.71828$

$$\forall \, a, \, b \, \sim \, Figure \, 1, 2, 3 \quad \longrightarrow \quad \frac{a}{b} \approx e$$
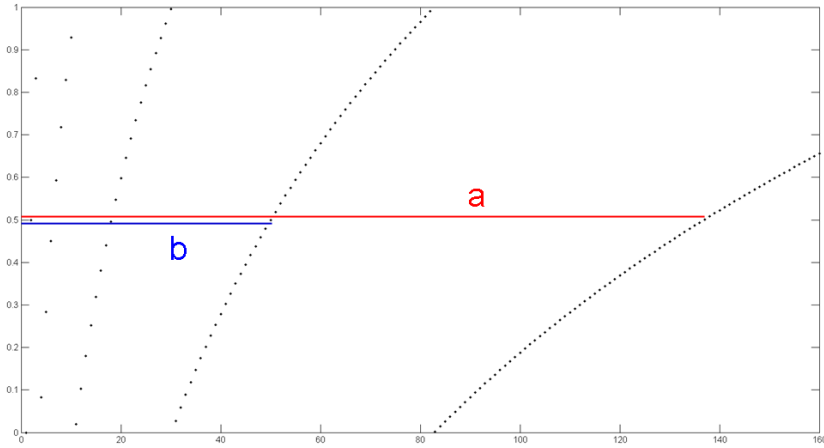


Figure 1:
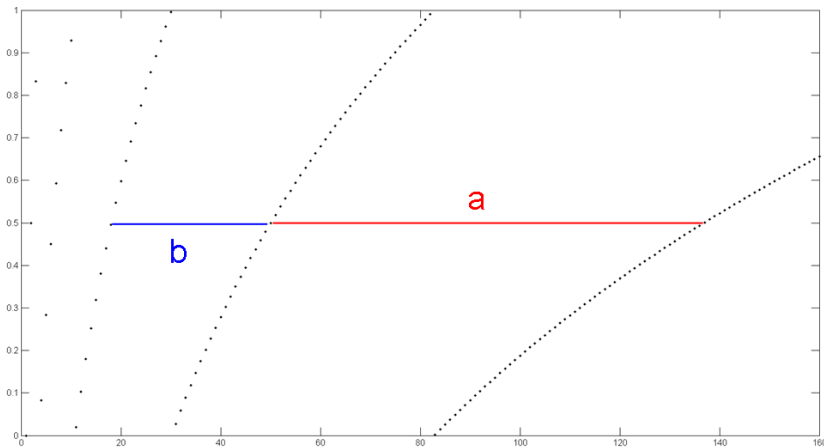


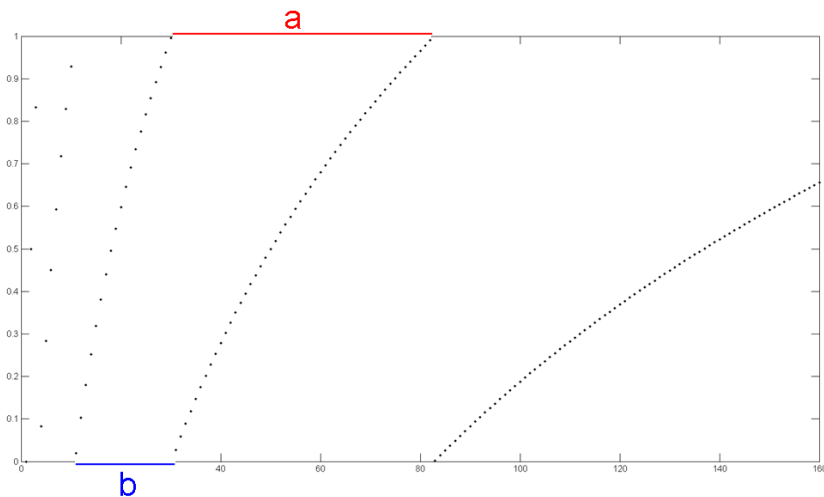Figure 2:



Figure 3:

Relationship between $B(n)$, $C(n)$ and $\gamma$

$$B(n) \approx D(n) := \begin{cases} C(n) + \gamma - 1 & \text{,if } B(n) < \gamma \\ C(n) + \gamma & \text{,if } B(n) > \gamma \end{cases}$$

| $n$ | $B(n)$ | $\approx$ | $D(n)$ |
|---|---|---|---|
| 1 | 0.000000000000000 | | -0.422784335098467 |
| 2 | 0.500000000000000 | $\approx$ | 0.270362845461478 |
| 3 | 0.833333333333333 | $\approx$ | 0.675827953569642 |
| 4 | 0.083333333333333 | $\approx$ | -0.036489973978577 |
| 5 | 0.283333333333333 | $\approx$ | 0.186653577335633 |
| 6 | 0.450000000000000 | $\approx$ | 0.368975134129588 |
| 7 | 0.592857142857143 | | 1.523125813956846 |
| 8 | 0.717857142857143 | $\approx$ | 0.656657206581369 |
| 9 | 0.828968253968254 | $\approx$ | 0.774440242237752 |
| 10 | 0.928968253968254 | $\approx$ | 0.879800757895579 |
| 20 | 0.597739657143682 | | 1.572947938455524 |
| 100 | 0.187377517639621 | $\approx$ | 0.182385850889625 |
| 148 | 0.577802512581241 | | 1.574427938665648 |
| 1000 | 0.485470860550343 | $\approx$ | 0.484970943883670 |
| 10000 | 0.787606036044348 | $\approx$ | 0.787556036877716 |
| 100000 | 0.090146129863335 | $\approx$ | 0.090141129871761 |
| 1000000 | 0.392726722864989 | $\approx$ | 0.392726222865806 |
| 10000000 | 0.695311365857272 | $\approx$ | 0.695311315859853 |
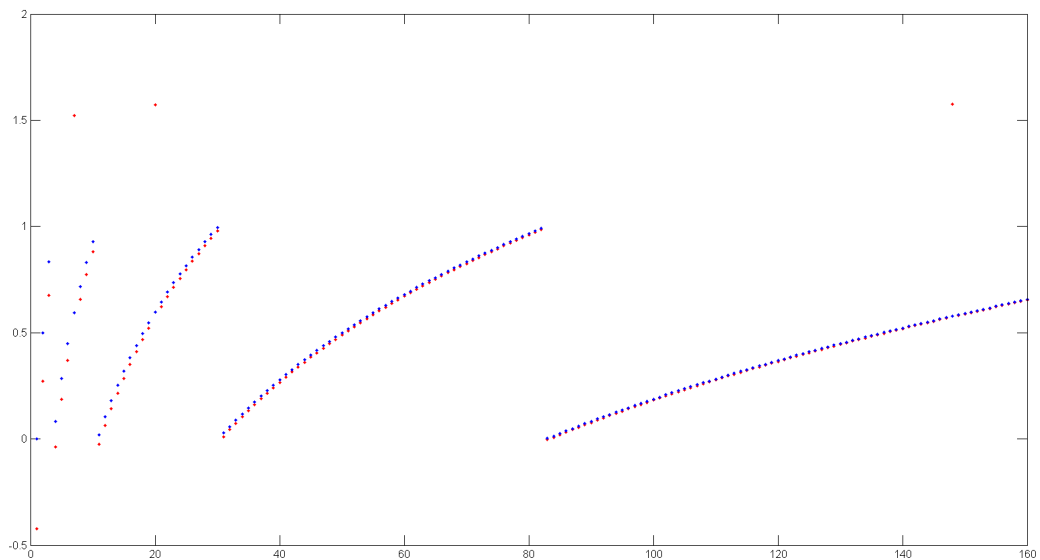


