Exact solution of ODEs
Including Bessel's

Just as I have shown earlier that $y = e^{kx}$ is not just the solution for the constant coefficients, linear homogeneous ordinary differential equation (LHODE), but for any LHODE of the form:
$y'' + Py' + [-k(P+k)]y = 0$; the solution to the general elementary HLODE is not limited to that LHODE.

My writing style is to leave math in there I feel is necessary to at least fairly easily follow the flow of derivation, rather than skip over it all and put some prose in there. I expect anyone who needs less, or simply wishes to skip over derivation portions is free to do so; and that is expected. But if nagging questions arise, the reader may return to dispel those. It is there for clarity, not either to impress nor to repel the reader. I feel that prose often makes it more difficult to reproduce derivation results, which I feel is as important as the results themselves. I write as if writing to the Krell of 'The Forbidden Planet', because I am. I am writing to higher dimensional beings and lower dimensional beings reaching and struggling for higher dimensionality; to those enraptured with a fascination and infatuation for elevation.

Again, as shown earlier:
$s = aP + b$:

$$y = e^{a\int Pdx+bx} \Rightarrow y'' + Py' + Qy = 0 \;;\; Q = -aP' - a(a+1)P^2 - b(2a+1)P - b^2$$

But, also:

$$y' = (aP+b)y$$
$$y'' = \left[(aP+b)^2 + aP'\right]y$$

So, for any function R (even non-continuous, or even non-integrable function R):

$$y'' + Ry' = \left[(aP+b)^2 + aP' + R(aP+b)\right]y$$
$$\Rightarrow y'' + Ry' + \left[-(aP+b)^2 - aP' - R(aP+b)\right]y = 0$$

When an ODE does not have these expressed forms, but is similar; the following technique may yield exact solution.

Recall that any 2nd order ODE has 2 linearly independent solutions, and that given one solution, the 2nd may be found by the reduction of order technique.

So, if a general elementary LODE is similar, 2 linearly independent solutions y_1 , y_2 are known.

Because the solutions are linearly independent, consider them as coordinate axes, and transform them to another coordinate system.

Let:

$$\left|\left.\boxed{\begin{array}{l} Y_\mu = uy_1 - vy_2 \\ \hline Y_\nu = vy_1 + uy_2 \end{array}}\right\} \Leftrightarrow \left\{ \boxed{\begin{array}{l} y_1 = \left(\frac{u}{u^2+v^2}\right)Y_\mu + \left(\frac{v}{u^2+v^2}\right)Y_\nu \\ \hline y_2 = -\left(\frac{v}{u^2+v^2}\right)Y_\mu + \left(\frac{u}{u^2+v^2}\right)Y_\nu \end{array}} \right.\right|$$

for u, v constants, or carefully chosen functions: Y_μ & Y_ν are linearly independent, as well.

When y_1 , y_2 satisfy:

$$\left\{ \boxed{\begin{array}{l} y_1' = r_1 y_1 + s_1 y_2 \\ \hline y_2' = r_2 y_1 + s_2 y_2 \end{array}} \right. \Bigg|$$

$$\Rightarrow \left\{ \boxed{\begin{array}{l} y_1'' = (r_1' + r_1^2 + r_2 s_1)y_1 + (s_1' + s_1 s_2 + r_1 s_1)y_2 \\ \hline y_2'' = (r_2' + r_1 r_2 + r_2 s_2)y_1 + (s_2' + s_2^2 + r_2 s_1)y_2 \end{array}} \right. \Bigg|$$

Examples of such functions:

(1):

Let:

$$\left.\boxed{\begin{array}{l} sing(z,w) \equiv \dfrac{e^{z\sqrt{-w}} - e^{-z\sqrt{-w}}}{2\sqrt{-w}} \\ \hline cong(z,w) \equiv \dfrac{e^{z\sqrt{-w}} + e^{-z\sqrt{-w}}}{2} \end{array}}\right\}$$

Note that:

$sing(z,1) \equiv \sin z$, $cong(z,1) \equiv \cos z$

$sing(z,-1) \equiv \sinh z$, $cong(z,-1) \equiv \cosh z$

Note, also, that:

$$sing(z,w) = \frac{1}{\sqrt{w}}\sin(z\sqrt{w}) \quad , \quad cong(z,w) = \cos(z\sqrt{w})$$

But these expressions don't express the fact that

$sing(z,w)$ and $cong(z,w)$ are real valued functions,

$\forall z, w \in \mathcal{R}$.

They can be used to more compactly write the possible solutions of $y'' + \lambda y = 0$

(whether λ is positive or negative):

$y = C_1 sing(z,\lambda) + C_2 cong(z,\lambda)$.

Real-valued functions;

with real period, whenever: $\lambda > 0$,

while, no real period, whenever: $\lambda < 0$.

(2):

Let:

$$\left\{\begin{array}{|l|}\hline m_1(u,v;z,w) \equiv ue^z + ve^w \\ \hline m_2(r,s;z,w) \equiv re^z + se^w \\ \hline\end{array}\right\}\left\{\begin{array}{|l|}\hline e^z = \dfrac{1}{(us-vr)}(sm_1 - vm_2) \\ \hline e^w = -\dfrac{1}{(us-vr)}(rm_1 - sm_2) \\ \hline\end{array}\right\}$$

$\therefore$

$$\begin{aligned} m_1' &= (u' + uz')e^z + (v' + vw')e^w \\ m_2' &= (r' + rz')e^z - (s' + sw')e^w \end{aligned}$$

so:

$$\begin{aligned} m_1' &= (u' + uz')e^z + (v' + vw')e^w \\ &= \left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]m_1 + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]m_2 \\ m_2' &= (r' + rz')e^z - (s' + sw')e^w \\ &= \left[\left(\frac{r'+rz'}{us-vr}\right)s + \left(\frac{s'+sw'}{us-vr}\right)r\right]m_1 + \left[-\left(\frac{r'+rz'}{us-vr}\right)v - \left(\frac{s'+sw'}{us-vr}\right)s\right]m_2 \end{aligned}$$

and:

$$\begin{aligned} m_1'' &= \left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]m_1' + \left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]' m_1 + \\ &\quad + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]m_2' + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]m_2 \\ &= \left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]\left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]m_1 + \\ &\quad + \left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]\left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]m_2 \\ &\quad + \left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]' m_1 + \\ &\quad + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]\left[\left(\frac{r'+rz'}{us-vr}\right)s + \left(\frac{s'+sw'}{us-vr}\right)r\right]m_1 + \\ &\quad + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]\left[-\left(\frac{r'+rz'}{us-vr}\right)v - \left(\frac{s'+sw'}{us-vr}\right)s\right]m_2 + \\ &\quad + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]m_2 \\ &= \left\{\left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]\left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right] + \right. \\ &\qquad + \left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right] + \\ &\qquad \left. + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]\left[\left(\frac{r'+rz'}{us-vr}\right)s + \left(\frac{s'+sw'}{us-vr}\right)r\right]\right\}m_1 + \\ &\quad + \left\{\left[\left(\frac{u'+uz'}{us-vr}\right)s - \left(\frac{v'+vw'}{us-vr}\right)r\right]\left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right] + \right. \\ &\qquad + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]\left[-\left(\frac{r'+rz'}{us-vr}\right)v - \left(\frac{s'+sw'}{us-vr}\right)s\right] + \\ &\qquad \left. + \left[-\left(\frac{u'+uz'}{us-vr}\right)v + \left(\frac{v'+vw'}{us-vr}\right)s\right]\right\}m_2 \\ m_2'' &= \left[\left(\frac{r'+rz'}{us-vr}\right)s + \left(\frac{s'+sw'}{us-vr}\right)r\right]m_1' + \left[\left(\frac{r'+rz'}{us-vr}\right)s + \left(\frac{s'+sw'}{us-vr}\right)r\right]' m_1 + \\ &\quad + \left[-\left(\frac{r'+rz'}{us-vr}\right)v - \left(\frac{s'+sw'}{us-vr}\right)s\right]m_2' + \left[-\left(\frac{r'+rz'}{us-vr}\right)v - \left(\frac{s'+sw'}{us-vr}\right)s\right]m_2 \end{aligned}$$

etc.

