

Some formulas and patterns

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Abstract

This document compiles some formulas and patterns.

Zeta function

Let

$$\zeta_A^B(x) := \sum_{n=A}^B \frac{1}{n^x} \quad (1)$$

where $A, B \in \mathbb{Z}$ and $x \in \mathbb{C}$.

Consider further $a_n := 1, 2, 4, 11, 31, 83, 227, \dots$ ([OEIS-A002387](#)) which is based on $\zeta_1^\infty(1)$.

Construction of e , γ and π

Based on the zeta function ζ and the sequence a_n , the constants γ , e and $\tau := 2\pi$ can be constructed as follows:

$$e = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_{n+0}}$$

$$\gamma = \lim_{n \rightarrow \infty} \zeta_1^{\lfloor (\frac{a_{n+1}}{a_{n+0}})^n \rfloor} (1) - n$$

$$\tau = \min(\{ x \in \mathbb{R}^+ : \lim_{n \rightarrow \infty} \frac{\zeta_{a_{n+0}}^{a_{n+1}}(1+ix)}{\zeta_{a_{n+1}}^{a_{n+2}}(1+ix)} \neq (\frac{a_{n+1}}{a_{n+0}})^{ix} \})$$

Note that even though numerical evidence indicates that the sequence a_n coincides with $[e^{n-\gamma}]$, this relationship is unproven and is not exploited in the above construction of e , γ and π .

In other words: Based on a_n it is possible to construct e and γ , but it is an open question, if it is possible to construct a_n out of e and γ .

Euler's identity

The previous construction of π is based on the formula:

$$\lim_{n \rightarrow \infty} \frac{\zeta_{a_{n+0}}^{a_{n+1}}(x)}{\zeta_{a_{n+1}}^{a_{n+2}}(x)} = e^{x-1},$$

which holds for all $x \in \mathbb{C}$ except those x of the form $2ki\pi + 1$ with $k \in \mathbb{Z} \setminus \{0\}$.

Euler's identity $e^{i\pi} = -1$ appears as special case of above formula for $x = 1i\pi + 1$.

x	$\cos(x) + i \cdot \sin(x)$		x	$\lim \frac{\zeta_{a_{n+0}}^{a_{n+1}}(x)}{\zeta_{a_{n+1}}^{a_{n+2}}(x)}$
5π	$e^{5i\pi}$		$5i\pi + 1$	$e^{5i\pi}$
4π	$e^{4i\pi}$		$4i\pi + 1$	\sim
3π	$e^{3i\pi}$		$3i\pi + 1$	$e^{3i\pi}$
2π	$e^{2i\pi}$		$2i\pi + 1$	\sim
1π	$e^{i\pi}$		$1i\pi + 1$	$e^{i\pi}$
0π	1		$0i\pi + 1$	1
-1π	$e^{-i\pi}$		$-1i\pi + 1$	$e^{-i\pi}$
-1π	$e^{-2i\pi}$		$-2i\pi + 1$	\sim
-3π	$e^{-3i\pi}$		$-3i\pi + 1$	$e^{-3i\pi}$
-4π	$e^{-4i\pi}$		$-4i\pi + 1$	\sim
-5π	$e^{-5i\pi}$		$-5i\pi + 1$	$e^{-5i\pi}$

The wave symbol (\sim) indicates that the limes does not exist

Definitions

$$A(n) := \sum_{i=1}^n \frac{1}{i} - \log(n)$$

$$B(n) := \sum_{i=1}^n \frac{1}{i} - \lfloor \sum_{i=1}^n \frac{1}{i} \rfloor$$

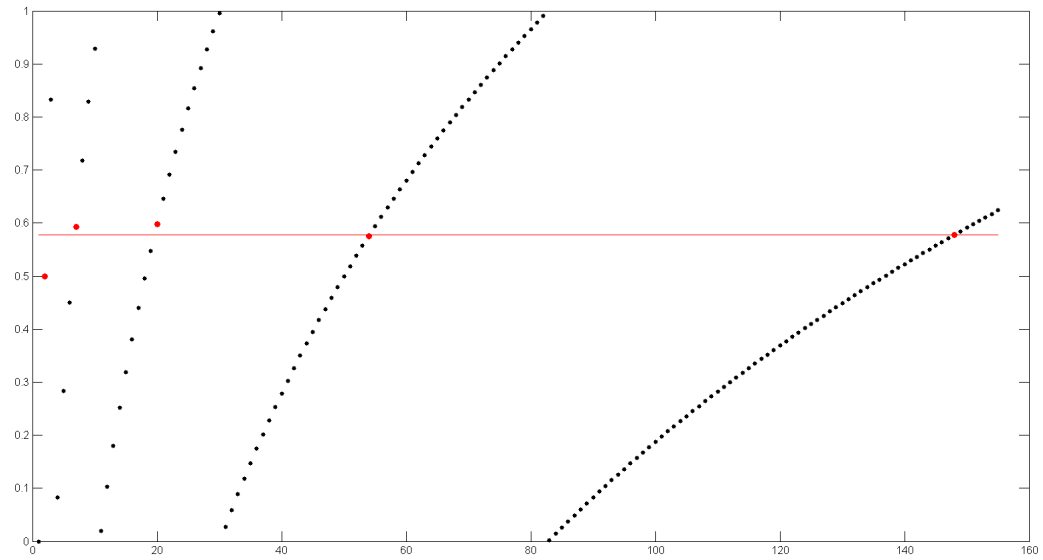
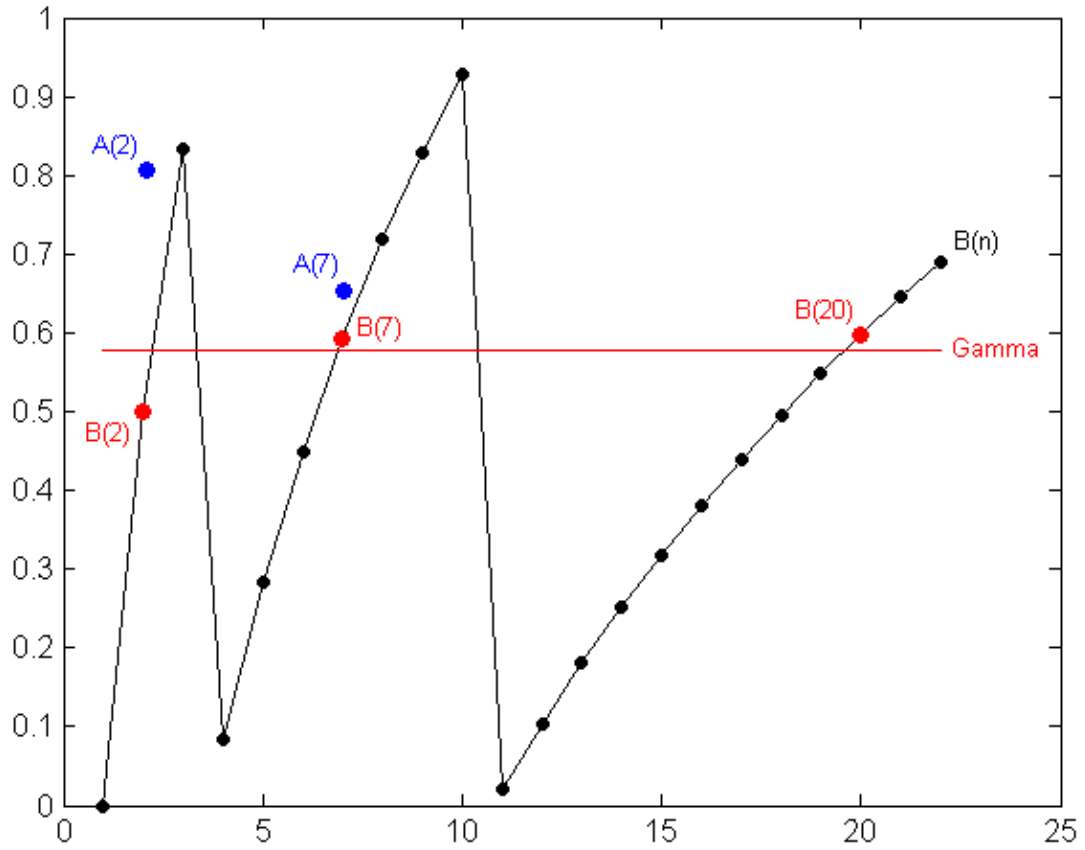
$$C(n) := \log(n) - \lfloor \log(n) \rfloor$$

$$\gamma := \lim_{n \rightarrow \infty} A(n)$$

In every period of oscillation of B there are n , so that $B(n)$ is closer to γ than $A(n)$

$$\exists n \in \mathbb{N}: B(n) - \gamma < A(n) - \gamma$$

n	$B(n)$	$B(n) - \gamma$	$<$	$A(n) - \gamma$	$A(n)$
2	0.500000000000000	-0.07721566490153	$<$	0.22963715453852	0.80685281944005
7	0.59285714285714	0.01564147795561	$<$	0.06973132890030	0.64694699380183
20	0.59773965714368	0.02052399224215	$<$	0.02479171868816	0.60200738358969
54	0.57543039374427	-0.00178527115727	$<$	0.00923068227846	0.58644634717999
148	0.57780251258124	0.00058684767971	$<$	0.00337457391559	0.58059023881713
403	0.57739240852976	0.00017674362823	$<$	0.00124018168154	0.57845584658308
1096	0.57709426741491	-0.00012139748662	$<$	0.00045613500542	0.57767179990695
2980	0.57706202025190	-0.00015364464963	$<$	0.00016777585092	0.57738344075245
8103	0.57726701163418	0.00005134673265	$<$	0.00006170427193	0.57727736917346
22026	0.57721721789959	0.00000155299805	$<$	0.00002270027317	0.57723836517470
59874	0.57722164886082	0.00000598395928	$<$	0.00000835084692	0.57722401574845
162754	0.57721387436125	-0.00000179054028	$<$	0.00000307211796	0.57721873701949
442413	0.57721590899706	0.00000024409553	$<$	0.00000113016521	0.57721679506674