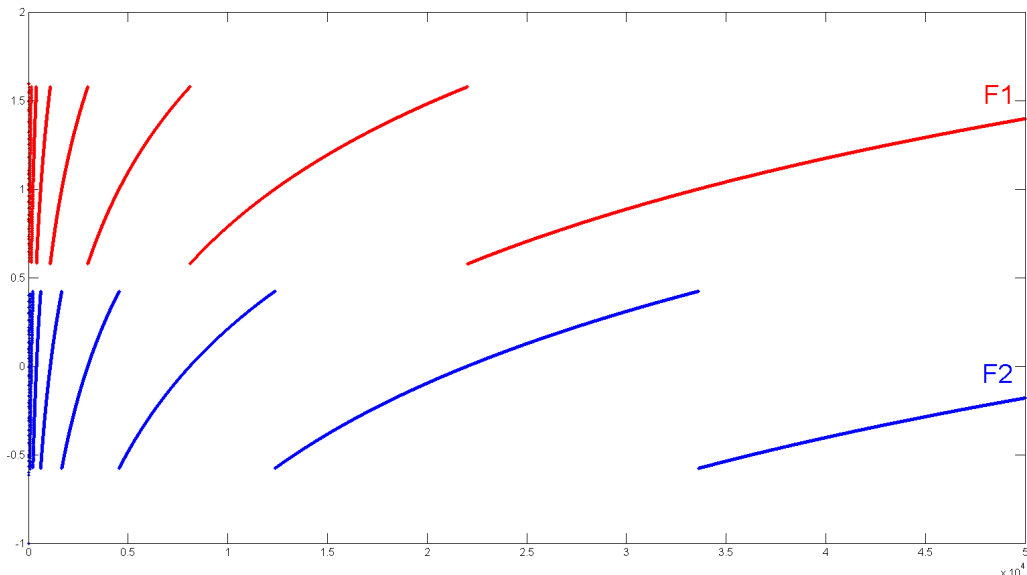Figure 4: Plot of $B(n)$ (blue) and $F(n)$ (red)
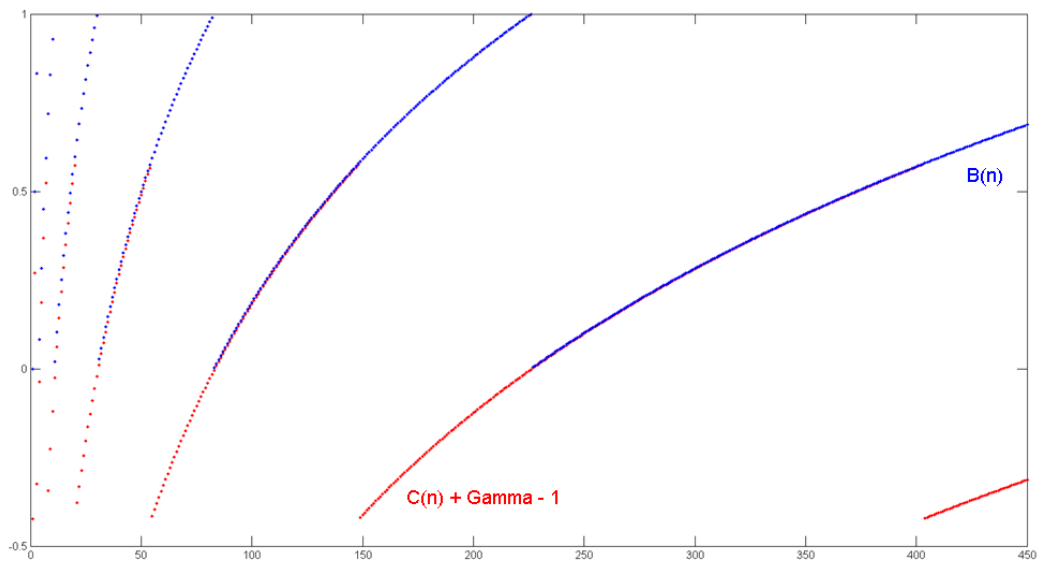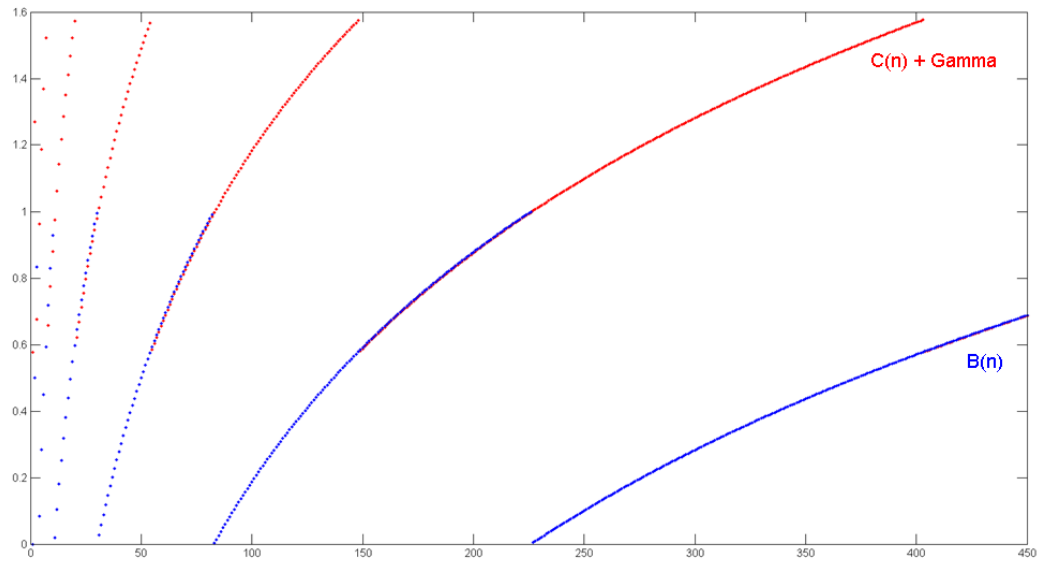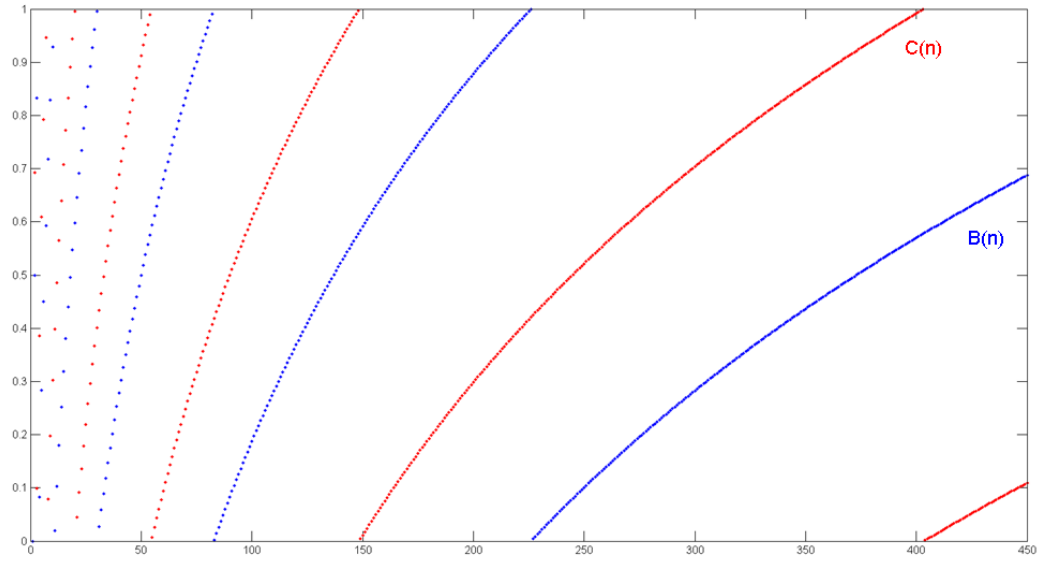
*Max/Min* Relationships

$$F_1(n) \;:=\; \sum_{i=1}^{n} \frac{1}{i} - \lfloor log(n) \rfloor \qquad (\in \mathbb{Q})$$

$$F_2(n) \;:=\; log(n) - \lfloor \sum_{i=1}^{n} \frac{1}{i} \rfloor \qquad (\in \mathbb{R})$$

Formula for $\gamma$:

$$\lim_{n \to \infty} \quad min\{ \; F_1 \; \} \;=\; \gamma$$

$$\lim_{n \to \infty} \quad max\{ \; F_2 \; \} \;=\; 1 - \gamma$$



| $n$ | $min\{ \; F_1 \; \}$ | $max\{ \; F_2 \; \}$ | $min\{ \; F_1 \; \} + max\{ \; F_2 \; \}$ |
|---|---|---|---|
| 1 | 1.000000000000000 | -1.000000000000000 | 0.000000000000000 |
| 2 | 1.000000000000000 | -0.306852819440055 | 0.693147180559945 |
| 3 | 0.833333333333333 | 0.098612288668110 | 0.931945622001443 |
| 4 | 0.833333333333333 | 0.098612288668110 | 0.931945622001443 |
| 5 | 0.833333333333333 | 0.098612288668110 | 0.931945622001443 |
| 6 | 0.833333333333333 | 0.098612288668110 | 0.931945622001443 |
| 7 | 0.833333333333333 | 0.098612288668110 | 0.931945622001443 |
| 8 | 0.717857142857143 | 0.098612288668110 | 0.816469431525252 |
| 9 | 0.717857142857143 | 0.19224577336220 | 0.915081720193362 |
| 10 | 0.717857142857143 | 0.302585092994046 | 1.020442235851188 |
| 100 | 0.593612211926086 | 0.406719247264253 | 1.000331459190339 |
| 1000 | 0.579867656054511 | 0.421622267806518 | 1.001489923861029 |
| 10000 | 0.577390407488084 | 0.422662707570003 | 1.000053115058087 |
| 100000 | 0.577238350322196 | 0.42277097174 3748 | 1.000009322065944 |

# 2   Some notes on the Euler-Mascheroni constant

Formula for $\gamma$

$$\gamma \;=\; \lim_{n\to\infty} \; \sum_{i=1}^{n} \frac{1}{i} \;-\; log(n)$$

$$=\; \lim_{n\to\infty} \; \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} \;-\; log(\lfloor e^n \rfloor)$$

$$=\; \lim_{n\to\infty} \; \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} \;-\; n$$

Note that the closer $e^n - \lfloor e^n \rfloor$ gets to $\frac{1}{2}$, the better is the approximation by the above formula. Therefore the subsequnece $n = 1, 2, 4, 5, 6, 10, 16, 21, 85, 115, ...$ (OEIS-A080053) gives a strictly monotonic approximation of $\gamma$ by the above formula.

Pattern

$$\sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} \;-\; n \;-\; \gamma \;<\; \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} \;-\; log(\lfloor e^n \rfloor) \;-\; \gamma$$

and

$$|\sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} \;-\; n \;-\; \gamma| \;<\; |\sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} \;-\; log(\lfloor e^n \rfloor) \;-\; \gamma|$$

Definition

$$A(n) \; := \; \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - log(\lfloor e^n \rfloor)$$

$$B(n) \; := \; \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n$$

$$T_A(n) \; := \; \frac{1}{2} \, \frac{n - log(\lfloor e^n \rfloor)}{e^n - \lfloor e^n \rfloor}$$

$$T_B(n) \; := \; \frac{\frac{1}{2} + \lfloor e^n \rfloor - e^n}{\lfloor e^n \rfloor}$$