Examples:

Define:

$$\left\{\begin{array}{|l|}\hline m_{10}(u,v;\psi) \equiv m_1(u,v;\psi,-\psi) \\ \hline m_{20}(r,s;\psi) \equiv m_2(r,s;\psi,-\psi) \\ \hline\end{array}\right\}$$

yields:

$$\left\{\begin{array}{|l|}\hline u = \frac{1}{2i} \quad , \quad v = -u \quad , \quad r = \frac{1}{2} \quad , \quad s = r \\ \hline z = i\varphi \quad , \quad w = -z \\ \hline\end{array}\right\} \Leftrightarrow \left\{\begin{array}{|l|}\hline m_{10}(\frac{1}{2i},-\frac{1}{2i};i\varphi) \equiv \sin\varphi \\ \hline m_{20}(\frac{1}{2},\frac{1}{2};i\varphi) \equiv \cos\varphi \\ \hline\end{array}\right\}$$

$$\left\{\begin{array}{|l|}\hline u = \frac{1}{2} \quad , \quad v = -u \quad , \quad r = \frac{1}{2} \quad , \quad s = r \\ \hline z = \varphi \quad , \quad w = -z \\ \hline\end{array}\right\} \Leftrightarrow \left\{\begin{array}{|l|}\hline m_{10}(\frac{1}{2},-\frac{1}{2};\varphi) \equiv \sinh\varphi \\ \hline m_{20}(\frac{1}{2},\frac{1}{2};\varphi) \equiv \cosh\varphi \\ \hline\end{array}\right\}$$

$$\left\{\begin{array}{|l|}\hline u = 1 \quad , \quad v = 0 \quad , \quad r = v \quad , \quad s = u \\ \hline z = \varphi \quad , \quad w = -z \\ \hline\end{array}\right\} \Leftrightarrow \left\{\begin{array}{|l|}\hline m_{10}(1,0;\varphi) \equiv e^{\varphi} \\ \hline m_{20}(0,1;\varphi) \equiv e^{-\varphi} \\ \hline\end{array}\right\}$$

So, returning to the generality:

$$\left|\begin{array}{|l|}\hline Y_\mu = uy_1 - vy_2 \\ \hline Y_\nu = vy_1 + uy_2 \\ \hline\end{array}\right\} \Leftrightarrow \left\{\begin{array}{|l|}\hline y_1 = \left(\frac{u}{u^2+v^2}\right)Y_\mu + \left(\frac{v}{u^2+v^2}\right)Y_\nu \\ \hline y_2 = -\left(\frac{v}{u^2+v^2}\right)Y_\mu + \left(\frac{u}{u^2+v^2}\right)Y_\nu \\ \hline\end{array}\right|$$

$$\left\{\begin{array}{|l|}\hline y_1' = r_1y_1 + s_1y_2 \\ \hline y_2' = r_2y_1 + s_2y_2 \\ \hline\end{array}\right|$$

$$\Rightarrow \left\{\begin{array}{|l|}\hline y_1'' = (r_1' + r_1^2 + r_2s_1)y_1 + (s_1' + s_1s_2 + r_1s_1)y_2 \\ \hline y_2'' = (r_2' + r_1r_2 + r_2s_2)y_1 + (s_2' + s_2^2 + r_2s_1)y_2 \\ \hline\end{array}\right|$$

$$\left\{\begin{array}{|l|}\hline Y_\mu'' + PY_\mu' = (uy_1 - vy_2)'' + P(uy_1 - vy_2)' \\ \hline Y_\nu'' + PY_\nu' = (vy_1 + uy_2)'' + P(vy_1 + uy_2)' \\ \hline\end{array}\right|$$

$$\Rightarrow \left\{\begin{array}{|l|}\hline Y_\mu'' + PY_\mu' = (u'y_1 + uy_1' - v'y_2 - vy_2')' + P(u'y_1 + uy_1' - v'y_2 - vy_2') \\ \hline Y_\nu'' + PY_\nu' = (v'y_1 + vy_1' + u'y_2 + uy_2')' + P(v'y_1 + vy_1' + u'y_2 + uy_2') \\ \hline\end{array}\right|$$

$$\Rightarrow \left\{\begin{array}{|c|}\hline Y_\mu'' + PY_\mu' = u''y_1 + 2u'y_1' + uy_1'' - v''y_2 - 2v'y_2' - vy_2'' + \\ \hline + P(u'y_1 + uy_1' - v'y_2 - vy_2') \\ \hline Y_\nu'' + PY_\nu' = v''y_1 + 2v'y_1' + vy_1'' + u''y_2 + 2u'y_2' + uy_2'' + \\ \hline + P(v'y_1 + vy_1' + u'y_2 + uy_2') \\ \hline\end{array}\right|$$

$$\Rightarrow \left\{ \begin{aligned} Y''_\mu + PY'_\mu &= [(u'' + Pu')y_1 + (2u' + Pu)y'_1 + uy''_1] + \\ &\quad - [(v'' + Pv')y_2 + (2v' + Pv)y'_2 + vy''_2] \\ Y''_\nu + PY'_\nu &= [(v'' + Pv')y_1 + (2v' + Pv)y'_1 + vy''_1] + \\ &\quad + [(u'' + Pu')y_2 + (2u' + Pu)y'_2 + uy''_2] \end{aligned} \right.$$

$$\Rightarrow \left\{ \begin{aligned} Y''_\mu + PY'_\mu &= \Big[(u'' + Pu')y_1 + (2u' + Pu)(r_1y_1 + s_1y_2) + \\ &\quad + u[(r'_1 + r_1^2 + r_2s_1)y_1 + (s'_1 + s_1s_2 + r_1s_1)y_2]\Big] + \\ &\quad - \Big[(v'' + Pv')y_2 + (2v' + Pv)(r_2y_1 + s_2y_2) + \\ &\quad + v[(r'_2 + r_1r_2 + r_2s_2)y_1 + (s'_2 + s_2^2 + r_2s_1)y_2]\Big] \\ Y''_\nu + PY'_\nu &= \Big[(v'' + Pv')y_1 + (2v' + Pv)(r_1y_1 + s_1y_2) + \\ &\quad + v[(r'_1 + r_1^2 + r_2s_1)y_1 + (s'_1 + s_1s_2 + r_1s_1)y_2]\Big] + \\ &\quad + \Big[(u'' + Pu')y_2 + (2u' + Pu)(r_2y_1 + s_2y_2) + \\ &\quad + u[(r'_2 + r_1r_2 + r_2s_2)y_1 + (s'_2 + s_2^2 + r_2s_1)y_2]\Big] \end{aligned} \right.$$