The periods of oscillation of B encode $e \approx 2.71828$

$$\forall a, b \sim \text{Figure 1, 2, 3} \longrightarrow \frac{a}{b} \approx e$$

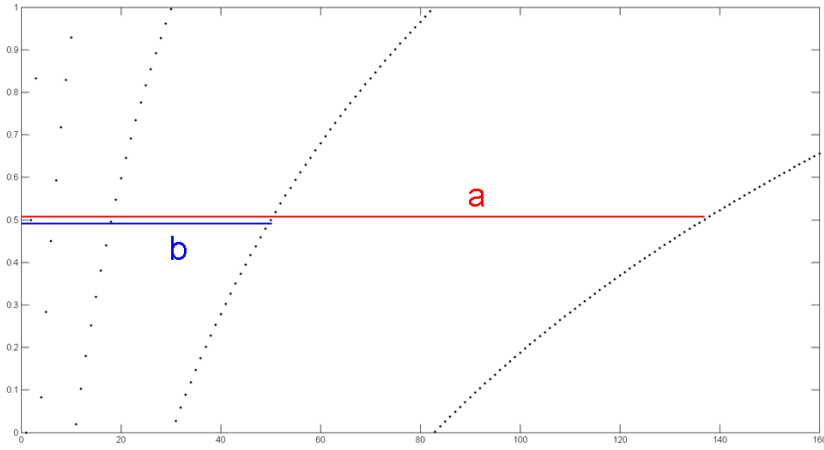


Figure 1:

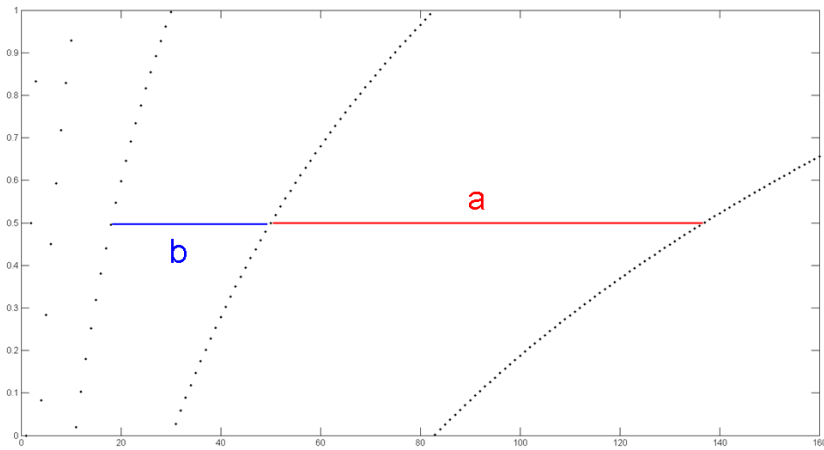


Figure 2:

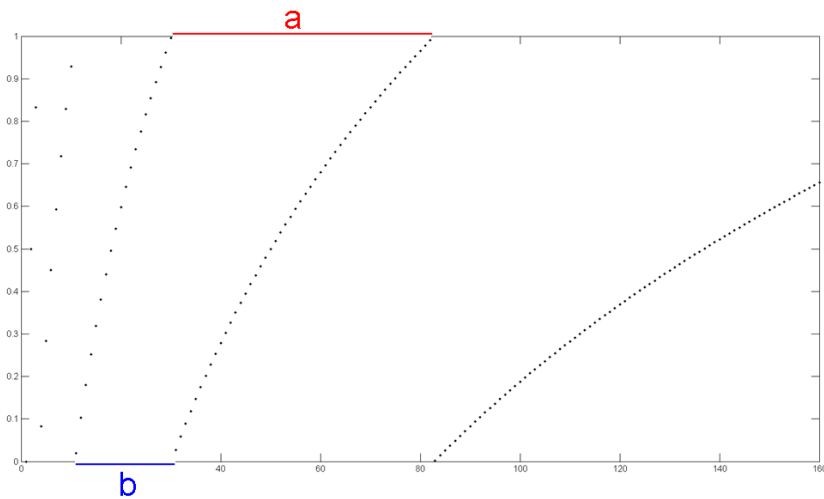


Figure 3:

Relationship between $B(n)$, $C(n)$ and γ

$$B(n) \approx D(n) := \begin{cases} C(n) + \gamma - 1 & ,\text{if } B(n) < \gamma \\ C(n) + \gamma & ,\text{if } B(n) > \gamma \end{cases}$$

n	$B(n)$	\approx	$D(n)$
1	0.0000000000000000		-0.422784335098467
2	0.5000000000000000	\approx	0.270362845461478
3	0.8333333333333333	\approx	0.675827953569642
4	0.0833333333333333	\approx	-0.036489973978577
5	0.2833333333333333	\approx	0.186653577335633
6	0.4500000000000000	\approx	0.368975134129588
7	0.592857142857143		1.523125813956846
8	0.717857142857143	\approx	0.656657206581369
9	0.828968253968254	\approx	0.774440242237752
10	0.928968253968254	\approx	0.879800757895579
20	0.597739657143682		1.572947938455524
100	0.187377517639621	\approx	0.182385850889625
148	0.577802512581241		1.574427938665648
1000	0.485470860550343	\approx	0.484970943883670
10000	0.787606036044348	\approx	0.787556036877716
100000	0.090146129863335	\approx	0.090141129871761
1000000	0.392726722864989	\approx	0.392726222865806
10000000	0.695311365857272	\approx	0.695311315859853