Pattern

$$A(n) - \gamma \; \approx \; T_A(n)$$

$$B(n) - \gamma \; \approx \; T_B(n)$$

| $n$ | $A(n) - \gamma$ | $\approx$ | $T_A(n)$ | $B(n) - \gamma$ | $\approx$ | $T_B(n)$ |
|---|---|---|---|---|---|---|
| 1 | 0.229637154538522 | $\approx$ | 0.213601964634381 | -0.077215664901533 | $\approx$ | -0.109140914229523 |
| 2 | 0.069731328900296 | $\approx$ | 0.069514205140797 | 0.015641477955610 | $\approx$ | 0.015849128724193 |
| 3 | 0.024791718688158 | $\approx$ | 0.024946691364183 | 0.020523992242149 | $\approx$ | 0.020723153840617 |
| 4 | 0.009230682278460 | $\approx$ | 0.009208353109854 | -0.001785271157266 | $\approx$ | -0.001817593206375 |
| 5 | 0.003374573915593 | $\approx$ | 0.003373671569258 | 0.000586847679708 | $\approx$ | 0.000586762820428 |
| 6 | 0.001240181681544 | $\approx$ | 0.001240035204982 | 0.000176743628226 | $\approx$ | 0.000176691085025 |
| 7 | 0.000456135005418 | $\approx$ | 0.000456072655816 | -0.000121397486621 | $\approx$ | -0.000121494916477 |
| 8 | 0.000167775850920 | $\approx$ | 0.000167758271537 | -0.000153644649630 | $\approx$ | -0.000153686926754 |
| 9 | 0.000061704271926 | $\approx$ | 0.000061705221596 | 0.000051346732651 | $\approx$ | 0.000051347948243 |
| 10 | 0.000022700273166 | $\approx$ | 0.000022700204904 | 0.000001552998054 | $\approx$ | 0.000001552946213 |
| 11 | 0.000008350846916 | $\approx$ | 0.000008350860275 | 0.000005983959284 | $\approx$ | 0.000005983979727 |
| 12 | 0.000003072117962 | $\approx$ | 0.000003072113647 | -0.000001790540282 | $\approx$ | -0.000001790548951 |
| 13 | 0.000001130165210 | $\approx$ | 0.000001130165204 | 0.000000244095527 | $\approx$ | 0.000000244095629 |
| 14 | 0.000000415763583 | $\approx$ | 0.000000415764407 | 0.000000179472383 | $\approx$ | 0.000000179473229 |
| 15 | 0.000000152950261 | $\approx$ | 0.000000152951169 | 0.000000039010172 | $\approx$ | 0.000000039011082 |
| 16 | 0.000000056265555 | $\approx$ | 0.000000056267590 | -0.000000002309892 | $\approx$ | -0.000000002307857 |
| 17 | 0.000000020695265 | $\approx$ | 0.000000020699688 | -0.000000010502283 | $\approx$ | -0.000000010497860 |
| 18 | 0.000000007612997 | $\approx$ | 0.000000007614984 | 0.000000005521458 | $\approx$ | 0.000000005523449 |
| 19 | 0.000000002802197 | $\approx$ | 0.000000002801399 | -0.000000002594347 | $\approx$ | -0.000000002595144 |
| 20 | 0.000000001031574 | $\approx$ | 0.000000001030576 | 0.000000000186934 | $\approx$ | 0.000000000185936 |

Generalization

$$n = \lfloor e^{\frac{v}{w}} \rfloor$$

$$v \;:=\; \lfloor \, log(n+1) \cdot (n+1) \, \rfloor$$

$$w \;:=\; n+1$$

| $n$ | $v/w$ | $\lfloor e^{\frac{v}{w}} \rfloor$ |
|---|---:|:---:|
| 1 | 1 / 2 = 0.50000 | 1 |
| 2 | 3 / 3 = 1.00000 | 2 |
| 3 | 5 / 4 = 1.25000 | 3 |
| 4 | 8 / 5 = 1.60000 | 4 |
| 5 | 10 / 6 = 1.66667 | 5 |
| 6 | 13 / 7 = 1.85714 | 6 |
| 7 | 16 / 8 = 2.00000 | 7 |
| 8 | 19 / 9 = 2.11111 | 8 |
| 9 | 23 / 10 = 2.30000 | 9 |
| 10 | 26 / 11 = 2.36364 | 10 |

Definition

$$\tilde{A}(n) \;:=\; \sum_{i=1}^{n} \frac{1}{i} - log(n)$$

$$\tilde{B}(n) \;:=\; \sum_{i=1}^{n} \frac{1}{i} - \frac{v}{w}$$

$$\tilde{T}_A(n) \;:=\; \frac{1}{2} \frac{\frac{v}{w} - log(n)}{e^{\frac{v}{w}} - n}$$

$$\tilde{T}_B(n) \;:=\; \frac{\frac{1}{2} + n - e^{\frac{v}{w}}}{n}$$

Pattern (generalized)
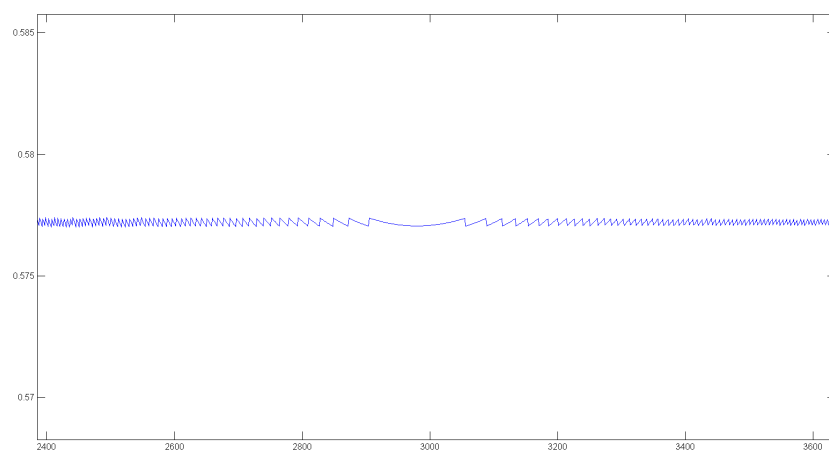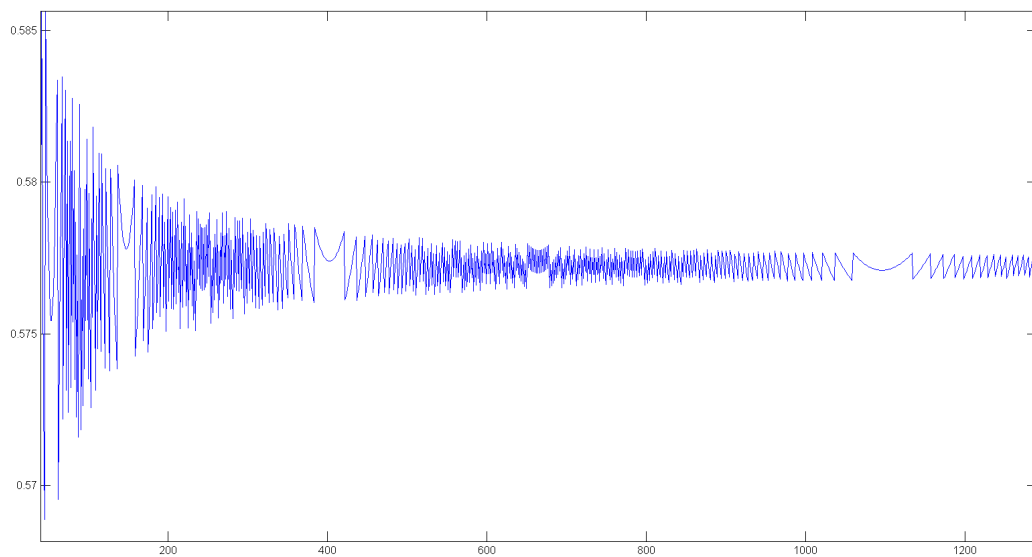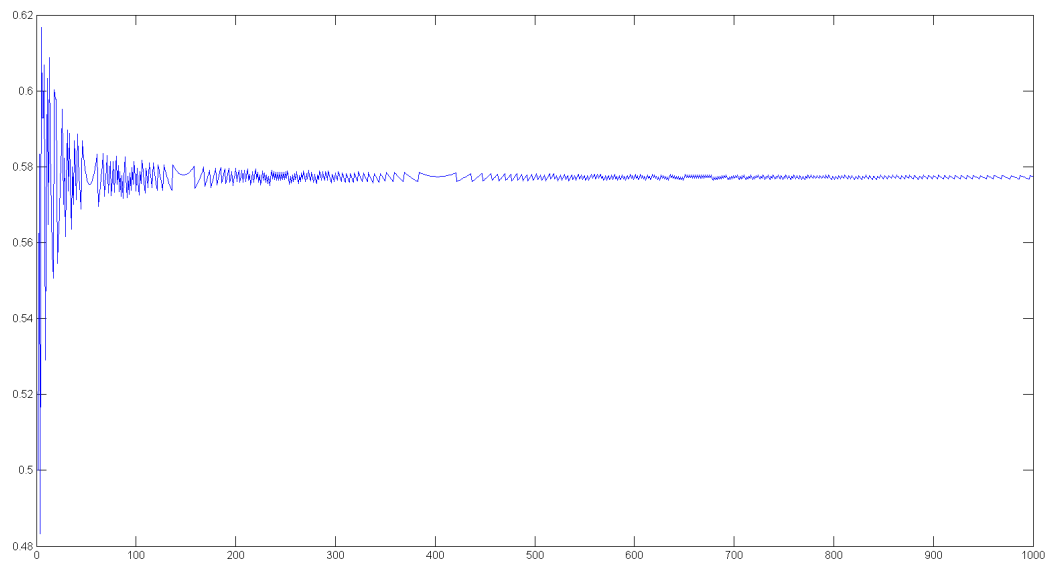
$$\tilde{A}(n) - \gamma \;\approx\; \tilde{T}_A(n)$$