$$\Rightarrow \left\{ \begin{aligned} Y''_\mu + PY'_\mu &= \Big[(u'' + Pu') + (2u' + Pu)r_1 + u(r'_1 + r_1^2 + r_2s_1) + \\ &\quad - (2v' + Pv)r_2 - v(r'_2 + r_1r_2 + r_2s_2)\Big]y_1 + \\ &\quad - \Big[(v'' + Pv') + (2v' + Pv)s_2 + v(s'_2 + s_2^2 + r_2s_1) + \\ &\quad - (2u' + Pu)s_1 - u(s'_1 + s_1s_2 + r_1s_1)\Big]y_2 \\ Y''_\nu + PY'_\nu &= \Big[(v'' + Pv') + (2v' + Pv)r_1 + v(r'_1 + r_1^2 + r_2s_1) + \\ &\quad + (2u' + Pu)r_2 + u(r'_2 + r_1r_2 + r_2s_2)\Big]y_1 + \\ &\quad + \Big[(u'' + Pu') + (2u' + Pu)s_2 + u(s'_2 + s_2^2 + r_2s_1) + \\ &\quad + (2v' + Pv)s_1 + v(s'_1 + s_1s_2 + r_1s_1)\Big]y_2 \end{aligned} \right.$$

So, let:

$$\begin{aligned} A_\mu &\equiv (u'' + Pu') + (2u' + Pu)r_1 + u(r'_1 + r_1^2 + r_2s_1) - (2v' + Pv)r_2 - v(r'_2 + r_1r_2 + r_2s_2) \\ B_\mu &\equiv (v'' + Pv') + (2v' + Pv)s_2 + v(s'_2 + s_2^2 + r_2s_1) - (2u' + Pu)s_1 - u(s'_1 + s_1s_2 + r_1s_1) \\ A_\nu &\equiv (v'' + Pv') + (2v' + Pv)r_1 + v(r'_1 + r_1^2 + r_2s_1) + (2u' + Pu)r_2 + u(r'_2 + r_1r_2 + r_2s_2) \\ B_\nu &\equiv (u'' + Pu') + (2u' + Pu)s_2 + u(s'_2 + s_2^2 + r_2s_1) + (2v' + Pv)s_1 + v(s'_1 + s_1s_2 + r_1s_1) \end{aligned}$$

$$\Rightarrow \left\{ \begin{aligned} Y''_\mu + PY'_\mu &= A_\mu y_1 - B_\mu y_2 \\ Y''_\nu + PY'_\nu &= A_\nu y_1 + B_\nu y_2 \end{aligned} \right.$$

$$\Rightarrow \left\{\begin{array}{|l|}\hline Y''_\mu + PY'_\mu = A_\mu\left[\left(\frac{u}{u^2+v^2}\right)Y_\mu + \left(\frac{v}{u^2+v^2}\right)Y_\nu\right] + \\ \hline \qquad - B_\mu\left[-\left(\frac{v}{u^2+v^2}\right)Y_\mu + \left(\frac{u}{u^2+v^2}\right)Y_\nu\right] \\ \hline Y''_\nu + PY'_\nu = A_\nu\left[\left(\frac{u}{u^2+v^2}\right)Y_\mu + \left(\frac{v}{u^2+v^2}\right)Y_\nu\right] + \\ \hline \qquad + B_\nu\left[-\left(\frac{v}{u^2+v^2}\right)Y_\mu + \left(\frac{u}{u^2+v^2}\right)Y_\nu\right] \\ \hline \end{array}\right|$$

$$\Rightarrow \left\{\begin{array}{|l|}\hline Y''_\mu + PY'_\mu = \left[A_\mu\left(\frac{u}{u^2+v^2}\right) + B_\mu\left(\frac{v}{u^2+v^2}\right)\right]Y_\mu + \\ \hline \qquad + \left[A_\mu\left(\frac{v}{u^2+v^2}\right) - B_\mu\left(\frac{u}{u^2+v^2}\right)\right]Y_\nu \\ \hline Y''_\nu + PY'_\nu = \left[A_\nu\left(\frac{v}{u^2+v^2}\right) - B_\nu\left(\frac{u}{u^2+v^2}\right)\right]Y_\nu + \\ \hline \qquad + \left[A_\nu\left(\frac{u}{u^2+v^2}\right) + B_\nu\left(\frac{v}{u^2+v^2}\right)\right]Y_\mu \\ \hline \end{array}\right|$$

If:

$$A_\mu\left(\frac{v}{u^2+v^2}\right) - B_\mu\left(\frac{u}{u^2+v^2}\right) = 0 = A_\nu\left(\frac{u}{u^2+v^2}\right) - B_\nu\left(\frac{v}{u^2+v^2}\right)$$

$$\Rightarrow B_\mu = A_\mu\frac{v}{u} \quad \& \quad B_\nu = A_\nu\frac{u}{v}$$

$$\Rightarrow \left\{\begin{array}{|l|}\hline Y''_\mu + PY'_\mu - A_\mu\frac{1}{u}Y_\mu = 0 \\ \hline Y''_\nu + PY'_\nu - A_\nu\frac{1}{v}Y_\nu = 0 \\ \hline \end{array}\right|$$

Now, as a remembering:

$$y = \left\{\begin{array}{|l|}\hline x^{-\frac{1}{2}}m_{10}(1,0;ix) \equiv x^{-\frac{1}{2}}e^{ix} \\ \hline x^{-\frac{1}{2}}m_{20}(0,1;ix) \equiv x^{-\frac{1}{2}}e^{-ix} \\ \hline \end{array}\right\} \Rightarrow y'' + \frac{1}{x}y' + \left[-\frac{(\frac{1}{2})^2}{x^2} + 1\right]y = 0$$

also:

$$y = \left\{\begin{array}{|l|}\hline x^{-\frac{1}{2}}m_{10}(\frac{1}{2i},-\frac{1}{2i};ix) \equiv x^{-\frac{1}{2}}\sin x \\ \hline x^{-\frac{1}{2}}m_{20}(\frac{1}{2},\frac{1}{2};ix) \equiv x^{-\frac{1}{2}}\cos x \\ \hline \end{array}\right\} \Rightarrow y'' + \frac{1}{x}y' + \left[-\frac{(\frac{1}{2})^2}{x^2} + 1\right]y = 0$$

Note correspondence to the Bessel functions:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\,x^{-\frac{1}{2}}\sin x = \sqrt{\frac{2}{\pi}}\,x^{-\frac{1}{2}}m_{10}(\tfrac{1}{2i},-\tfrac{1}{2i};ix)$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\,x^{-\frac{1}{2}}\cos x = \sqrt{\frac{2}{\pi}}\,x^{-\frac{1}{2}}m_{20}(\tfrac{1}{2},-\tfrac{1}{2};ix)$$