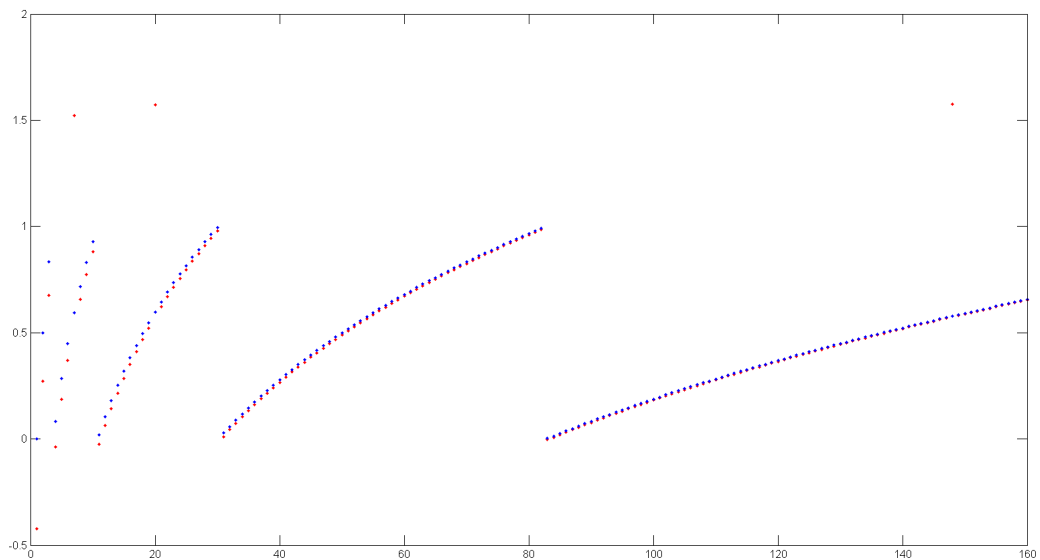
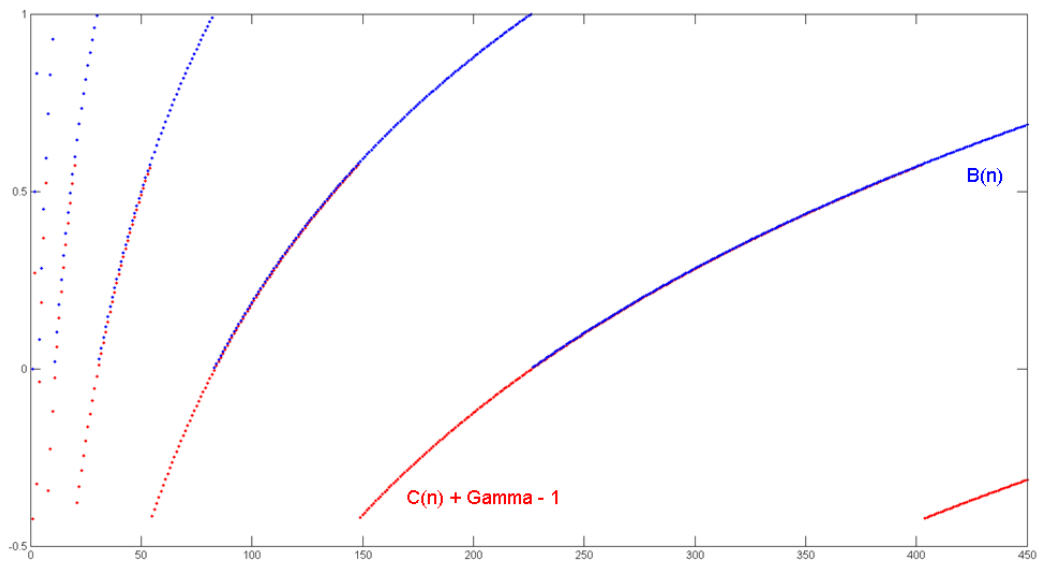
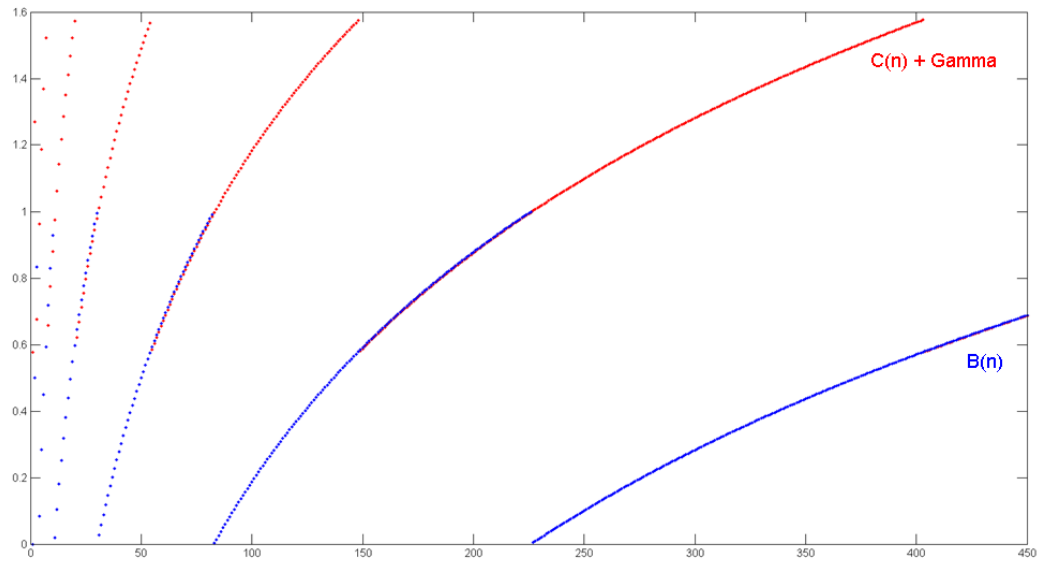
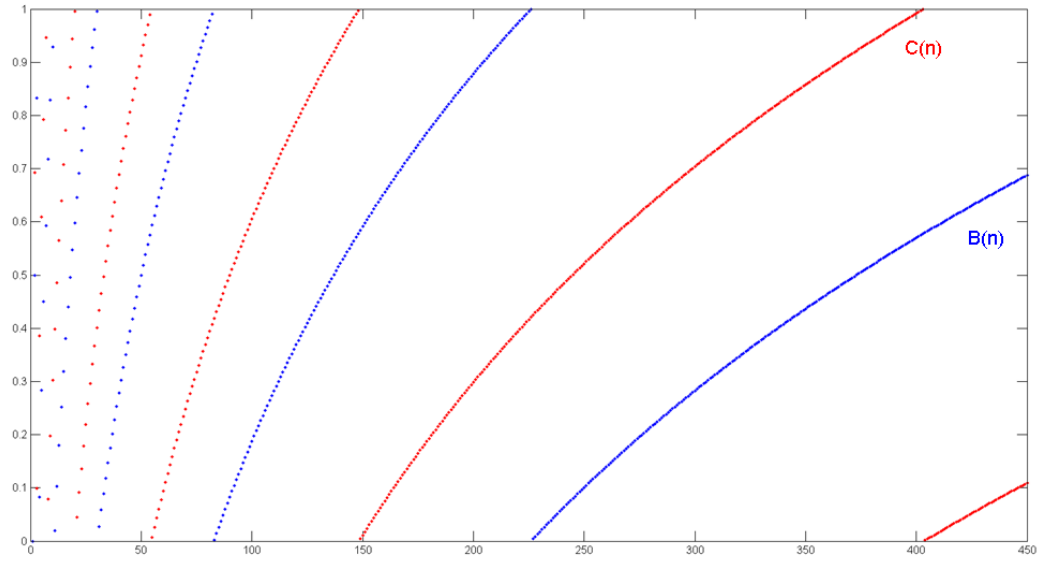


Figure 4: Plot of $B(n)$ (blue) and $F(n)$ (red)



Max/Min Relationships

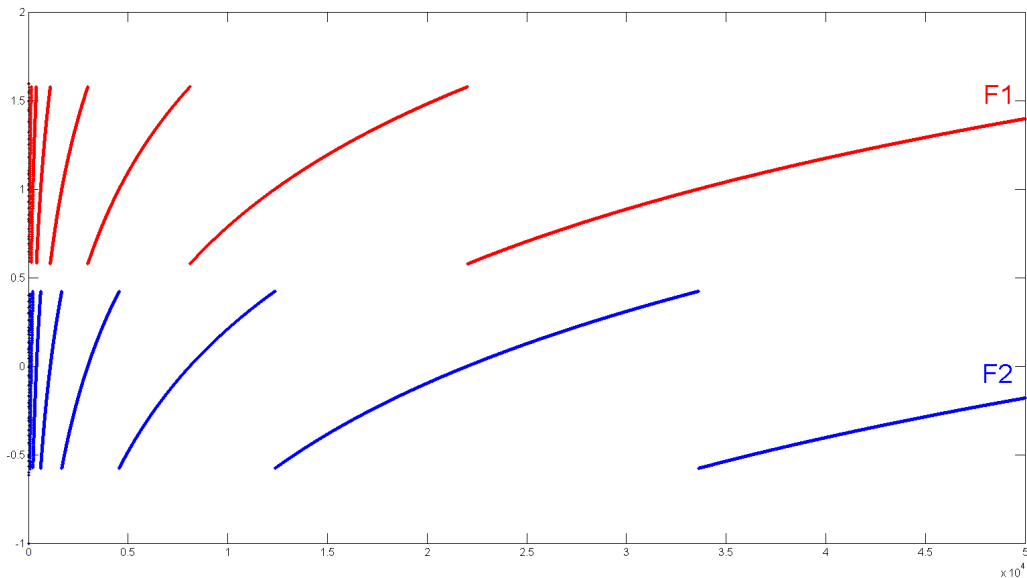
$$F_1(n) := \sum_{i=1}^n \frac{1}{i} - \lfloor \log(n) \rfloor \quad (\in \mathbb{Q})$$

$$F_2(n) := \log(n) - \left\lfloor \sum_{i=1}^n \frac{1}{i} \right\rfloor \quad (\in \mathbb{R})$$

Formula for γ :

$$\lim_{n \rightarrow \infty} \min\{ F_1 \} = \gamma$$

$$\lim_{n \rightarrow \infty} \max\{ F_2 \} = 1 - \gamma$$



n	$\min\{ F_1 \}$	$\max\{ F_2 \}$	$\min\{ F_1 \} + \max\{ F_2 \}$
1	1.0000000000000000	-1.0000000000000000	0.0000000000000000
2	1.0000000000000000	-0.306852819440055	0.693147180559945
3	0.8333333333333333	0.098612288668110	0.931945622001443
4	0.8333333333333333	0.098612288668110	0.931945622001443
5	0.8333333333333333	0.098612288668110	0.931945622001443
6	0.8333333333333333	0.098612288668110	0.931945622001443
7	0.8333333333333333	0.098612288668110	0.931945622001443
8	0.717857142857143	0.098612288668110	0.816469431525252
9	0.717857142857143	0.197224577336220	0.915081720193362
10	0.717857142857143	0.302585092994046	1.020442235851188
100	0.593612211926086	0.406719247264253	1.000331459190339
1000	0.579867656054511	0.421622267806518	1.001489923861029
10000	0.577390407488084	0.422662707570003	1.000053115058087
100000	0.577238350322196	0.422770971743748	1.000009322065944

Formula for γ

$$\begin{aligned}\gamma &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{i} - \log(n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - \log(\lfloor e^n \rfloor) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n\end{aligned}$$

Pattern

$$\sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n - \gamma < \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - \log(\lfloor e^n \rfloor) - \gamma$$

and

$$\left| \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n - \gamma \right| < \left| \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - \log(\lfloor e^n \rfloor) - \gamma \right|$$

Definition

$$A(n) := \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - \log(\lfloor e^n \rfloor)$$

$$B(n) := \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n$$

$$T_A(n) := \frac{1}{2} \frac{n - \log(\lfloor e^n \rfloor)}{e^n - \lfloor e^n \rfloor}$$

$$T_B(n) := \frac{\frac{1}{2} + \lfloor e^n \rfloor - e^n}{\lfloor e^n \rfloor}$$