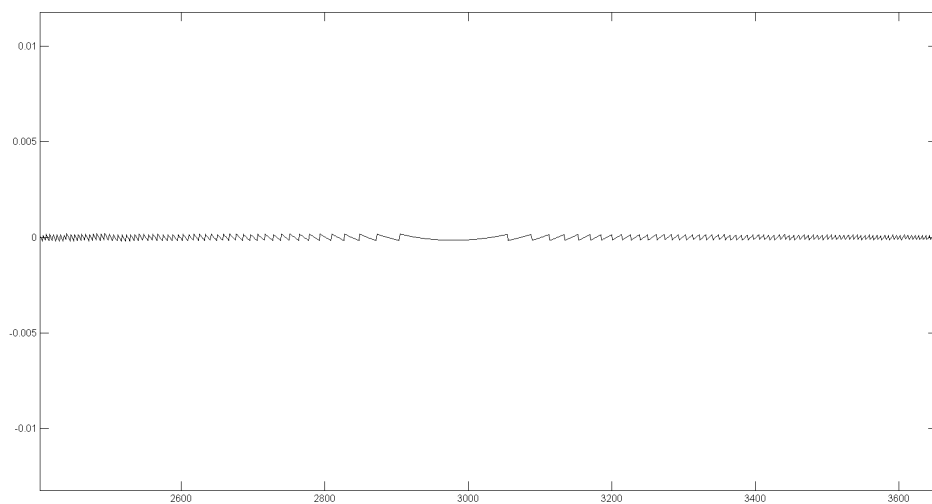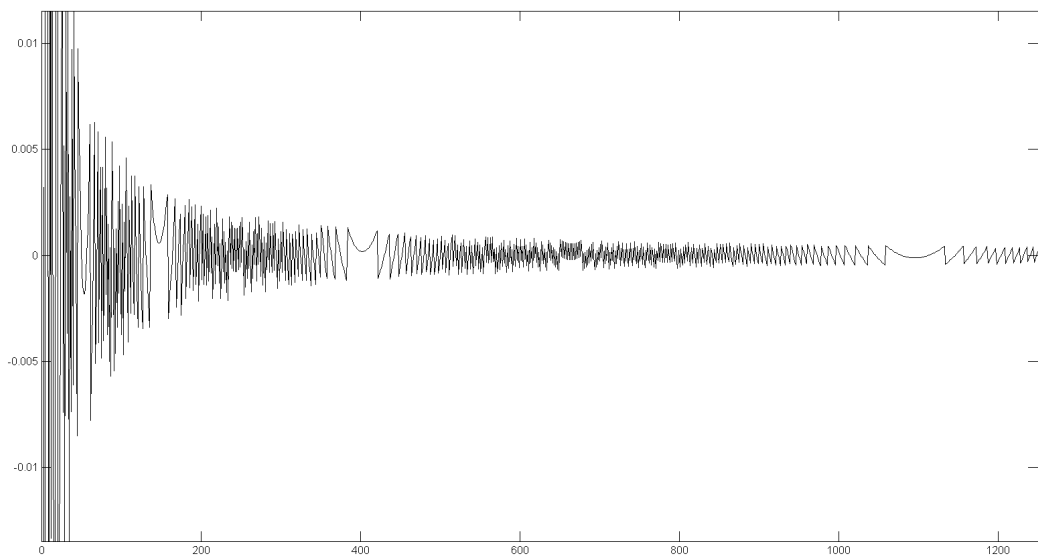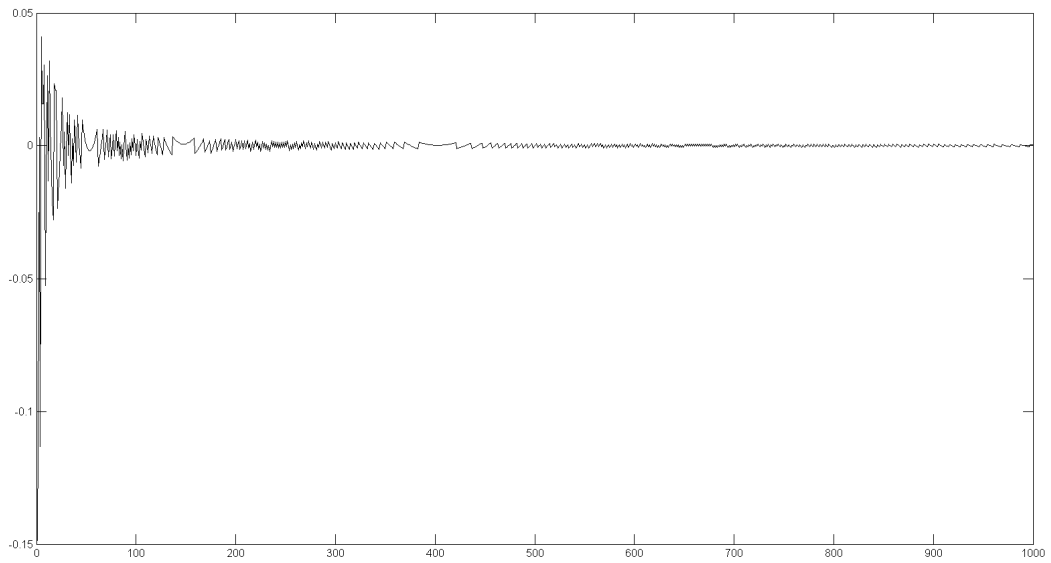$$\tilde{B}(n) - \gamma \;\approx\; \tilde{T}_B(n)$$

| $n$ | $\tilde{A}(n) - \gamma$ | $\approx$ | $\tilde{T}_A(n)$ | $\tilde{B}(n) - \gamma$ | $\approx$ | $\tilde{T}_B(n)$ | $\frac{v}{w}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.42278434 | $\approx$ | 0.38537352 | **-0.07721566** | $\approx$ | -0.14872127 | 0.5000 |
| 2 | 0.22963715 | $\approx$ | 0.21360196 | **-0.07721566** | $\approx$ | -0.10914091 | **1.0000** |
| 3 | 0.15750538 | $\approx$ | 0.15436921 | 0.00611767 | $\approx$ | 0.00321901 | 1.2500 |
| 4 | 0.11982331 | $\approx$ | 0.11211877 | -0.09388233 | $\approx$ | -0.11325811 | 1.6000 |
| 5 | 0.09667976 | $\approx$ | 0.09716585 | 0.03945100 | $\approx$ | 0.04110199 | 1.6667 |
| 6 | 0.08102487 | $\approx$ | 0.08063871 | **0.01564148** | $\approx$ | 0.01576509 | 1.8571 |
| 7 | 0.06973133 | $\approx$ | 0.06951421 | **0.01564148** | $\approx$ | 0.01584913 | **2.0000** |
| 8 | 0.06119994 | $\approx$ | 0.06151555 | 0.02953037 | $\approx$ | 0.03032361 | 2.1111 |
| 9 | 0.05452801 | $\approx$ | 0.05274958 | -0.04824741 | $\approx$ | -0.05268694 | 2.3000 |
| 10 | 0.04916750 | $\approx$ | 0.04848925 | -0.01188377 | $\approx$ | -0.01295341 | 2.3636 |
| 11 | 0.04476641 | $\approx$ | 0.04502926 | 0.02599501 | $\approx$ | 0.02650586 | 2.4167 |
| 12 | 0.04108836 | $\approx$ | 0.04056090 | -0.01246653 | $\approx$ | -0.01334823 | 2.5385 |
| 13 | 0.03796873 | $\approx$ | 0.03833707 | 0.03148952 | $\approx$ | 0.03196129 | 2.5714 |
| 14 | 0.03528933 | $\approx$ | 0.03522353 | 0.00767999 | $\approx$ | 0.00772028 | 2.6667 |
| 15 | 0.03296313 | $\approx$ | 0.03263906 | -0.00898667 | $\approx$ | -0.00950879 | 2.7500 |
| 16 | 0.03092461 | $\approx$ | 0.03046081 | -0.02001608 | $\approx$ | -0.02101048 | 2.8235 |
| 17 | 0.02912351 | $\approx$ | 0.02860060 | -0.02655203 | $\approx$ | -0.02784283 | 2.8889 |
| 18 | 0.02752066 | $\approx$ | 0.02771720 | 0.02315557 | $\approx$ | 0.02340315 | 2.8947 |
| 19 | 0.02608501 | $\approx$ | 0.02624269 | **0.02052399** | $\approx$ | 0.02073928 | 2.9500 |
| 20 | 0.02479172 | $\approx$ | 0.02494669 | **0.02052399** | $\approx$ | 0.02072315 | **3.0000** |
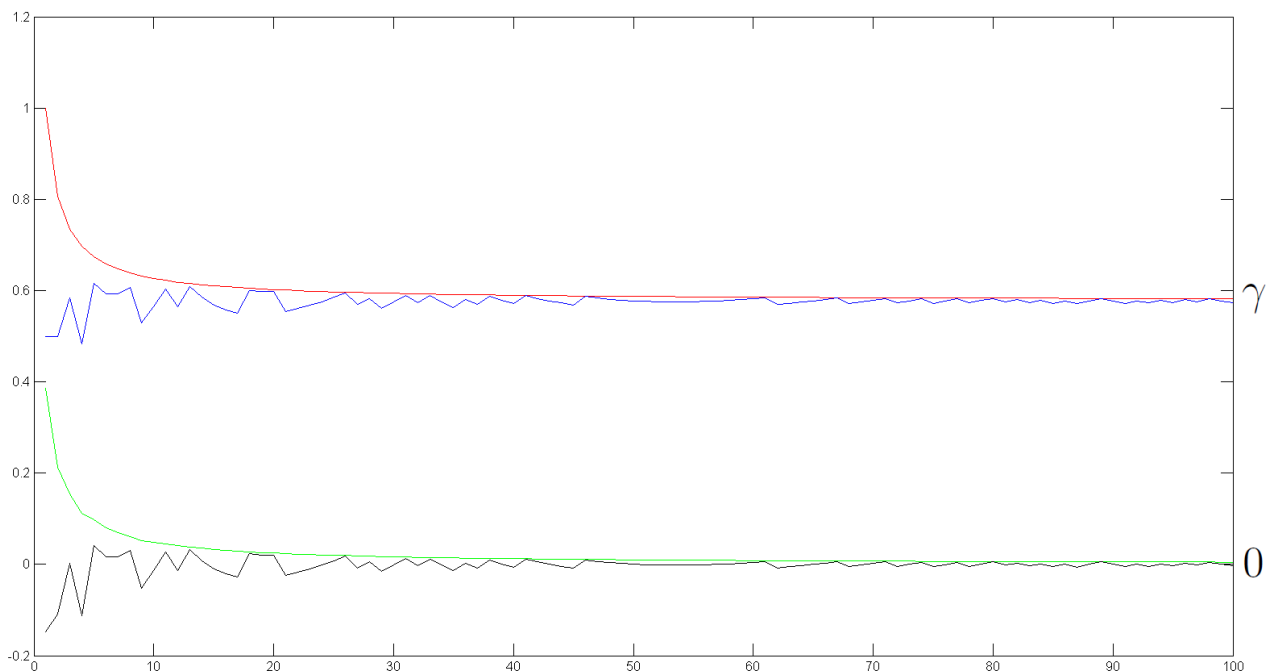


$\tilde{A}(n) = $ red, $\tilde{B}(n) = $ blue, $\tilde{T}_A(n) = $ green, $\tilde{T}_B(n) = $ black $\qquad (n = 1, ..., 100)$

Note: $\tilde{T}_A(n)$ and $\tilde{T}_B(n)$ appears to be some kind of copy of $\tilde{A}(n)$ and $\tilde{B}(n)$

Plots of $\tilde{B}(n)$

(Note: $\tilde{B}(n)$ converges to $\gamma$)

Plots of $\tilde{T}_B(n)$ $\qquad\qquad\qquad\qquad$ (Note: $\tilde{T}_B(n)$ converges to zero)