Now, consider transformations:

$$\left\{\begin{array}{|l|}\hline Y_\mu = \frac{1}{x}J_{\frac{1}{2}} - J_{-\frac{1}{2}} \\ \hline Y_\nu = J_{\frac{1}{2}} + \frac{1}{x}J_{-\frac{1}{2}} \\ \hline \end{array}\right| \Leftrightarrow \left\{\begin{array}{|l|l|l|}\hline u = x^{-1} & u' = -x^{-2} & u'' = 2x^{-3} \\ \hline v = 1 & v' = 0 & v'' = 0 \\ \hline \end{array}\right|$$

$$\left\{\begin{array}{|l|}\hline J'_{\frac{1}{2}} = -\frac{1}{2}\frac{1}{x}J_{\frac{1}{2}} + J_{-\frac{1}{2}} \\ \hline J'_{-\frac{1}{2}} = -J_{\frac{1}{2}} - \frac{1}{2}\frac{1}{x}J_{-\frac{1}{2}} \\ \hline\end{array}\right| \Leftrightarrow \left\{\begin{array}{|l|l|l|l|}\hline r_1 = -\frac{1}{2}x^{-1} & s_1 = 1 & r'_1 = \frac{1}{2}x^{-2} & r''_1 = -x^{-3} \\ \hline r_2 = -1 & s_2 = -\frac{1}{2}x^{-1} & s'_2 = \frac{1}{2}x^{-2} & s''_2 = -x^{-3} \\ \hline\end{array}\right|$$

So, for: $P = \frac{1}{x}$:

$$\left\{\begin{array}{|l|l|}\hline A_\mu \equiv \left(\frac{9}{4}x^{-2} - 1\right)x^{-1} & B_\mu \equiv \frac{9}{4}x^{-2} - 1 \\ \hline A_\nu \equiv \frac{9}{4}x^{-2} - 1 & B_\nu \equiv \left(\frac{9}{4}x^{-2} - 1\right)x^{-1} \\ \hline\end{array}\right|$$

$$\Rightarrow B_\mu = A_\mu x \quad \& \quad B_\nu = A_\nu \frac{1}{x}$$

$$\Rightarrow \left\{\begin{array}{|l|}\hline Y''_\mu + PY'_\mu + \left[-\frac{(\frac{3}{2})^2}{x^2} + 1\right]Y_\mu = 0 \\ \hline Y''_\nu + PY'_\nu + \left[-\frac{(\frac{3}{2})^2}{x^2} + 1\right]Y_\nu = 0 \\ \hline\end{array}\right|$$

Thus: $\mu = \frac{3}{2}$, $\nu = -\frac{3}{2}$ $i.e.$:

$$\left\{\begin{array}{|l|}\hline J_{\frac{3}{2}} = \frac{1}{x}J_{\frac{1}{2}} - J_{-\frac{1}{2}} \\ \hline J_{-\frac{3}{2}} = J_{\frac{1}{2}} + \frac{1}{x}J_{-\frac{1}{2}} \\ \hline\end{array}\right\}$$

So, so far: $J_{\pm\frac{1}{2}}$, $J_{\pm\frac{3}{2}}$ have been determined without resort to infinite series methods.

Now, generalizing, for:

$$\left|\begin{array}{|l|}\hline Y_\mu = u_1y_1 + v_1y_2 \\ \hline Y_\nu = u_2y_1 + v_2y_2 \\ \hline\end{array}\right\} \Leftrightarrow \left\{\begin{array}{|l|}\hline y_1 = \left(\frac{-v_2}{v_1u_2 - u_1v_2}\right)Y_\mu + \left(\frac{v_1}{v_1u_2 - u_1v_2}\right)Y_\nu \\ \hline y_2 = \left(\frac{u_2}{v_1u_2 - u_1v_2}\right)Y_\mu + \left(\frac{-u_1}{v_1u_2 - u_1v_2}\right)Y_\nu \\ \hline\end{array}\right|$$

for u_1, u_2, v_1, v_2 constants, or carefully chosen functions: Y_μ & Y_ν are linearly independent, as well.

When y_1 , y_2 satisfy:

$$\left\{\begin{array}{|l|}\hline y'_1 = r_1y_1 + s_1y_2 \\ \hline y'_2 = r_2y_1 + s_2y_2 \\ \hline\end{array}\right|$$

$$\Rightarrow \left\{\begin{array}{|l|}\hline y''_1 = (r'_1 + r_1^2 + r_2s_1)y_1 + (s'_1 + s_1s_2 + r_1s_1)y_2 \\ \hline y''_2 = (r'_2 + r_1r_2 + r_2s_2)y_1 + (s'_2 + s_2^2 + r_2s_1)y_2 \\ \hline\end{array}\right|$$

$$\left\{\begin{array}{|l|}\hline Y''_\mu + PY'_\mu = (u_1y_1 + v_1y_2)'' + P(u_1y_1 + v_1y_2)' \\ \hline Y''_\nu + PY'_\nu = (u_2y_1 + v_2y_2)'' + P(u_2y_1 + v_2y_2)' \\ \hline\end{array}\right|$$