Pattern

$$A(n) - \gamma \approx T_A(n)$$

$$B(n) - \gamma \approx T_B(n)$$

n	$A(n) - \gamma$	\approx	$T_A(n)$	$B(n) - \gamma$	\approx	$T_B(n)$
1	0.229637154538522	\approx	0.213601964634381	-0.077215664901533	\approx	-0.109140914229523
2	0.069731328900296	\approx	0.069514205140797	0.015641477955610	\approx	0.015849128724193
3	0.024791718688158	\approx	0.024946691364183	0.020523992242149	\approx	0.020723153840617
4	0.009230682278460	\approx	0.009208353109854	-0.001785271157266	\approx	-0.001817593206375
5	0.003374573915593	\approx	0.003373671569258	0.000586847679708	\approx	0.000586762820428
6	0.001240181681544	\approx	0.001240035204982	0.000176743628226	\approx	0.000176691085025
7	0.000456135005418	\approx	0.000456072655816	-0.000121397486621	\approx	-0.000121494916477
8	0.000167775850920	\approx	0.000167758271537	-0.000153644649630	\approx	-0.000153686926754
9	0.000061704271926	\approx	0.000061705221596	0.000051346732651	\approx	0.000051347948243
10	0.000022700273166	\approx	0.000022700204904	0.000001552998054	\approx	0.000001552946213
11	0.000008350846916	\approx	0.000008350860275	0.000005983959284	\approx	0.000005983979727
12	0.000003072117962	\approx	0.000003072113647	-0.000001790540282	\approx	-0.000001790548951
13	0.000001130165210	\approx	0.000001130165204	0.000000244095527	\approx	0.000000244095629
14	0.000000415763583	\approx	0.000000415764407	0.000000179472383	\approx	0.000000179473229
15	0.000000152950261	\approx	0.000000152951169	0.000000039010172	\approx	0.000000039011082
16	0.000000056265555	\approx	0.000000056267590	-0.000000002309892	\approx	-0.000000002307857
17	0.000000020695265	\approx	0.000000020699688	-0.000000010502283	\approx	-0.000000010497860
18	0.000000007612997	\approx	0.000000007614984	0.000000005521458	\approx	0.000000005523449
19	0.000000002802197	\approx	0.000000002801399	-0.000000002594347	\approx	-0.000000002595144
20	0.000000001031574	\approx	0.000000001030576	0.000000000186934	\approx	0.000000000185936

Generalization

$$n = \lfloor e^{\frac{v}{w}} \rfloor$$

$$v := \lfloor \log(n+1) \cdot (n+1) \rfloor$$

$$w := n+1$$

n	v/w	$\lfloor e^{\frac{v}{w}} \rfloor$
1	$1 / 2 = 0.50000$	1
2	$3 / 3 = 1.00000$	2
3	$5 / 4 = 1.25000$	3
4	$8 / 5 = 1.60000$	4
5	$10 / 6 = 1.66667$	5
6	$13 / 7 = 1.85714$	6
7	$16 / 8 = 2.00000$	7
8	$19 / 9 = 2.11111$	8
9	$23 / 10 = 2.30000$	9
10	$26 / 11 = 2.36364$	10

Definition

$$\tilde{A}(n) := \sum_{i=1}^n \frac{1}{i} - \log(n)$$

$$\tilde{B}(n) := \sum_{i=1}^n \frac{1}{i} - \frac{v}{w}$$

$$\tilde{T}_A(n) := \frac{1}{2} \frac{\frac{v}{w} - \log(n)}{e^{\frac{v}{w}} - n}$$

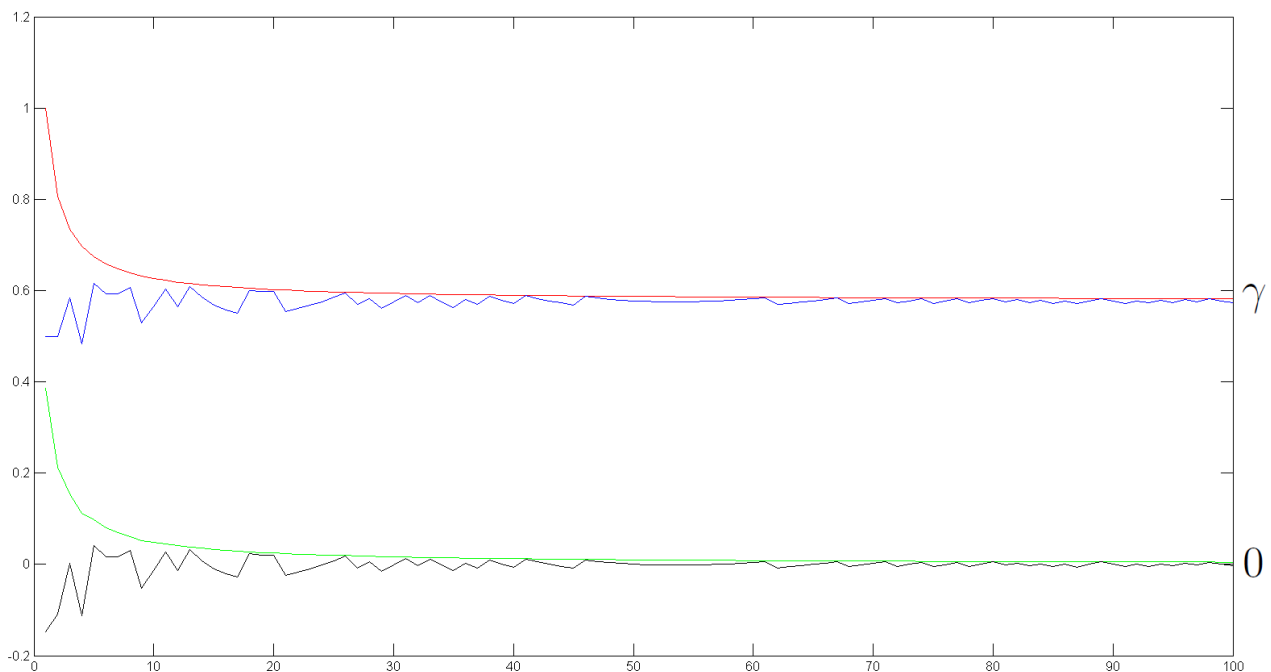
$$\tilde{T}_B(n) := \frac{\frac{1}{2} + n - e^{\frac{v}{w}}}{n}$$

Pattern (generalized)

$$\tilde{A}(n) - \gamma \approx \tilde{T}_A(n)$$

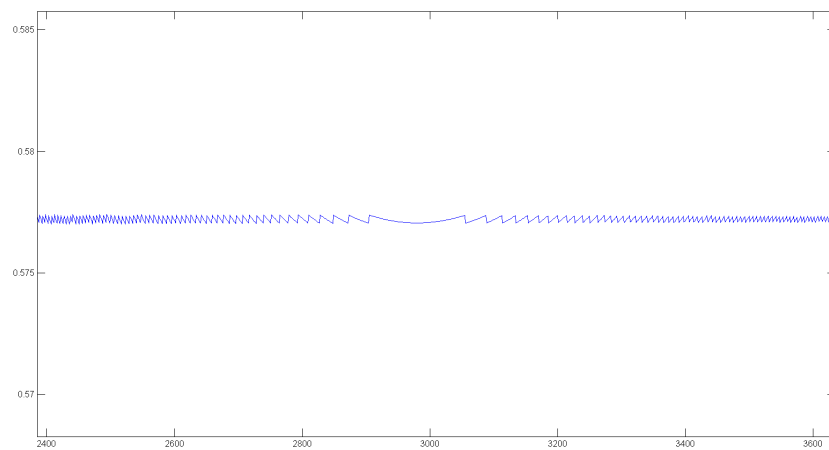
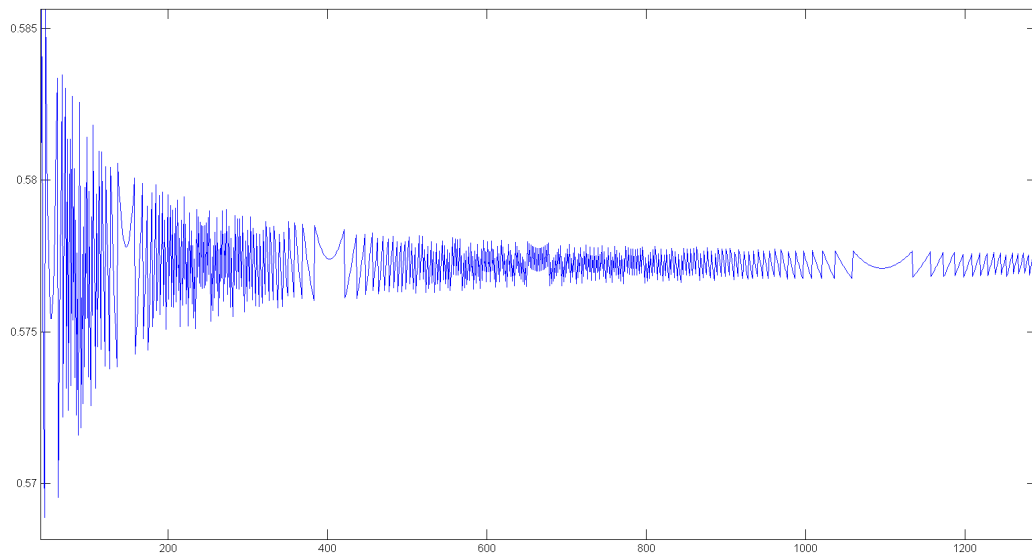
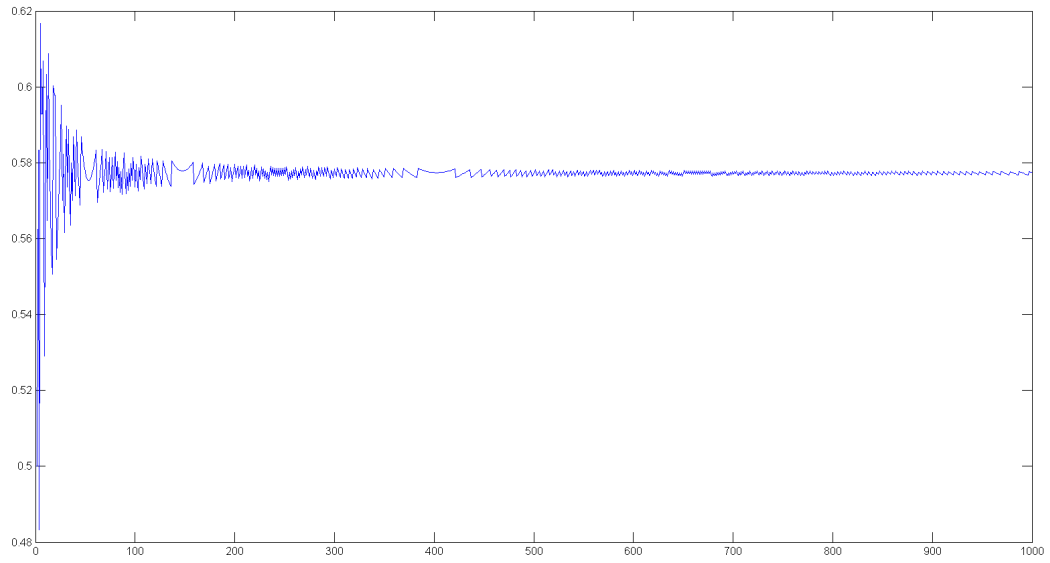
$$\tilde{B}(n) - \gamma \approx \tilde{T}_B(n)$$