$\tilde{T}_A(n)$ and $\tilde{T}_B(n)$ can be used as correction terms for $\tilde{A}(n)$ and $\tilde{B}(n)$

| $n$ | $|\tilde{A}(n) - \gamma|$ | $|\tilde{A}(n) - \tilde{T}_A(n) - \gamma|$ | $|\tilde{B}(n) - \gamma|$ | $|\tilde{B}(n) - \tilde{T}_B(n) - \gamma|$ | $\frac{v}{w}$ |
|---|---|---|---|---|---|
| 1 | 4.22784335e-001 | 3.74108145e-002 | **7.72156649e-002** | 7.15056058e-002 | 0.5000 |
| 2 | 2.29637155e-001 | 1.60351899e-002 | **7.72156649e-002** | 3.19252493e-002 | **1.0000** |
| 3 | 1.57505380e-001 | 3.13616835e-003 | 6.11766843e-003 | 2.89865425e-003 | 1.2500 |
| 4 | 1.19823307e-001 | 7.70454125e-003 | 9.38823316e-002 | 1.93757745e-002 | 1.6000 |
| 5 | 9.66797560e-002 | 4.86097554e-004 | 3.94510018e-002 | 1.65098814e-003 | 1.6667 |
| 6 | 8.10248659e-002 | 3.86155069e-004 | **1.56414780e-002** | 1.23616682e-004 | 1.8571 |
| 7 | 6.97313289e-002 | 2.17123759e-004 | **1.56414780e-002** | 2.07650769e-004 | **2.0000** |
| 8 | 6.11999363e-002 | 3.15613351e-004 | 2.95303668e-002 | 7.93246836e-004 | 2.1111 |
| 9 | 5.45280117e-002 | 1.77843584e-003 | 4.82474109e-002 | 4.43952849e-003 | 2.3000 |
| 10 | 4.91674961e-002 | 6.78248563e-004 | 1.18837746e-002 | 1.06963652e-003 | 2.3636 |
| 11 | 4.47664072e-002 | 2.62850398e-004 | 2.59950133e-002 | 5.10848073e-004 | 2.4167 |
| 12 | 4.10883635e-002 | 5.27465407e-004 | 1.24665252e-002 | 8.81706728e-004 | 2.5385 |
| 13 | 3.79687328e-002 | 3.68339974e-004 | 3.14895188e-002 | 4.71770177e-004 | 2.5714 |
| 14 | 3.52893320e-002 | 6.58015589e-005 | 7.67999499e-003 | 4.02839237e-005 | 2.6667 |
| 15 | 3.29631272e-002 | 3.24069057e-004 | 8.98667167e-003 | 5.22120607e-004 | 2.7500 |
| 16 | 3.09246061e-002 | 4.63796961e-004 | 2.00160834e-002 | 9.94397926e-004 | 2.8235 |
| 17 | 2.91235137e-002 | 5.22909915e-004 | 2.65520311e-002 | 1.29080051e-003 | 2.8889 |
| 18 | 2.75206554e-002 | 1.96540316e-004 | 2.31555712e-002 | 2.47581522e-004 | 2.8947 |
| 19 | 2.60850131e-002 | 1.57672889e-004 | **2.05239922e-002** | 2.15285219e-004 | 2.9500 |
| 20 | 2.47917187e-002 | 1.54972676e-004 | **2.05239922e-002** | 1.99161598e-004 | **3.0000** |
| 21 | 2.36206021e-002 | 3.59031254e-004 | 2.27660510e-002 | 9.03769035e-004 | 3.0909 |
| 22 | 2.25551320e-002 | 2.72560480e-004 | 1.68371973e-002 | 6.14026037e-004 | 3.1304 |
| 23 | 2.15816303e-002 | 1.79570471e-004 | 9.59082048e-003 | 3.33448745e-004 | 3.1667 |
| 24 | 2.06886825e-002 | 8.31189405e-005 | 1.25748715e-003 | 9.79377315e-005 | 3.2000 |
| 25 | 1.98666880e-002 | 1.46137118e-005 | 7.97328208e-003 | 6.23042354e-005 | 3.2308 |
| 26 | 1.91075133e-002 | 1.12078093e-004 | 1.79447921e-002 | 1.22579716e-004 | 3.2593 |
| 27 | 1.84042223e-002 | 1.21653371e-004 | 7.18748308e-003 | 2.15982971e-004 | 3.3214 |
| 28 | 1.77508639e-002 | 6.18995297e-006 | 5.12778786e-003 | 2.62716515e-005 | 3.3448 |
| 29 | 1.71423027e-002 | 1.81319188e-004 | 1.55618673e-002 | 4.41582596e-004 | 3.4000 |
| 30 | 1.65740844e-002 | 5.82719273e-005 | 1.58337269e-003 | 7.32665906e-005 | 3.4194 |
| 31 | 1.60423260e-002 | 5.83937978e-005 | 1.25295305e-002 | 8.05291112e-005 | 3.4375 |
| 32 | 1.55436277e-002 | 6.74691464e-005 | 3.56895431e-003 | 1.02442321e-004 | 3.4848 |
| 33 | 1.50749994e-002 | 5.00733127e-005 | 1.15825608e-002 | 7.04101107e-005 | 3.5000 |
| 34 | 1.46338009e-002 | 4.88837369e-005 | 1.86281731e-003 | 6.47391048e-005 | 3.5429 |
| 35 | 1.42176926e-002 | 1.30940800e-004 | 1.37675792e-002 | 3.27244657e-004 | 3.5833 |
| 36 | 1.38245934e-002 | 1.24768478e-005 | 2.74893730e-003 | 2.73330197e-006 | 3.5946 |
| 37 | 1.34526463e-002 | 7.82536329e-005 | 7.20838845e-003 | 1.54049523e-004 | 3.6316 |
| 38 | 1.31001887e-002 | 3.50908745e-005 | 9.66070737e-003 | 5.17842662e-005 | 3.6410 |
| 39 | 1.27657279e-002 | 1.83980842e-005 | 1.32737403e-003 | 1.08831938e-005 | 3.6750 |
| 40 | 1.24479199e-002 | 6.28009318e-005 | 5.98969914e-003 | 1.18942284e-004 | 3.7073 |

## A Further Generalization

For all $k \in \mathbb{Z}$, let the set $\mathbb{X}_k$ be defined as

$$\mathbb{X}_k := \{n \in \mathbb{Z} \mid n > -k\}$$

Consider the following family of functions

$$\Gamma := \{\hat{A}_k, \hat{B}_k : k \in \mathbb{Z}\}$$

mapping from $\mathbb{X}_k$ to $\mathbb{R}$ and $\mathbb{Q}$ respectively

$$\hat{A}_k : \mathbb{X}_k \to \mathbb{R}$$

$$\hat{B}_k : \mathbb{X}_k \to \mathbb{Q}$$

where $\hat{A}_k$ and $\hat{B}_k$ are defined as

$$\hat{A}_k(n) := \begin{cases} -\sum_{i=1}^{-n} \frac{1}{i} - log(n+k) & \text{,if } n < 0 \\ - log(n+k) & \text{,if } n = 0 \\ \sum_{i=1}^{n} \frac{1}{i} - log(n+k) & \text{,if } n > 0 \end{cases}$$

$$\hat{B}_k(n) := \begin{cases} -\sum_{i=1}^{-n} \frac{1}{i} - \frac{\lfloor (n+k) \ log(n+k) \rfloor}{n+k} & \text{,if } n < 0 \\ - \frac{\lfloor (n+k) \ log(n+k) \rfloor}{n+k} & \text{,if } n = 0 \\ \sum_{i=1}^{n} \frac{1}{i} - \frac{\lfloor (n+k) \ log(n+k) \rfloor}{n+k} & \text{,if } n > 0 \end{cases}$$
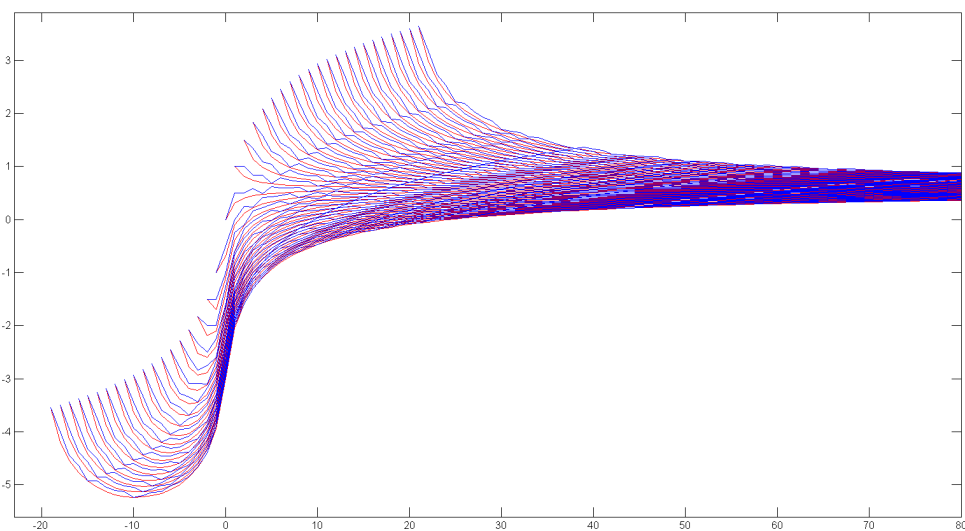
Then