$$\Rightarrow \begin{cases} Y''_\mu + PY'_\mu = (u'_1 y_1 + u_1 y'_1 + v'_1 y_2 + v_1 y'_2)' + P(u'_1 y_1 + u_1 y'_1 + v'_1 y_2 + v_1 y'_2) \\ Y''_\nu + PY'_\nu = (u'_2 y_1 + u_2 y'_1 + v'_2 y_2 + v_2 y'_2)' + P(u'_2 y_1 + u_2 y'_1 + v'_2 y_2 + v_2 y'_2) \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + PY'_\mu = u''_1 y_1 + 2u'_1 y'_1 + u_1 y''_1 - v''_1 y_2 - 2v'_1 y'_2 - v_1 y''_2 + \\ \qquad + P(u'_1 y_1 + u_1 y'_1 + v'_1 y_2 + v_1 y'_2) \\ Y''_\nu + PY'_\nu = u''_2 y_1 + 2u'_2 y'_1 + u_2 y''_1 + v''_2 y_2 + 2v'_2 y'_2 + v_2 y''_2 + \\ \qquad + P(u'_2 y_1 + u_2 y'_1 + v'_2 y_2 + v_2 y'_2) \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + PY'_\mu = [(u''_1 + Pu'_1)y_1 + (2u'_1 + Pu_1)y'_1 + u_1 y''_1] + \\ \qquad + [(v''_1 + Pv'_1)y_2 + (2v'_1 + Pv_1)y'_2 + v_1 y''_2] \\ Y''_\nu + PY'_\nu = [(u''_2 + Pu'_2)y_1 + (2u'_2 + Pu_2)y'_1 + u_2 y''_1] + \\ \qquad + [(v''_2 + Pv'_2)y_2 + (2v'_2 + Pv_2)y'_2 + v_2 y''_2] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + PY'_\mu = \Big[(u''_1 + Pu'_1)y_1 + (2u'_1 + Pu_1)(r_1 y_1 + s_1 y_2) + \\ \qquad + u_1[(r'_1 + r_1^2 + r_2 s_1)y_1 + (s'_1 + s_1 s_2 + r_1 s_1)y_2]\Big] + \\ \qquad + \Big[(v''_1 + Pv'_1)y_2 + (2v'_1 + Pv_1)(r_2 y_1 + s_2 y_2) + \\ \qquad + v_1[(r'_2 + r_1 r_2 + r_2 s_2)y_1 + (s'_2 + s_2^2 + r_2 s_1)y_2]\Big] \\ Y''_\nu + PY'_\nu = \Big[(u''_2 + Pu'_2)y_1 + (2u'_2 + Pu_2)(r_1 y_1 + s_1 y_2) + \\ \qquad + u_2[(r'_1 + r_1^2 + r_2 s_1)y_1 + (s'_1 + s_1 s_2 + r_1 s_1)y_2]\Big] + \\ \qquad + \Big[(v''_2 + Pv'_2)y_2 + (2v'_2 + Pv_2)(r_2 y_1 + s_2 y_2) + \\ \qquad + v_2[(r'_2 + r_1 r_2 + r_2 s_2)y_1 + (s'_2 + s_2^2 + r_2 s_1)y_2]\Big] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + PY'_\mu = \Big[(u''_1 + Pu'_1) + (2u'_1 + Pu_1)r_1 + u_1(r'_1 + r_1^2 + r_2 s_1) + \\ \qquad + (2v'_1 + Pv_1)r_2 + v_1(r'_2 + r_1 r_2 + r_2 s_2)\Big]y_1 + \\ \qquad + \Big[(v''_1 + Pv'_1) + (2v'_1 + Pv_1)s_2 + v_1(s'_2 + s_2^2 + r_2 s_1) + \\ \qquad + (2u'_1 + Pu_1)s_1 + u_1(s'_1 + s_1 s_2 + r_1 s_1)\Big]y_2 \\ Y''_\nu + PY'_\nu = \Big[(u''_2 + Pu'_2) + (2u'_2 + Pu_2)r_1 + u_2(r'_1 + r_1^2 + r_2 s_1) + \\ \qquad + (2v'_2 + Pv_2)r_2 + v_2(r'_2 + r_1 r_2 + r_2 s_2)\Big]y_1 + \\ \qquad + \Big[(v''_2 + Pv'_2) + (2v'_2 + Pv_2)s_2 + v_2(s'_2 + s_2^2 + r_2 s_1) + \\ \qquad + (2u'_2 + Pu_2)s_1 + u_2(s'_1 + s_1 s_2 + r_1 s_1)\Big]y_2 \end{cases}$$

So, let:

$$\begin{aligned}
A_\mu &\equiv (u_1'' + Pu_1') + (2u_1' + Pu_1)r_1 + u_1(r_1' + r_1^2 + r_2 s_1) + (2v_1' + Pv_1)r_2 + v_1(r_2' + r_1 r_2 + r_2 s_2) \\
B_\mu &\equiv (v_1'' + Pv_1') + (2v_1' + Pv_1)s_2 + v_1(s_2' + s_2^2 + r_2 s_1) + (2u_1' + Pu_1)s_1 + u_1(s_1' + s_1 s_2 + r_1 s_1) \\
A_\nu &\equiv (u_2'' + Pu_2') + (2u_2' + Pu_2)r_1 + u_2(r_1' + r_1^2 + r_2 s_1) + (2v_2' + Pv_2)r_2 + v_2(r_2' + r_1 r_2 + r_2 s_2) \\
B_\nu &\equiv (v_2'' + Pv_2') + (2v_2' + Pv_2)s_2 + v_2(s_2' + s_2^2 + r_2 s_1) + (2u_2' + Pu_2)s_1 + u_2(s_1' + s_1 s_2 + r_1 s_1)
\end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y_\mu'' + PY_\mu' = A_\mu y_1 + B_\mu y_2 \\ \hline Y_\nu'' + PY_\nu' = A_\nu y_1 + B_\nu y_2 \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y_\mu'' + PY_\mu' = A_\mu\Big[\Big(\frac{-v_2}{v_1u_2 - v_2u_1}\Big)Y_\mu + \Big(\frac{v_1}{v_1u_2 - v_2u_1}\Big)Y_\nu \Big] + \\ \hline \qquad\qquad + B_\mu\Big[\Big(\frac{u_2}{v_1u_2 - u_1v_2}\Big)Y_\mu + \Big(\frac{-u_1}{v_1u_2 - u_1v_2}\Big)Y_\nu\Big] \\ \hline Y_\nu'' + PY_\nu' = A_\nu\Big[\Big(\frac{-v_2}{v_1u_2 - v_2u_1}\Big)Y_\mu + \Big(\frac{v_1}{v_1u_2 - v_2u_1}\Big)Y_\nu \Big] + \\ \hline \qquad\qquad + B_\nu\Big[\Big(\frac{u_2}{v_1u_2 - u_1v_2}\Big)Y_\mu + \Big(\frac{-u_1}{v_1u_2 - u_1v_2}\Big)Y_\nu\Big] \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y_\mu'' + PY_\mu' = \Big[-A_\mu\Big(\frac{v_2}{v_1u_2 - v_2u_1}\Big) + B_\mu\Big(\frac{u_2}{v_1u_2 - v_2u_1}\Big)\Big]Y_\mu + \\ \hline \qquad\qquad + \Big[A_\mu\Big(\frac{v_1}{v_1u_2 - v_2u_1}\Big) - B_\mu\Big(\frac{u_1}{v_1u_2 - v_2u_1}\Big)\Big]Y_\nu \\ \hline Y_\nu'' + PY_\nu' = \Big[-A_\nu\Big(\frac{v_2}{v_1u_2 - v_2u_1}\Big) + B_\nu\Big(\frac{u_2}{v_1u_2 - v_2u_1}\Big)\Big]Y_\mu + \\ \hline \qquad\qquad + \Big[A_\nu\Big(\frac{v_1}{v_1u_2 - v_2u_1}\Big) - B_\nu\Big(\frac{u_1}{v_1u_2 - v_2u_1}\Big)\Big]Y_\nu \\ \hline \end{array} \right.$$

If:

$$A_\mu\Big(\frac{v_1}{v_1u_2 - v_2u_1}\Big) - B_\mu\Big(\frac{u_1}{v_1u_2 - v_2u_1}\Big) = 0 = -A_\nu\Big(\frac{v_2}{v_1u_2 - v_2u_1}\Big) + B_\nu\Big(\frac{u_2}{v_1u_2 - v_2u_1}\Big)$$

$$\Rightarrow B_\mu = A_\mu \frac{v_1}{u_1} \quad \& \quad B_\nu = A_\nu \frac{v_2}{u_2}$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y_\mu'' + PY_\mu' - A_\mu \frac{1}{u_1} Y_\mu = 0 \\ \hline Y_\nu'' + PY_\nu' - A_\nu \frac{1}{u_2} Y_\nu = 0 \\ \hline \end{array} \right.$$