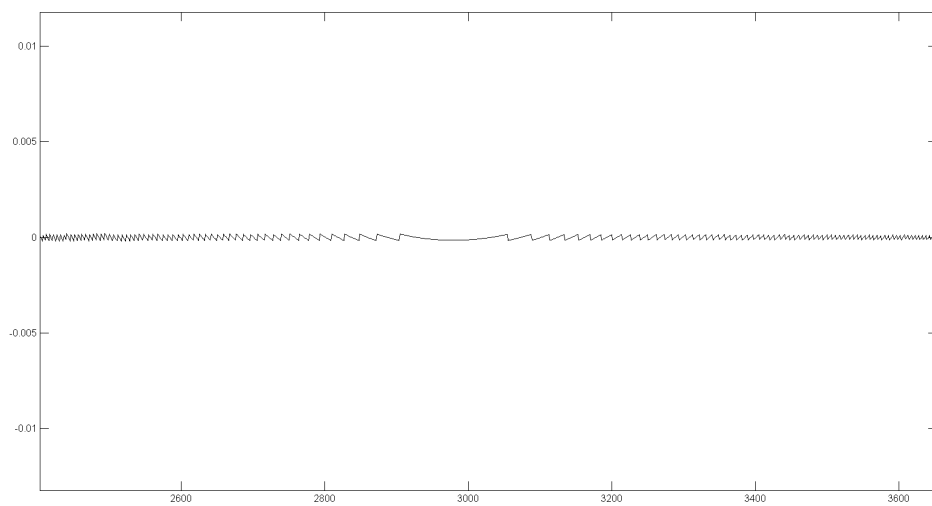
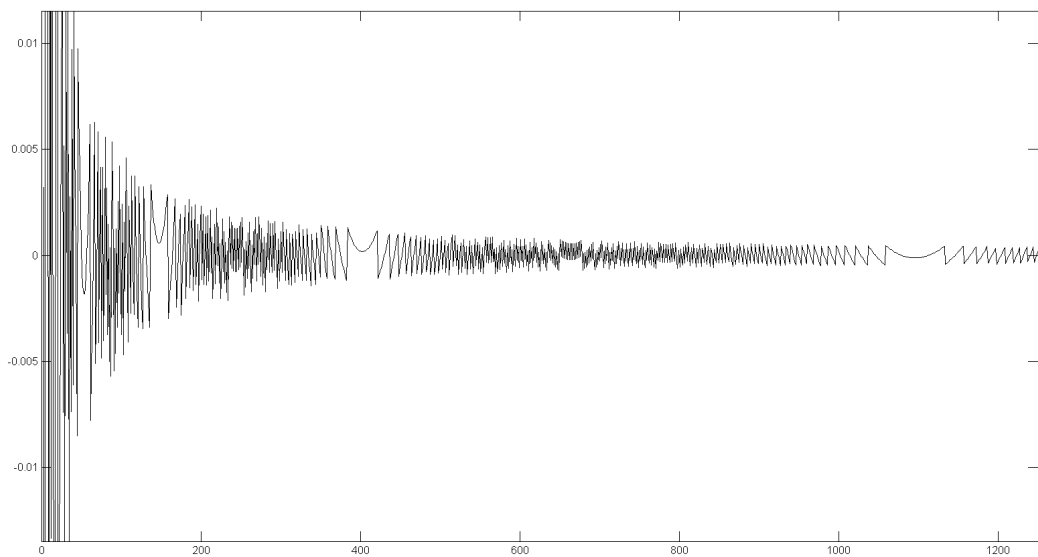
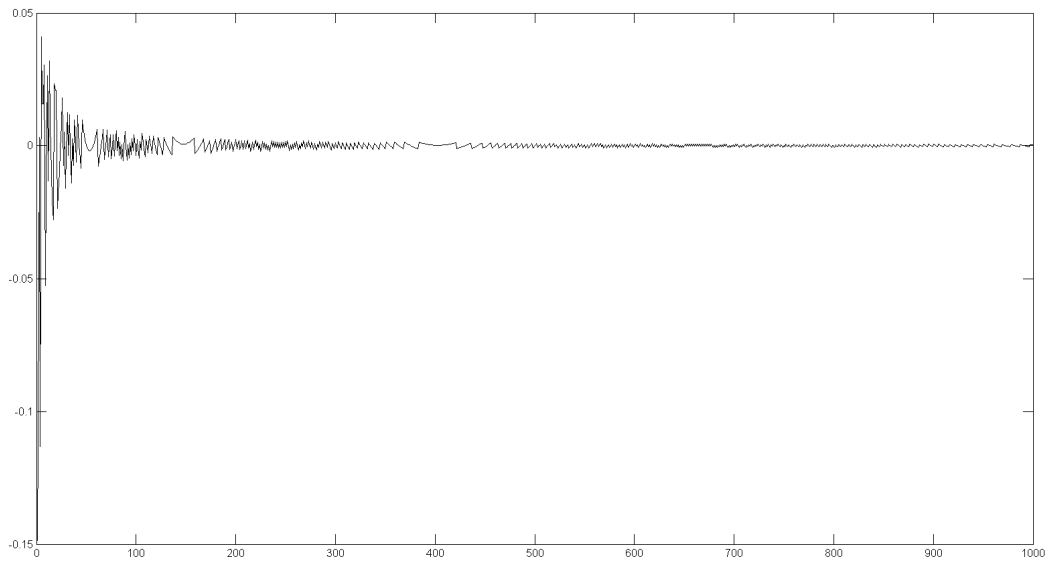
n	$\tilde{A}(n) - \gamma$	\approx	$\tilde{T}_A(n)$	$\tilde{B}(n) - \gamma$	\approx	$\tilde{T}_B(n)$	$\frac{v}{w}$
1	0.42278434	\approx	0.38537352	-0.07721566	\approx	-0.14872127	0.5000
2	0.22963715	\approx	0.21360196	-0.07721566	\approx	-0.10914091	1.0000
3	0.15750538	\approx	0.15436921	0.00611767	\approx	0.00321901	1.2500
4	0.11982331	\approx	0.11211877	-0.09388233	\approx	-0.11325811	1.6000
5	0.09667976	\approx	0.09716585	0.03945100	\approx	0.04110199	1.6667
6	0.08102487	\approx	0.08063871	0.01564148	\approx	0.01576509	1.8571
7	0.06973133	\approx	0.06951421	0.01564148	\approx	0.01584913	2.0000
8	0.06119994	\approx	0.06151555	0.02953037	\approx	0.03032361	2.1111
9	0.05452801	\approx	0.05274958	-0.04824741	\approx	-0.05268694	2.3000
10	0.04916750	\approx	0.04848925	-0.01188377	\approx	-0.01295341	2.3636
11	0.04476641	\approx	0.04502926	0.02599501	\approx	0.02650586	2.4167
12	0.04108836	\approx	0.04056090	-0.01246653	\approx	-0.01334823	2.5385
13	0.03796873	\approx	0.03833707	0.03148952	\approx	0.03196129	2.5714
14	0.03528933	\approx	0.03522353	0.00767999	\approx	0.00772028	2.6667
15	0.03296313	\approx	0.03263906	-0.00898667	\approx	-0.00950879	2.7500
16	0.03092461	\approx	0.03046081	-0.02001608	\approx	-0.02101048	2.8235
17	0.02912351	\approx	0.02860060	-0.02655203	\approx	-0.02784283	2.8889
18	0.02752066	\approx	0.02771720	0.02315557	\approx	0.02340315	2.8947
19	0.02608501	\approx	0.02624269	0.02052399	\approx	0.02073928	2.9500
20	0.02479172	\approx	0.02494669	0.02052399	\approx	0.02072315	3.0000



$\tilde{A}(n)$ = red, $\tilde{B}(n)$ = blue, $\tilde{T}_A(n)$ = green, $\tilde{T}_B(n)$ = black ($n = 1, \dots, 100$)

Note: $\tilde{T}_A(n)$ and $\tilde{T}_B(n)$ appears to be some kind of copy of $\tilde{A}(n)$ and $\tilde{B}(n)$

Plots of $\tilde{B}(n)$ (Note: $\tilde{B}(n)$ converges to γ)

Plots of $\tilde{T}_B(n)$ (Note: $\tilde{T}_B(n)$ converges to zero)