$$\tilde{A}(n) = \sum_{i=1}^{n} \frac{1}{i} - log(n)$$

and

$$\tilde{B}(n) = \sum_{i=1}^{n} \frac{1}{i} - \frac{v}{w}$$

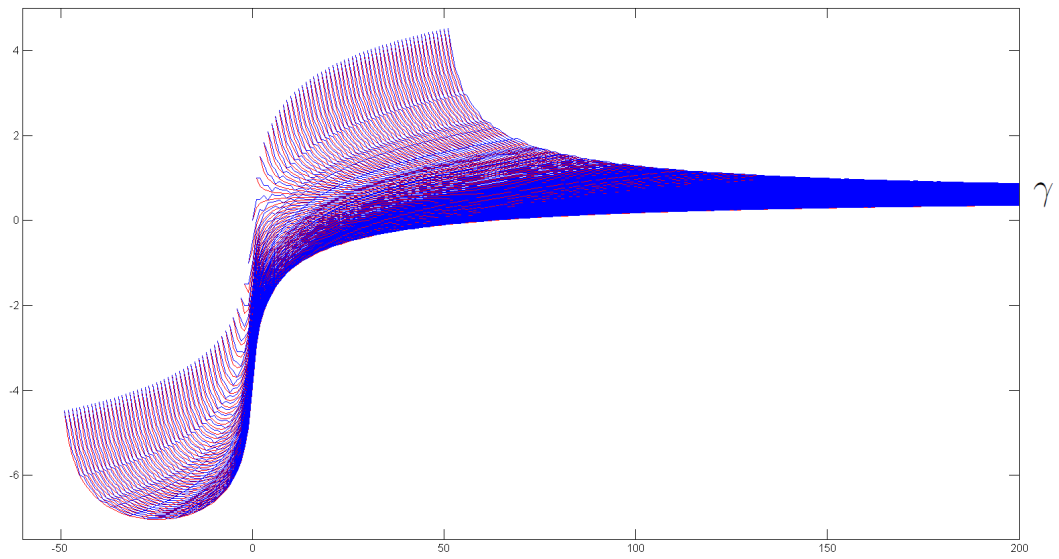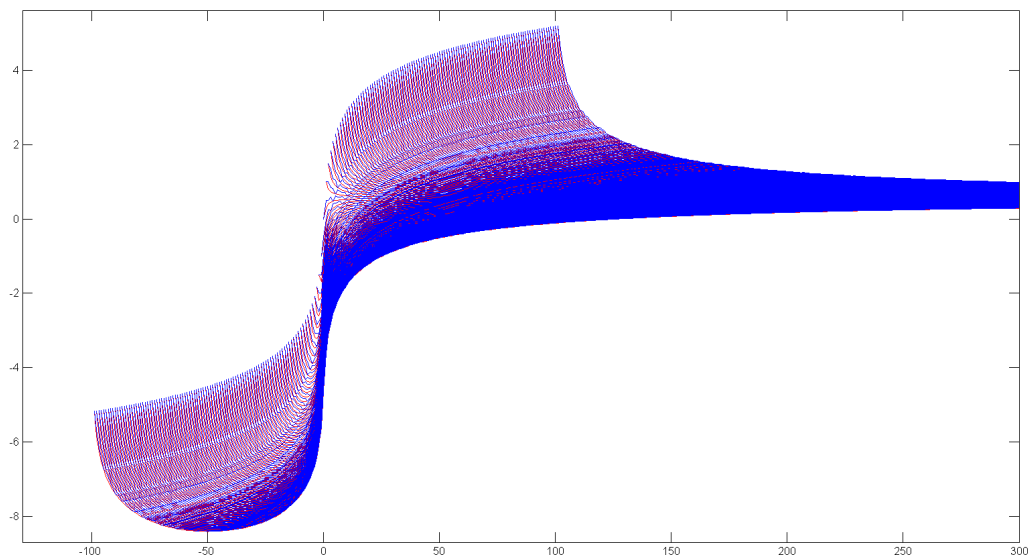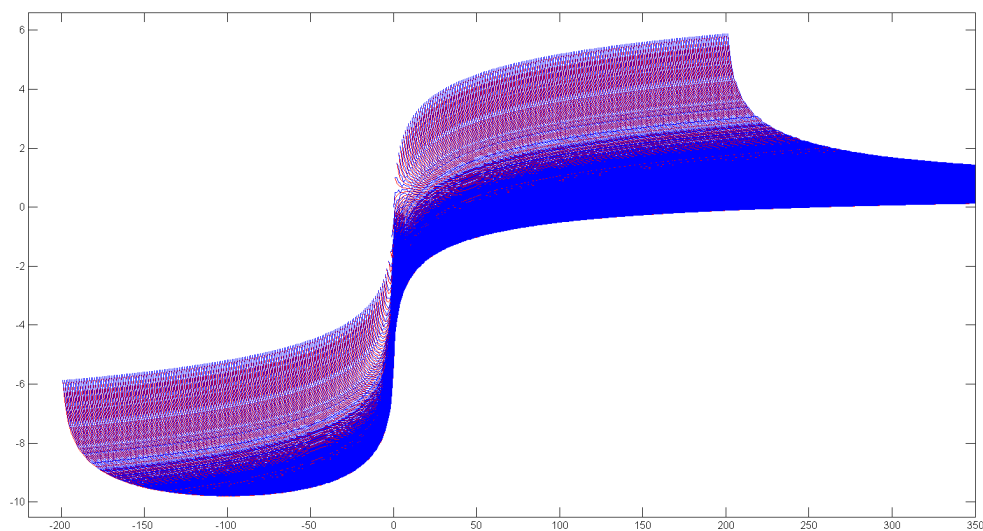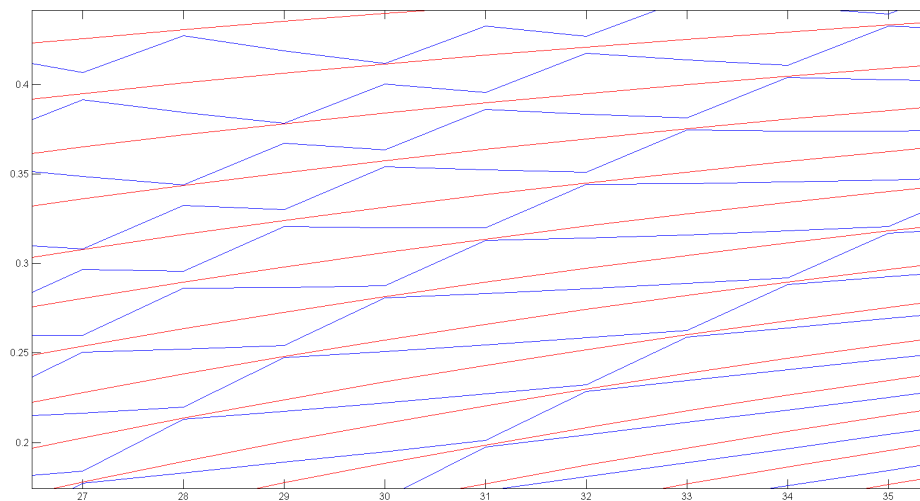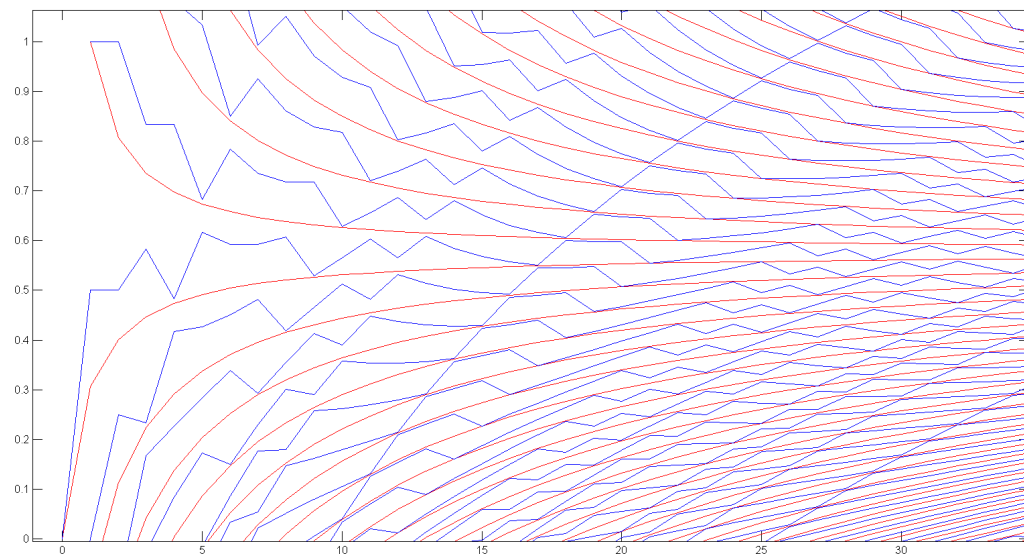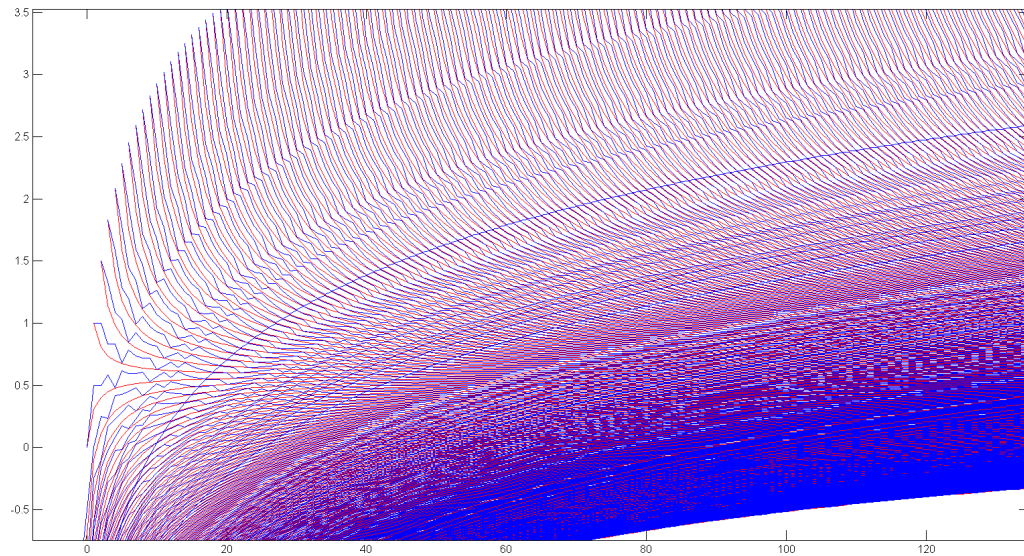are members of $\Gamma$, because

$$\hat{A}_0(n) = \tilde{A}(n) \qquad\qquad (\mathbb{X}_0 = 1, 2, 3, ...)$$

$$\hat{B}_1(n) = \tilde{B}(n) \qquad\qquad (\mathbb{X}_1 = 0, 1, 2, 3, ...)$$

Plot of $\hat{A}_1(n)$ (red) and $\hat{B}_1(n)$ (blue)