Under the specific transformations:

$$\left. \begin{array}{|l|l|} \hline u_1 = u & v_1 = -v \\ \hline u_2 = v & v_2 = u \\ \hline \end{array} \right\}$$

$$\left. \begin{array}{|l|} \hline Y_\mu = uy_1 - vy_2 \\ \hline Y_\nu = vy_1 + uy_2 \\ \hline \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{|l|} \hline y_1 = \quad \Big(\frac{u}{u^2 + v^2}\Big)Y_\mu + \Big(\frac{v}{u^2 + v^2}\Big)Y_\nu \\ \hline y_2 = -\Big(\frac{v}{u^2 + v^2}\Big)Y_\mu + \Big(\frac{u}{u^2 + v^2}\Big)Y_\nu \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y''_\mu + PY'_\mu = \Big[(u'' + Pu') + (2u' + Pu)r_1 + u(r'_1 + r_1^2 + r_2 s_1) + \\ \hline \qquad\qquad - (2v' + Pv)r_2 - v(r'_2 + r_1 r_2 + r_2 s_2)\Big]y_1 + \\ \hline \quad - \Big[(v'' + Pv') + (2v' + Pv)s_2 + v(s'_2 + s_2^2 + r_2 s_1) + \\ \hline \qquad\qquad - (2u' + Pu)s_1 - u(s'_1 + s_1 s_2 + r_1 s_1)\Big]y_2 \\ \hline Y''_\nu + PY'_\nu = \Big[(v'' + Pv') + (2v' + Pv)r_1 + v(r'_1 + r_1^2 + r_2 s_1) + \\ \hline \qquad\qquad + (2u' + Pu)r_2 + u(r'_2 + r_1 r_2 + r_2 s_2)\Big]y_1 + \\ \hline \quad + \Big[(u'' + Pu') + (2u' + Pu)s_2 + u(s'_2 + s_2^2 + r_2 s_1) + \\ \hline \qquad\qquad + (2v' + Pv)s_1 + v(s'_1 + s_1 s_2 + r_1 s_1)\Big]y_2 \\ \hline \end{array} \right.$$

$$\begin{aligned} A_\mu &\equiv (u'' + Pu') + (2u' + Pu)r_1 + u(r'_1 + r_1^2 + r_2 s_1) - (2v' + Pv)r_2 - v(r'_2 + r_1 r_2 + r_2 s_2) \\ -B_\mu &\equiv (v'' + Pv') + (2v' + Pv)s_2 + v(s'_2 + s_2^2 + r_2 s_1) - (2u' + Pu)s_1 - u(s'_1 + s_1 s_2 + r_1 s_1) \\ A_\nu &\equiv (v'' + Pv') + (2v' + Pv)r_1 + v(r'_1 + r_1^2 + r_2 s_1) + (2u' + Pu)r_2 + u(r'_2 + r_1 r_2 + r_2 s_2) \\ B_\nu &\equiv (u'' + Pu') + (2u' + Pu)s_2 + u(s'_2 + s_2^2 + r_2 s_1) + (2v' + Pv)s_1 + v(s'_1 + s_1 s_2 + r_1 s_1) \end{aligned}$$

NOTE: $s_2 = r_1$ & $r_2 = -s_1 \Rightarrow A_\nu = -B_\mu$ & $B_\nu = A_\mu$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y''_\mu + PY'_\mu = A_\mu y_1 + B_\mu y_2 \\ \hline Y''_\nu + PY'_\nu = A_\nu y_1 + B_\nu y_2 \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y''_\mu + PY'_\mu = A_\mu\left[\left(\frac{u}{u^2+v^2}\right)Y_\mu + \left(\frac{v}{u^2+v^2}\right)Y_\nu \right] + \\ \hline \qquad\qquad + B_\mu\left[-\left(\frac{v}{u^2+v^2}\right)Y_\mu + \left(\frac{u}{u^2+v^2}\right)Y_\nu \right] \\ \hline Y''_\nu + PY'_\nu = A_\nu\left[\left(\frac{u}{u^2+v^2}\right)Y_\mu + \left(\frac{v}{u^2+v^2}\right)Y_\nu \right] + \\ \hline \qquad\qquad + B_\nu\left[-\left(\frac{v}{u^2+v^2}\right)Y_\mu + \left(\frac{u}{u^2+v^2}\right)Y_\nu \right] \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y''_\mu + PY'_\mu = \left[A_\mu\left(\frac{u}{u^2+v^2}\right) - B_\mu\left(\frac{v}{u^2+v^2}\right) \right]Y_\mu + \\ \hline \qquad\qquad + \left[A_\mu\left(\frac{v}{u^2+v^2}\right) + B_\mu\left(\frac{u}{u^2+v^2}\right) \right]Y_\nu \\ \hline Y''_\nu + PY'_\nu = \left[A_\nu\left(\frac{u}{u^2+v^2}\right) - B_\nu\left(\frac{v}{u^2+v^2}\right) \right]Y_\mu + \\ \hline \qquad\qquad + \left[A_\nu\left(\frac{v}{u^2+v^2}\right) + B_\nu\left(\frac{u}{u^2+v^2}\right) \right]Y_\nu \\ \hline \end{array} \right.$$

If:

$$A_\mu\left(\frac{v}{u^2+v^2}\right) + B_\mu\left(\frac{u}{u^2+v^2}\right) = 0 = A_\nu\left(\frac{u}{u^2+v^2}\right) - B_\nu\left(\frac{v}{u^2+v^2}\right)$$

$$\Rightarrow -B_\mu = A_\mu \frac{v}{u} \quad \& \quad B_\nu = A_\nu \frac{u}{v}$$

$$\Rightarrow \left\{ \begin{array}{|c|} \hline Y''_\mu + PY'_\mu - A_\mu \frac{1}{u} Y_\mu = 0 \\ \hline Y''_\nu + PY'_\nu - A_\nu \frac{1}{v} Y_\nu = 0 \\ \hline \end{array} \right|$$

Now, as a remembering:

$$y = \left\{ \begin{array}{|c|} \hline x^{-\frac{1}{2}} m_{10}(1,0;ix) \equiv x^{-\frac{1}{2}} e^{ix} \\ \hline x^{-\frac{1}{2}} m_{20}(0,1;ix) \equiv x^{-\frac{1}{2}} e^{-ix} \\ \hline \end{array} \right\} \Rightarrow y'' + \frac{1}{x} y' + \left[-\frac{(\frac{1}{2})^2}{x^2} + 1 \right] y = 0$$

also:

$$y = \left\{ \begin{array}{|c|} \hline x^{-\frac{1}{2}} m_{10}(\tfrac{1}{2i}, -\tfrac{1}{2i}; ix) \equiv x^{-\frac{1}{2}} \sin x \\ \hline x^{-\frac{1}{2}} m_{20}(\tfrac{1}{2}, \tfrac{1}{2}; ix) \equiv x^{-\frac{1}{2}} \cos x \\ \hline \end{array} \right\} \Rightarrow y'' + \frac{1}{x} y' + \left[-\frac{(\frac{1}{2})^2}{x^2} + 1 \right] y = 0$$