$\tilde{T}_A(n)$ and $\tilde{T}_B(n)$ can be used as correction terms for $\tilde{A}(n)$ and $\tilde{B}(n)$

n	$ \tilde{A}(n) - \gamma $	$ \tilde{A}(n) - \tilde{T}_A(n) - \gamma $	$ \tilde{B}(n) - \gamma $	$ \tilde{B}(n) - \tilde{T}_B(n) - \gamma $	$\frac{v}{w}$
1	4.22784335e-001	3.74108145e-002	7.72156649e-002	7.15056058e-002	0.5000
2	2.29637155e-001	1.60351899e-002	7.72156649e-002	3.19252493e-002	1.0000
3	1.57505380e-001	3.13616835e-003	6.11766843e-003	2.89865425e-003	1.2500
4	1.19823307e-001	7.70454125e-003	9.38823316e-002	1.93757745e-002	1.6000
5	9.66797560e-002	4.86097554e-004	3.94510018e-002	1.65098814e-003	1.6667
6	8.10248659e-002	3.86155069e-004	1.56414780e-002	1.23616682e-004	1.8571
7	6.97313289e-002	2.17123759e-004	1.56414780e-002	2.07650769e-004	2.0000
8	6.11999363e-002	3.15613351e-004	2.95303668e-002	7.93246836e-004	2.1111
9	5.45280117e-002	1.77843584e-003	4.82474109e-002	4.43952849e-003	2.3000
10	4.91674961e-002	6.78248563e-004	1.18837746e-002	1.06963652e-003	2.3636
11	4.47664072e-002	2.62850398e-004	2.59950133e-002	5.10848073e-004	2.4167
12	4.10883635e-002	5.27465407e-004	1.24665252e-002	8.81706728e-004	2.5385
13	3.79687328e-002	3.68339974e-004	3.14895188e-002	4.71770177e-004	2.5714
14	3.52893320e-002	6.58015589e-005	7.67999499e-003	4.02839237e-005	2.6667
15	3.29631272e-002	3.24069057e-004	8.98667167e-003	5.22120607e-004	2.7500
16	3.09246061e-002	4.63796961e-004	2.00160834e-002	9.94397926e-004	2.8235
17	2.91235137e-002	5.22909915e-004	2.65520311e-002	1.29080051e-003	2.8889
18	2.75206554e-002	1.96540316e-004	2.31555712e-002	2.47581522e-004	2.8947
19	2.60850131e-002	1.57672889e-004	2.05239922e-002	2.15285219e-004	2.9500
20	2.47917187e-002	1.54972676e-004	2.05239922e-002	1.99161598e-004	3.0000
21	2.36206021e-002	3.59031254e-004	2.27660510e-002	9.03769035e-004	3.0909
22	2.25551320e-002	2.72560480e-004	1.68371973e-002	6.14026037e-004	3.1304
23	2.15816303e-002	1.79570471e-004	9.59082048e-003	3.33448745e-004	3.1667
24	2.06886825e-002	8.31189405e-005	1.25748715e-003	9.79377315e-005	3.2000
25	1.98666880e-002	1.46137118e-005	7.97328208e-003	6.23042354e-005	3.2308
26	1.91075133e-002	1.12078093e-004	1.79447921e-002	1.22579716e-004	3.2593
27	1.84042223e-002	1.21653371e-004	7.18748308e-003	2.15982971e-004	3.3214
28	1.77508639e-002	6.18995297e-006	5.12778786e-003	2.62716515e-005	3.3448
29	1.71423027e-002	1.81319188e-004	1.55618673e-002	4.41582596e-004	3.4000
30	1.65740844e-002	5.82719273e-005	1.58337269e-003	7.32665906e-005	3.4194
31	1.60423260e-002	5.83937978e-005	1.25295305e-002	8.05291112e-005	3.4375
32	1.55436277e-002	6.74691464e-005	3.56895431e-003	1.02442321e-004	3.4848
33	1.50749994e-002	5.00733127e-005	1.15825608e-002	7.04101107e-005	3.5000
34	1.46338009e-002	4.88837369e-005	1.86281731e-003	6.47391048e-005	3.5429
35	1.42176926e-002	1.30940800e-004	1.37675792e-002	3.27244657e-004	3.5833
36	1.38245934e-002	1.24768478e-005	2.74893730e-003	2.73330197e-006	3.5946
37	1.34526463e-002	7.82536329e-005	7.20838845e-003	1.54049523e-004	3.6316
38	1.31001887e-002	3.50908745e-005	9.66070737e-003	5.17842662e-005	3.6410
39	1.27657279e-002	1.83980842e-005	1.32737403e-003	1.08831938e-005	3.6750
40	1.24479199e-002	6.28009318e-005	5.98969914e-003	1.18942284e-004	3.7073

A Further Generalization

For all $k \in \mathbb{Z}$, let the set \mathbb{X}_k be defined as

$$\mathbb{X}_k := \{n \in \mathbb{Z} \mid n > -k\}$$

Consider the following family of functions

$$\Gamma := \{\hat{A}_k, \hat{B}_k : k \in \mathbb{Z}\}$$

mapping from \mathbb{X}_k to \mathbb{R} and \mathbb{Q} respectively

$$\hat{A}_k : \mathbb{X}_k \rightarrow \mathbb{R}$$

$$\hat{B}_k : \mathbb{X}_k \rightarrow \mathbb{Q}$$

where \hat{A}_k and \hat{B}_k are defined as

$$\hat{A}_k(n) := \begin{cases} -\sum_{i=1}^{-n} \frac{1}{i} - \log(n+k) & , \text{if } n < 0 \\ -\log(n+k) & , \text{if } n = 0 \\ \sum_{i=1}^n \frac{1}{i} - \log(n+k) & , \text{if } n > 0 \end{cases}$$

$$\hat{B}_k(n) := \begin{cases} -\sum_{i=1}^{-n} \frac{1}{i} - \frac{\lfloor (n+k) \log(n+k) \rfloor}{n+k} & , \text{if } n < 0 \\ -\frac{\lfloor (n+k) \log(n+k) \rfloor}{n+k} & , \text{if } n = 0 \\ \sum_{i=1}^n \frac{1}{i} - \frac{\lfloor (n+k) \log(n+k) \rfloor}{n+k} & , \text{if } n > 0 \end{cases}$$

Then

$$\tilde{A}(n) = \sum_{i=1}^n \frac{1}{i} - \log(n)$$

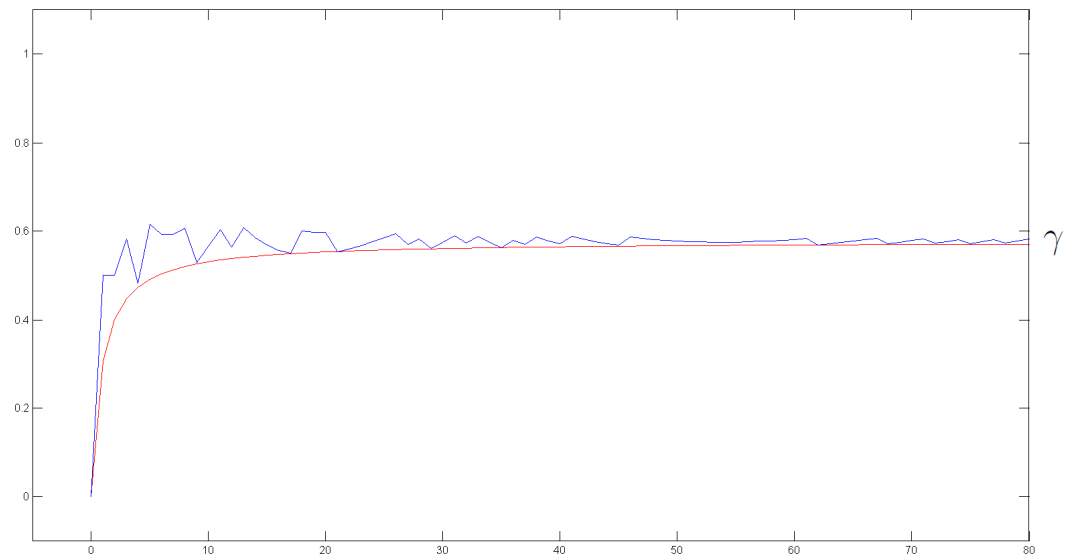
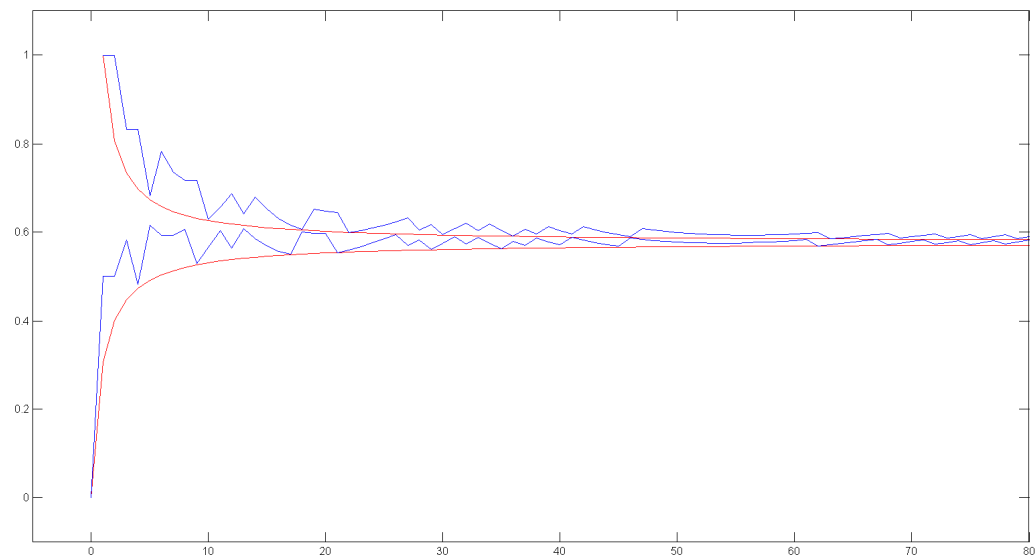
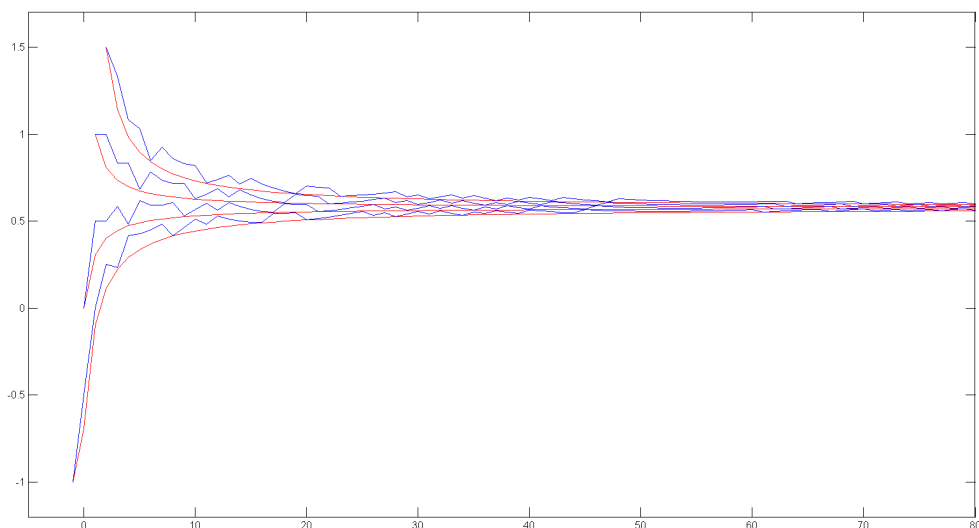
and