$\gamma$
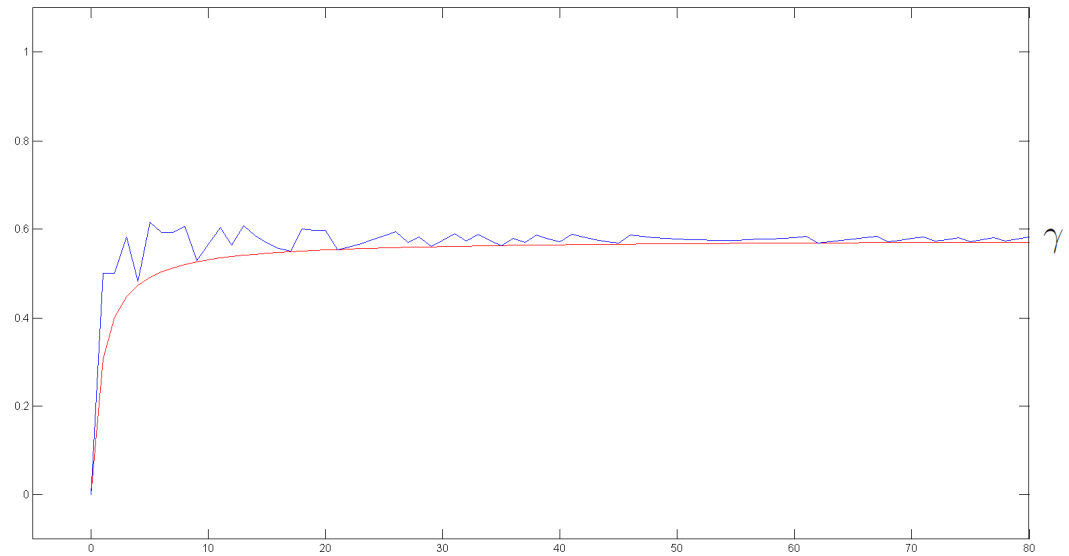
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{0, 1\}$
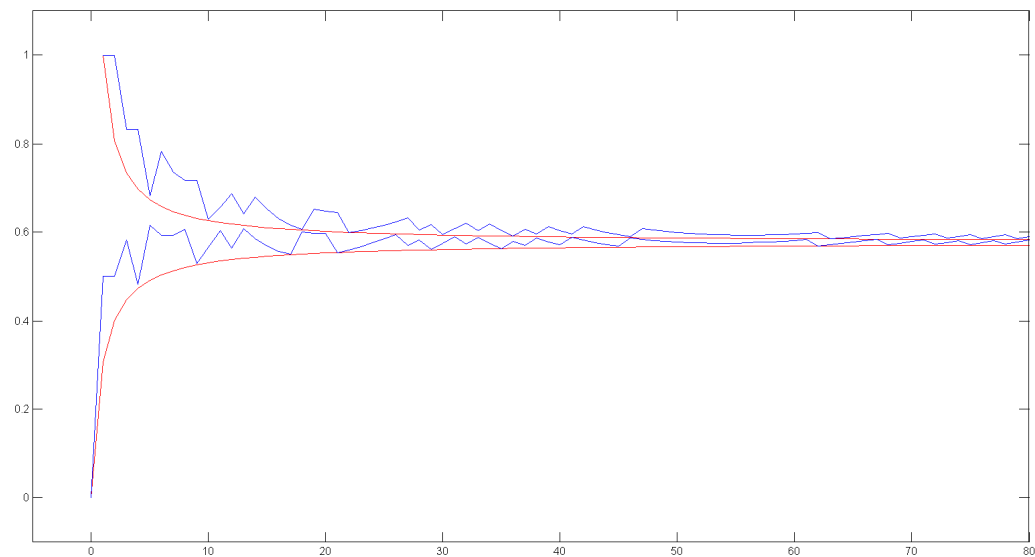


Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-1, 0, 1, 2\}$
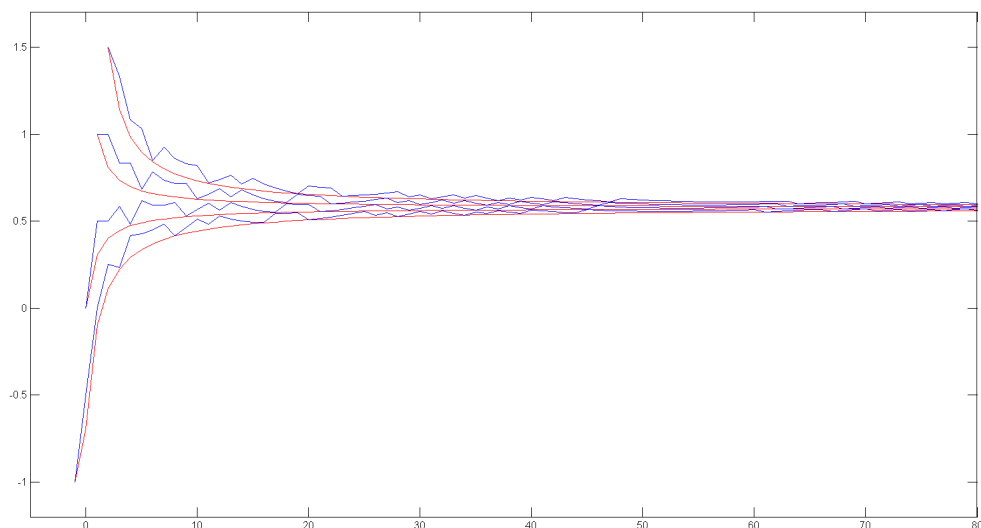
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-5, -4, ..., 4, 5\}$



$\gamma$
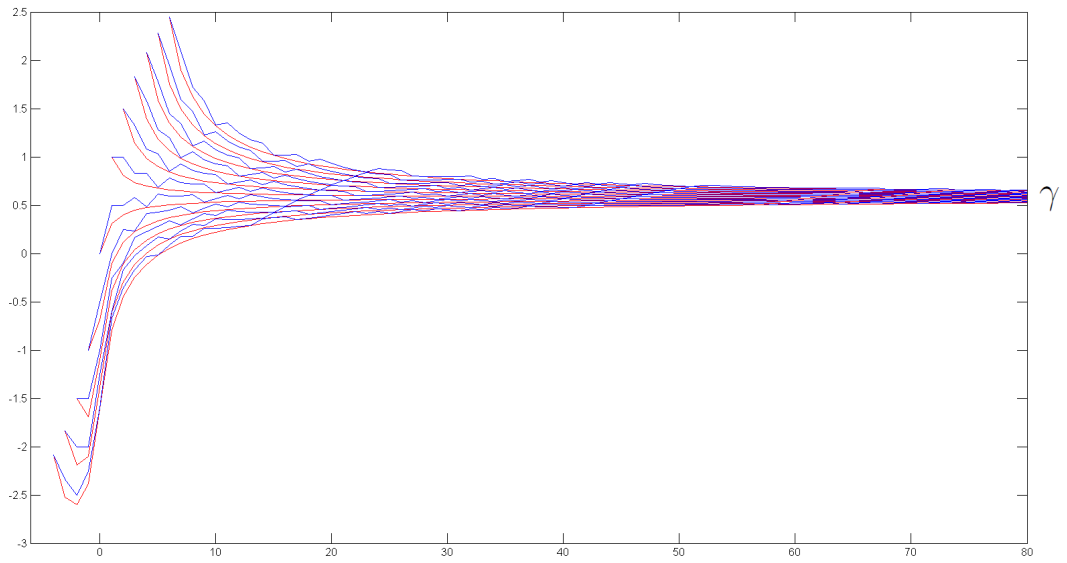
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-10, ..., 10\}$
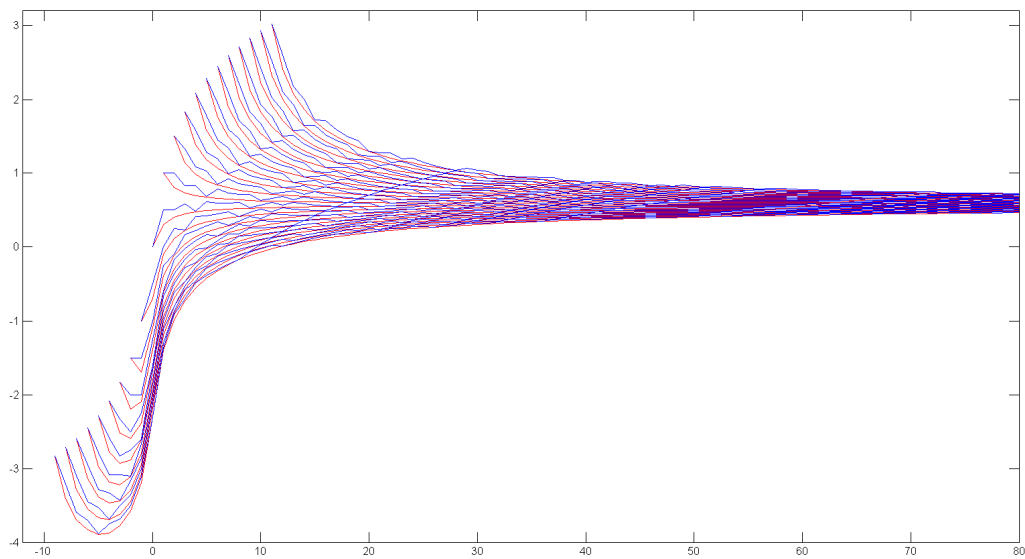


Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-20, ..., 20\}$

Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-50, ..., 50\}$



Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-100, ..., 100\}$



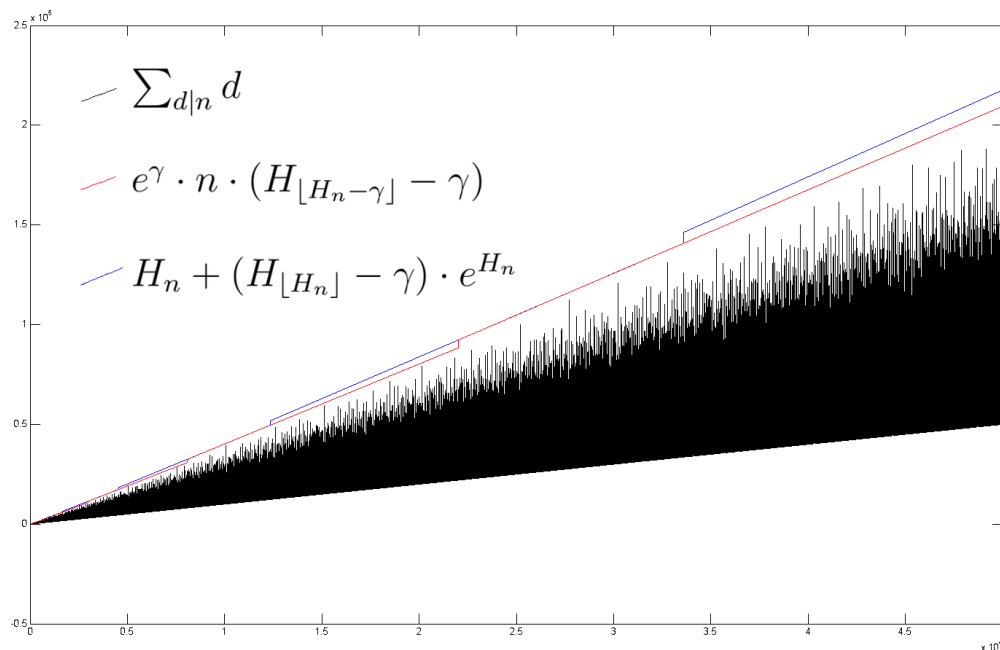Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-200, ..., 200\}$
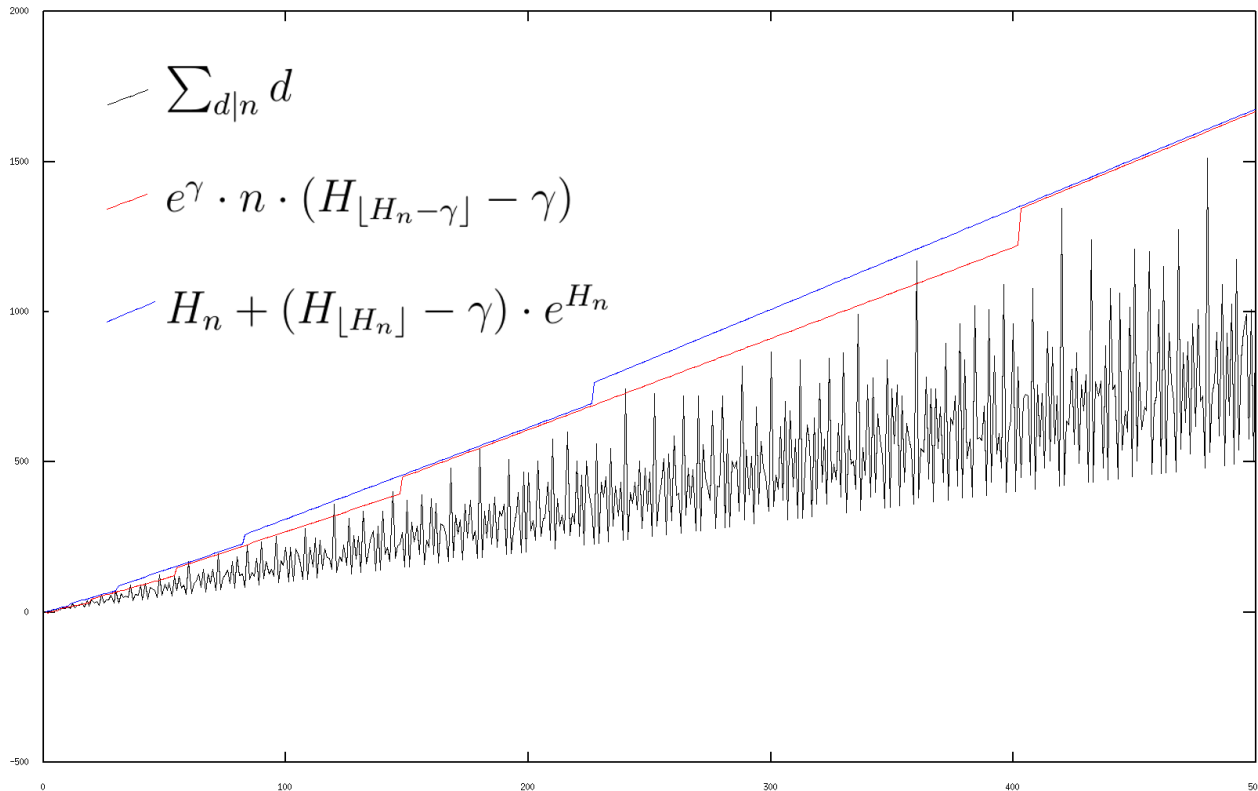
Pattern: $\hat{A}_k(n) < \hat{B}_k(n) < \hat{A}_{k-1}(n) \quad \forall\, k \in \mathbb{Z},\ n \in \mathbb{X}_{k-1}$
(In words: The red and blue lines never touch or cross each other)

# 3  Pattern related to divisor function

Substituting $log(x)$ by $H_{\lfloor x \rfloor} - \gamma$ in Lagarias criterion for the RH and Groenwall's theorem, gives the following pattern:

$$e^\gamma \cdot n \cdot (H_{\lfloor H_n - \gamma \rfloor} - \gamma) \quad < \quad H_n + (H_{\lfloor H_n \rfloor} - \gamma) \cdot e^{H_n}$$





Note that the "steps" of $H_n + (H_{\lfloor H_n \rfloor} - \gamma) \cdot e^{H_n}$ correspond with $a_n \overset{?}{=} [e^{n-\gamma}]$ (see Page 1).

# 4  Note on the indeterminate form $0^0$

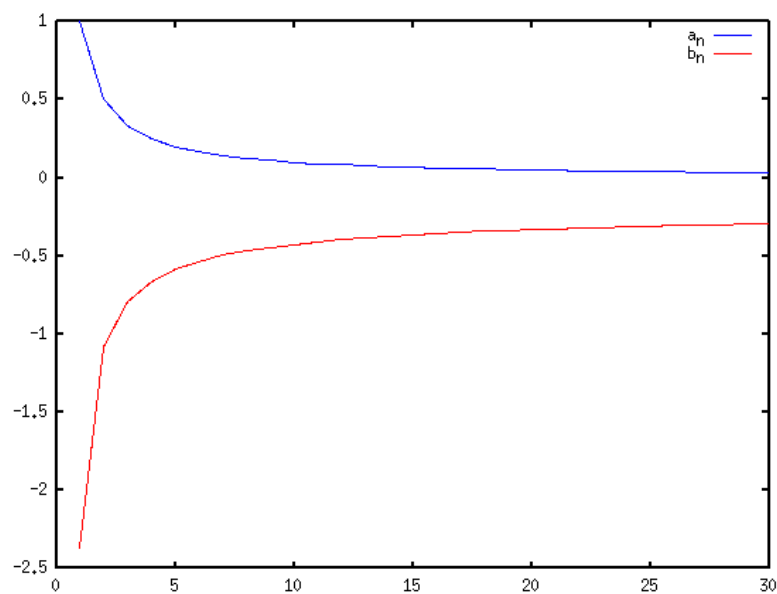While $0^0$ is *"normally"* one, there is the following exception:

$$a_n := \frac{1}{n} \qquad\qquad (a_n \longrightarrow 0^+)$$

$$b_n := \frac{1}{\gamma - H_n} \qquad\qquad (b_n \longrightarrow 0^-)$$

then

$$\lim_{n\to\infty} \quad a_n^{b_n} \longrightarrow 0^0 = e$$

$$\lim_{n\to\infty} \quad b_n^{a_n} \longrightarrow 0^0 = 1 + 0 \cdot \pi i$$



Matlab/Octave source code:

```
function zeropowerzero
gamma = 0.5772156649;
n = 100000;
a = 1/n;
b = 1/(gamma-H(n));
printf('\n %10.9f'          ,a^b);
printf('\n %10.9f + %10.9f * i',b^a,imag(b^a));
% Harmonic Number
function [h] = H(n)
  h = 0;
  for i = 1 : n
      h = h + 1 / i;
  end
-------------------------------------
Output:
>> 2.718280648
>> 0.999975565 + 0.000031415 * i
```

# 5  Appendix: Source code to construct $e$,$\gamma$ and $\pi$

Programming language: Matlab/Octave

```matlab
function construct % This program numerically approximates 'e', 'gamma' and 'pi'
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  n    = 1; % Initialize index 'n' for sequence 'a(n)'
  a(n) = 1;  % Define first element of 'a(n)' as 1
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  for k = 1:50000 % Generate sequence 'a(n)' which is OEIS-A004080 for 'n>=1'

    if  Zeta(1,k ,1) - floor(Zeta(1,k ,1)) < ...
        Zeta(1,k-1,1) - floor(Zeta(1,k-1,1)) % Check if 'k' is part of 'a(n)'

        n = n + 1; % Increase index 'n'
        a(n) = k; % Update sequence 'a(n)'

    end
  end
  a % Print sequence 'a(n)'
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  EULER = a(n) / a(n-1)                   % Print approximation of 'e'
  GAMMA = Zeta(1,floor(EULER^n),1) - n % Print approximation of 'gamma'
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  TAU      = 0;       % Initialize 'tau' as zero
  x        = 0;       % Initialize 'x' as zero
  tolerance = 1;      % Tolerance for equality 'exp()=lim zeta()/zeta()'
  stepsize = 0.001;   % Stepsize to increase 'x' from zero to 'infinity'
  i        = sqrt(-1); % Define 'i' as imaginary unit sqrt(-1)
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  while (TAU == 0)  % Loop that increases 'x' and checks equality

    x = x + stepsize; % Increase 'x' by stepsize increment

    ratio = Zeta(a(n-2), a(n-1)-1, 1+i*x) / ...
            Zeta(a(n-1), a(n-0)-1, 1+i*x); % zeta()/zeta()

    % Check if equality 'exp()=lim zeta()/zeta()' fails
    if ( abs(ratio-EULER^(i*x)) > tolerance )

        TAU = x % If equality fails, 'tau' is found and printed

    end
  end
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  PI = TAU / 2 % Print 'pi' as half of 'tau'


% Zetafunction from 'A' to 'B' with input argument 'x'
function [z] = Zeta( A, B, x)
  z = 0;
  for n = A : B
     z = z + 1 / n^x;
  end
```