Note correspondence to the Bessel functions:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\, x^{-\frac{1}{2}} \sin x = \sqrt{\frac{2}{\pi}}\, x^{-\frac{1}{2}} m_{10}(\tfrac{1}{2i}, -\tfrac{1}{2i}; ix)$$
$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\, x^{-\frac{1}{2}} \cos x = \sqrt{\frac{2}{\pi}}\, x^{-\frac{1}{2}} m_{20}(\tfrac{1}{2}, -\tfrac{1}{2}; ix)$$

Now, consider transformations:

$$\left\{ \begin{array}{|c|} \hline Y_\mu = \frac{a}{x} J_{\frac{1}{2}} - bJ_{-\frac{1}{2}} \\ \hline Y_\nu = bJ_{\frac{1}{2}} + \frac{a}{x} J_{-\frac{1}{2}} \\ \hline \end{array} \right| \Leftrightarrow \left\{ \begin{array}{|c|c|c|} \hline u = ax^{-1} & u' = -ax^{-2} & u'' = 2ax^{-3} \\ \hline v = b & v' = 0 & v'' = 0 \\ \hline \end{array} \right|$$

$$\left\{ \begin{array}{|c|} \hline J'_{\frac{1}{2}} = -\frac{1}{2}\frac{1}{x} J_{\frac{1}{2}} + J_{-\frac{1}{2}} \\ \hline J'_{-\frac{1}{2}} = -J_{\frac{1}{2}} - \frac{1}{2}\frac{1}{x} J_{-\frac{1}{2}} \\ \hline \end{array} \right| \Leftrightarrow \left\{ \begin{array}{|c|c|c|c|} \hline r_1 = -\frac{1}{2}x^{-1} & s_1 = 1 & r'_1 = \frac{1}{2}x^{-2} & r''_1 = -x^{-3} \\ \hline r_2 = -1 & s_2 = -\frac{1}{2}x^{-1} & s'_2 = \frac{1}{2}x^{-2} & s''_2 = -x^{-3} \\ \hline \end{array} \right|$$

$$A_\mu \equiv (u'' + Pu') + (2u' + Pu)r_1 + u(r'_1 + r_1^2 + r_2 s_1) - (2v' + Pv)r_2 - v(r'_2 + r_1 r_2 + r_2 s_2)$$
$$-B_\mu \equiv (v'' + Pv') + (2v' + Pv)s_2 + v(s'_2 + s_2^2 + r_2 s_1) - (2u' + Pu)s_1 - u(s'_1 + s_1 s_2 + r_1 s_1)$$
$$A_\nu \equiv (v'' + Pv') + (2v' + Pv)r_1 + v(r'_1 + r_1^2 + r_2 s_1) + (2u' + Pu)r_2 + u(r'_2 + r_1 r_2 + r_2 s_2)$$
$$B_\nu \equiv (u'' + Pu') + (2u' + Pu)s_2 + u(s'_2 + s_2^2 + r_2 s_1) + (2v' + Pv)s_1 + v(s'_1 + s_1 s_2 + r_1 s_1)$$

NOTE: $s_2 = r_1$ & $r_2 = -s_1 \Rightarrow A_\nu = -B_\mu$ & $B_\nu = A_\mu$

So, for: $P = x^{-1}$:

$$A_\mu \equiv ax^{-3} + (-ax^{-2})\left(-\frac{1}{2}x^{-1}\right) + (ax^{-1})\left(\frac{3}{4}x^{-2} - 1\right) + (bx^{-1}) - b(x^{-1})$$
$$-B_\mu \equiv (bx^{-1})\left(-\frac{1}{2}x^{-1}\right) + b\left(\frac{3}{4}x^{-2} - 1\right) - (-2ax^{-2} + ax^{-2}) - (ax^{-1})(-x^{-1})$$
$$A_\nu \equiv (x^{-1}b)\left(-\frac{1}{2}x^{-1}\right) + b\left(\frac{1}{2}x^{-2} + \frac{1}{4}x^{-2} - 1\right) - (-2ax^{-2} + ax^{-2}) + ax^{-1}(x^{-1})$$
$$B_\nu \equiv ax^{-3} + (-ax^{-2})\left(-\frac{1}{2}x^{-1}\right) + (ax^{-1})\left(\frac{3}{4}x^{-2} - 1\right) + bx^{-1} + b(-x^{-1})$$
$$A_\mu \equiv \left[\left(\frac{3}{2}\right)^2 x^{-2} - 1\right] u$$
$$-B_\mu \equiv \left[\left(\frac{1}{4} + 2\frac{a}{b}\right) x^{-2} - 1\right] v = A_\mu \frac{v}{u}$$
$$A_\nu \equiv \left[\left(\frac{1}{4} + 2\frac{a}{b}\right) x^{-2} - 1\right] v = B_\mu$$

$$B_\nu \equiv \left[\left(\frac{3}{2}\right)^2 x^{-2} - 1\right]u = A_\mu$$
$$B_\nu = A_\nu \frac{u}{v} \Leftrightarrow a = b = v$$
$$-B_\mu = A_\mu \frac{v}{u} \Leftrightarrow a = b = v$$
$$\Rightarrow \left\{ \begin{array}{|l|} \hline Y_\mu'' + \frac{1}{x} Y_\mu' + \left[-\frac{\left(\frac{3}{2}\right)^2}{x^2} + 1 \right] Y_\mu = 0 \\ \hline Y_\nu'' + \frac{1}{x} Y_\nu' + \left[-\frac{\left(\frac{3}{2}\right)^2}{x^2} + 1 \right] Y_\nu = 0 \\ \hline \end{array} \right|$$

Since a & b are equal constants, they are irrelevant, so choose $a = 1$ to define $J_{\pm\frac{3}{2}}$.

Thus: $\mu = \frac{3}{2}$, $\nu = -\frac{3}{2}$ $i.e.$:

$$\left\{ \begin{array}{|l|} \hline J_{\frac{3}{2}} \equiv \frac{1}{x} J_{\frac{1}{2}} - J_{-\frac{1}{2}} \\ \hline J_{-\frac{3}{2}} \equiv J_{\frac{1}{2}} + \frac{1}{x} J_{-\frac{1}{2}} \\ \hline \end{array} \right|$$

So, so far: $J_{\pm\frac{1}{2}}$, $J_{\pm\frac{3}{2}}$; $\left(J_{\pm\frac{2n+1}{2}} : n \in \{0,1\} \right)$

have been determined without resort to infinite series methods.

Any $J_{\pm\frac{2n+1}{2}}$ may be determined in this manner, as follows.