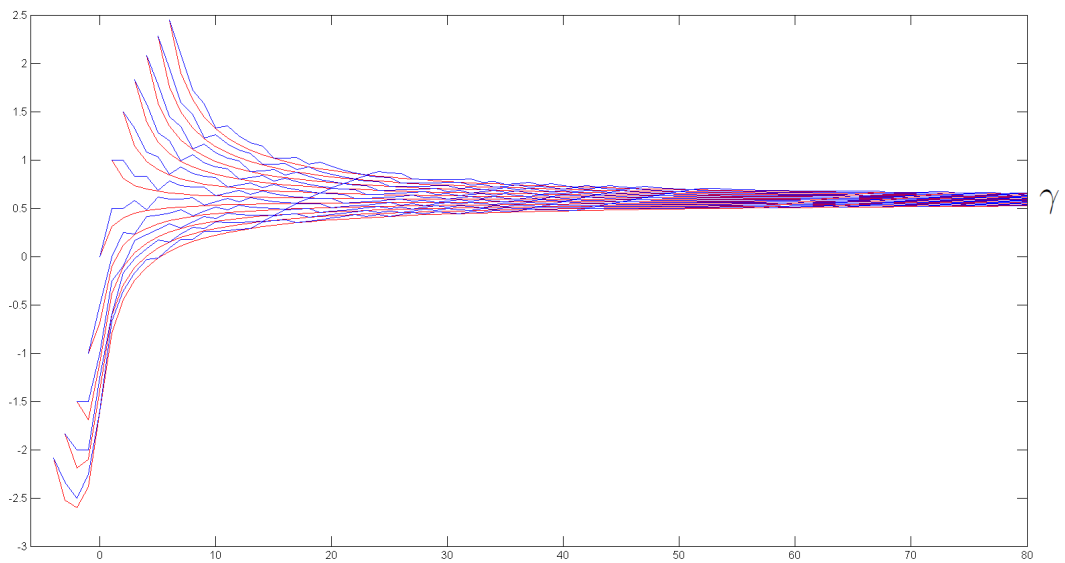
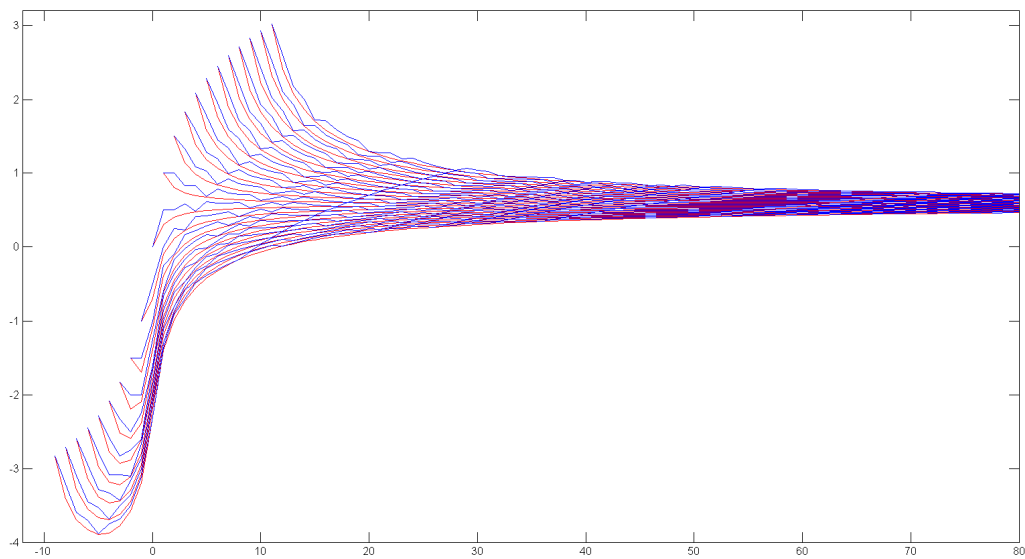
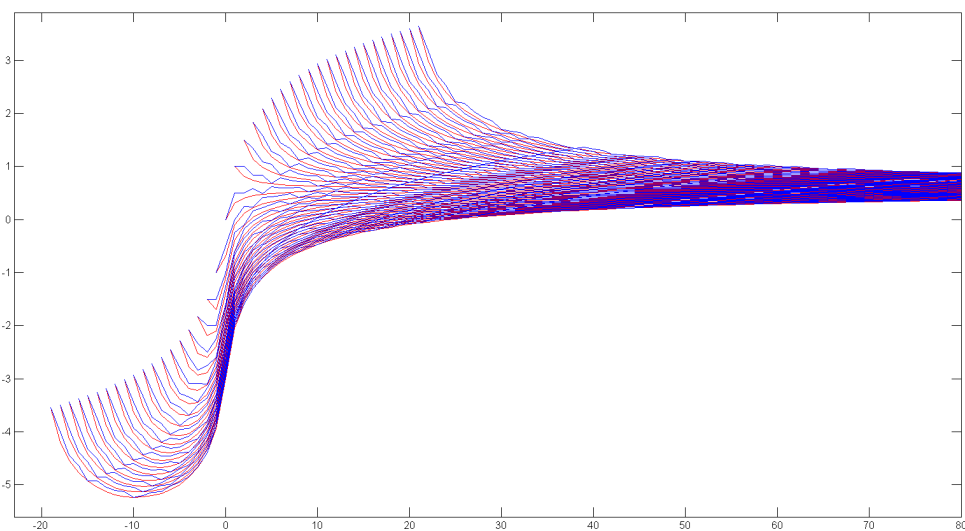
$$\tilde{B}(n) = \sum_{i=1}^n \frac{1}{i} - \frac{v}{w}$$

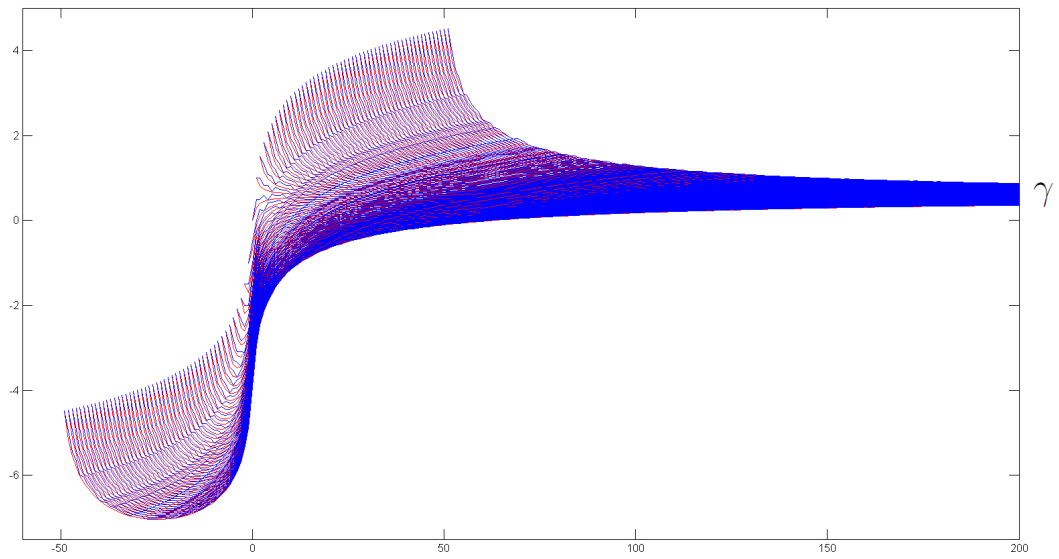
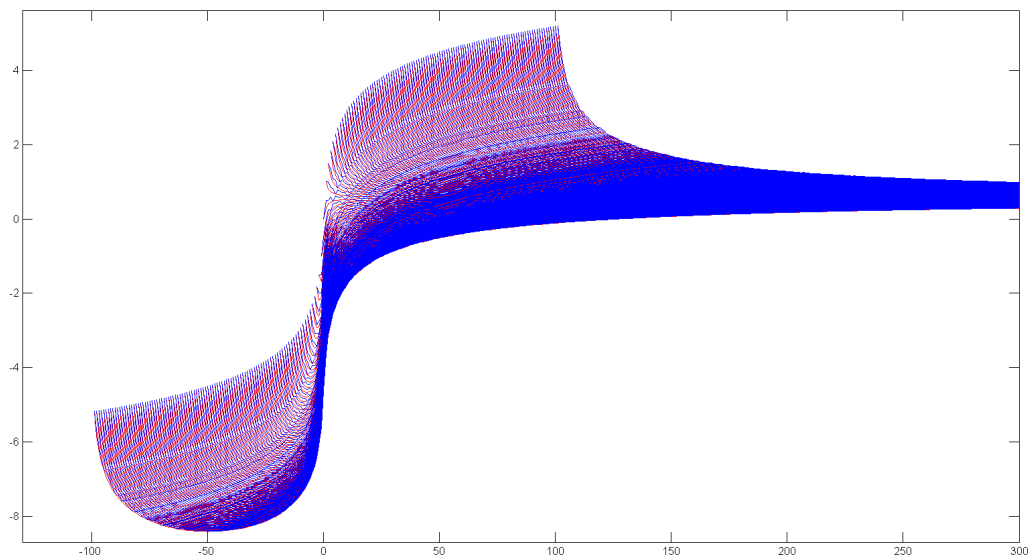
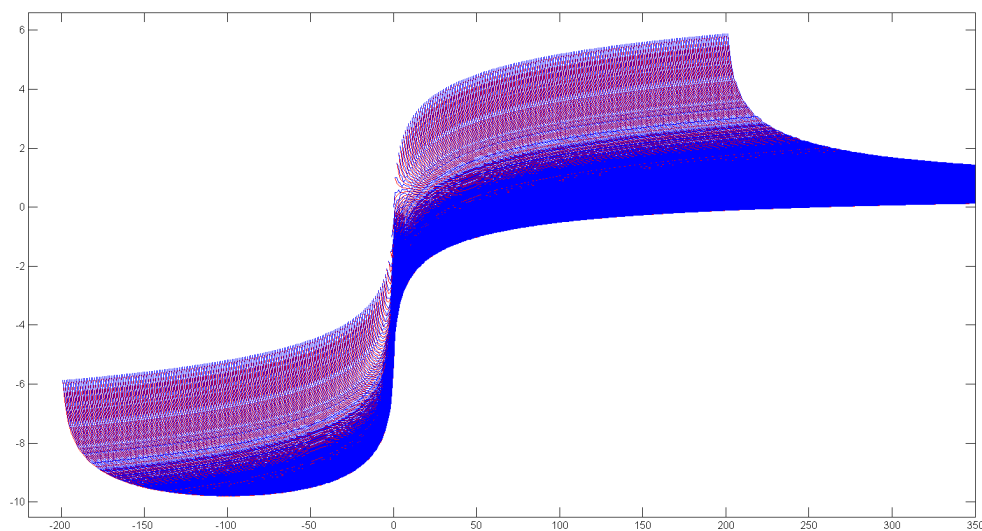
are members of Γ , because

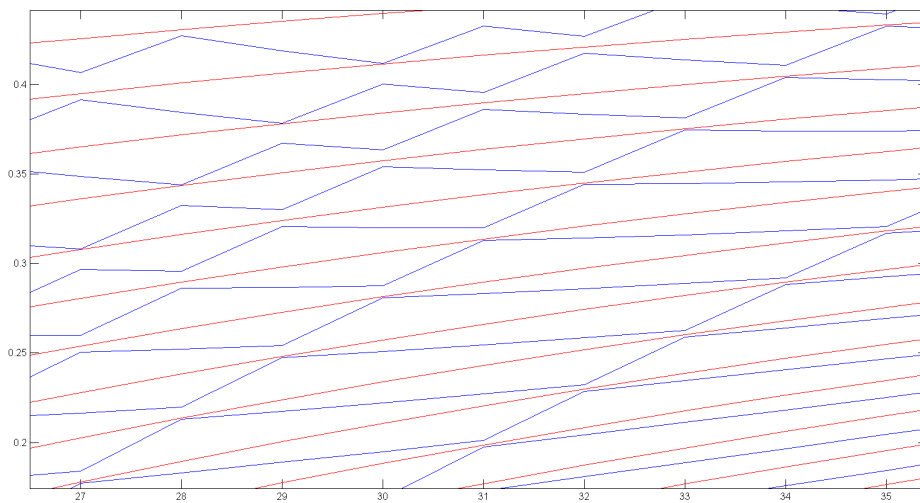
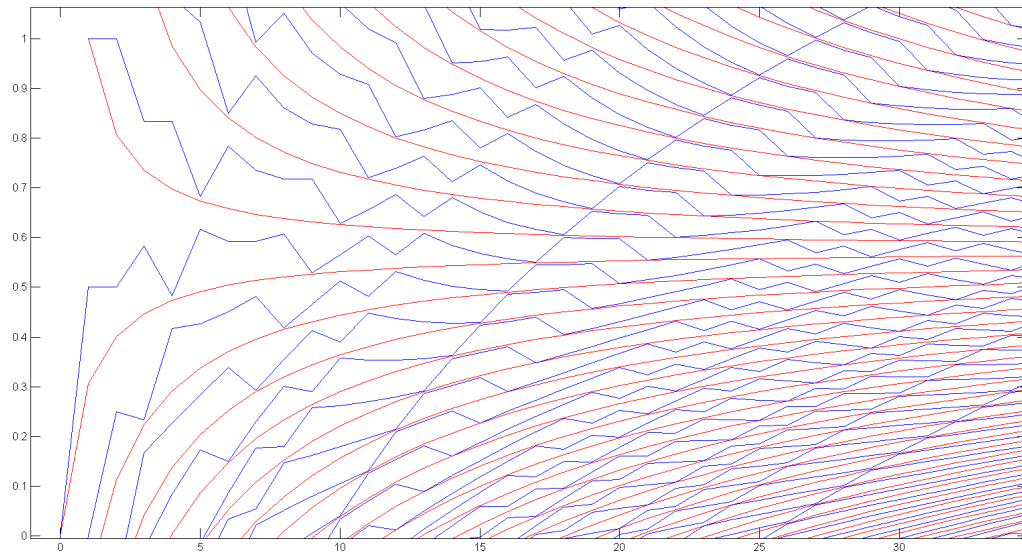
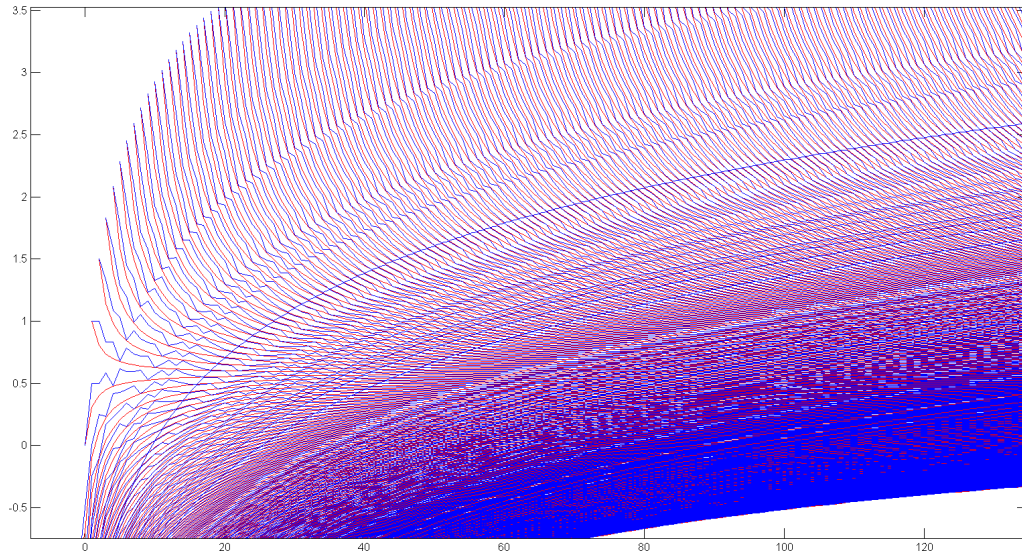
$$\hat{A}_0(n) = \tilde{A}(n) \quad (\mathbb{X}_0 = 1, 2, 3, \dots)$$

$$\hat{B}_1(n) = \tilde{B}(n) \quad (\mathbb{X}_1 = 0, 1, 2, 3, \dots)$$

Plot of $\hat{A}_1(n)$ (red) and $\hat{B}_1(n)$ (blue)Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{0, 1\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-1, 0, 1, 2\}$ 

Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-5, -4, \dots, 4, 5\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-10, \dots, 10\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-20, \dots, 20\}$ 

Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-50, \dots, 50\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-100, \dots, 100\}$ Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-200, \dots, 200\}$ 



Pattern: $\hat{A}_k(n) < \hat{B}_k(n) < \hat{A}_{k-1}(n) \quad \forall k \in \mathbb{Z}, n \in \mathbb{X}_{k-1}$
 (In words: The red and blue lines never touch or cross each other)