Define: $pol(n) \equiv \dfrac{1-(-1)^n}{2} = \begin{cases} 1 \ ; \ iff \ \ n \ \ is\ odd \\ 0 \ ; \ iff \ \ n \ \ is\ even \end{cases}$

Let:

$$\left\{ \begin{array}{|l|l|} \hline J_{\frac{2n+1}{2}} \equiv u_n J_{\frac{1}{2}} - v_n J_{-\frac{1}{2}} & u_n \equiv \sum_{i=0}^{\frac{n-pol(n)}{2}} \frac{a_{2i+pol(n)}}{x^{2i+pol(n)}} \\ \hline J_{-\frac{2n+1}{2}} \equiv v_n J_{\frac{1}{2}} + u_n J_{-\frac{1}{2}} & v_n \equiv \sum_{i=0}^{\frac{n-2+pol(n)}{2}} \frac{b_{2i+1-pol(n)}}{x^{2i+1-pol(n)}} \\ \hline \end{array} \right|$$

$$\left\{\begin{array}{|l|}\hline u_n \equiv \sum_{i=0}^{\frac{n-pol(n)}{2}} a_{2i+pol(n)}x^{-[2i+pol(n)]} \\ \hline v_n \equiv \sum_{i=0}^{\frac{n-2+pol(n)}{2}} b_{2i+1-pol(n)}x^{-[2i+1-pol(n)]} \\ \hline u_n' \equiv \sum_{i=0}^{\frac{n-pol(n)}{2}} -[2i+pol(n)]a_{2i+pol(n)}x^{-[2i+1+pol(n)]} \\ \hline u_n'' \equiv \sum_{i=0}^{\frac{n-pol(n)}{2}} [2i+pol(n)][2i+1+pol(n)]a_{2i+pol(n)}x^{-[2i+2+pol(n)]} \\ \hline v_n' \equiv \sum_{i=0}^{\frac{n-2+pol(n)}{2}} -[2i+1-pol(n)]b_{2i+1-pol(n)}x^{-[2i+2-pol(n)]} \\ \hline v_n'' \equiv \sum_{i=0}^{\frac{n-2+pol(n)}{2}} [2i+1-pol(n)][2i+2-pol(n)]b_{2i+1-pol(n)}x^{-[2i+3-pol(n)]} \\ \hline \end{array}\right|$$

$$\left\{\begin{array}{|l|}\hline J'_{\frac{1}{2}} = -\frac{1}{2}\frac{1}{x}J_{\frac{1}{2}} + J_{-\frac{1}{2}} \\ \hline J'_{-\frac{1}{2}} = -J_{\frac{1}{2}} - \frac{1}{2}\frac{1}{x}J_{-\frac{1}{2}} \\ \hline \end{array}\right| \Leftrightarrow \left\{\begin{array}{|l|l|l|l|}\hline r_1 = -\frac{1}{2}x^{-1} & s_1 = 1 & r_1' = \frac{1}{2}x^{-2} & r_1'' = -x^{-3} \\ \hline r_2 = -1 & s_2 = -\frac{1}{2}x^{-1} & s_2' = \frac{1}{2}x^{-2} & s_2'' = -x^{-3} \\ \hline \end{array}\right|$$

$$A_\mu = (u_n'' + Pu_n') + (2u_n' + Pu_n)\left(-\frac{1}{2}x^{-1}\right) + u_n\left(\frac{3}{4}x^{-2} - 1\right) + (2v_n' + Pv_n) - v_n(x^{-1})$$
$$-B_\mu = (v_n'' + Pv_n') + (2v_n' + Pv_n)\left(-\frac{1}{2}x^{-1}\right) + v_n\left(\frac{3}{4}x^{-2} - 1\right) - (2u_n' + Pu_n) - u_n(-x^{-1})$$

Since: $s_2 = r_1$ & $r_2 = -s_1 \Rightarrow A_\nu = -B_\mu$ & $B_\nu = A_\mu$; this is sufficient.

So, for: $P = x^{-1}$:

$$A_\mu = u_n'' + \frac{1}{4}u_n x^{-2} - u_n + 2v_n'$$
$$-B_\mu = v_n'' + \frac{1}{4}v_n x^{-2} - v_n - 2u_n'$$

For targets of Bessel LHODEs:

$$(\lambda^2 x^{-2} - 1)u_n = A_\mu = u_n'' + \frac{1}{4}u_n x^{-2} - u_n + 2v_n'$$
$$(\lambda^2 x^{-2} - 1)v_n = -B_\mu = v_n'' + \frac{1}{4}v_n x^{-2} - v_n - 2u_n'$$

So:

$$\left(\lambda^2 - \frac{1}{4}\right)x^{-2}u_n - u_n'' = 2v_n'$$
$$\left(\lambda^2 - \frac{1}{4}\right)x^{-2}v_n - v_n'' = -2u_n'$$

So:

$$\sum_{i=0}^{\frac{n-pol(n)}{2}} \left[\left(\lambda^2 - \frac{1}{4}\right) - [2i+pol(n)][2i+1+pol(n)]\right]a_{2i+pol(n)}x^{-[2i+2+pol(n)]} =$$

$$= -2 \sum_{i=0}^{\frac{n-2+pol(n)}{2}} [2i+1-pol(n)] b_{2i+1-pol(n)} x^{-[2i+2-pol(n)]}$$

Separating into evens and odds,
for even $n > 0$:

$$\sum_{i=0}^{\frac{n}{2}} \left[\left(\lambda^2 - \frac{1}{4}\right) - [2i][2i+1] \right] a_{2i} x^{-[2i+2]} = -2 \sum_{i=0}^{\frac{n}{2}-1} [2i+1] b_{2i+1} x^{-[2i+2]}$$

$$\Rightarrow \left[\left(\lambda^2 - \frac{1}{4}\right) - [n][n+1] \right] a_n x^{-[n+2]} +$$

$$+ \sum_{i=0}^{\frac{n}{2}-1} \left[\left(\lambda^2 - \frac{1}{4}\right) - [2i][2i+1] \right] a_{2i} x^{-[2i+2]} = -2 \sum_{i=0}^{\frac{n}{2}-1} [2i+1] b_{2i+1} x^{-[2i+2]}$$

$$\Rightarrow \left(\lambda^2 - \frac{1}{4}\right) = [n][n+1] = 0 \Rightarrow \lambda = \pm \frac{2n+1}{2}$$

and:

$$[n(n+1) - 2i(2i+1)] a_{2i} = -2[2i+1] b_{2i+1} \qquad , \quad \left\{ i \in \mathbb{I} \mid 0 \le i \le \frac{n}{2} - 1 \right\}$$

for odd $n > 0$:

$$\sum_{i=0}^{\frac{n-1}{2}} [n(n+1) - [2i+1][2i+1+1]] a_{2i+1} x^{-[2i+3]} =$$

$$= -2 \sum_{i=0}^{\frac{n-2+1}{2}} [2i+1-1] b_{2i+1-1} x^{-[2i+1]} = -4 \sum_{i=0}^{\frac{n-1}{2}} i b_{2i} x^{-[2i+1]}$$

$$\Rightarrow [n(n+1) - [n][n+1]] a_n x^{-[n+2]} +$$

$$+ \sum_{i=0}^{\frac{n-1}{2}-1} [n(n+1) - [2i+1][2i+2]] a_{2i+1} x^{-[2i+3]} = -4 \sum_{i=0}^{\frac{n-1}{2}} i b_{2i} x^{-[2i+1]}$$

$$\Rightarrow \sum_{j=1}^{\frac{n-1}{2}} [n(n+1) - [2(j-1)+1][2(j-1)+2]] a_{2(j-1)+1} x^{-[2(j-1)+3]} = -4 \sum_{i=0}^{\frac{n-1}{2}} i b_{2i} x^{-[2i+1]}$$

$$\Rightarrow \sum_{j=1}^{\frac{n-1}{2}} [n(n+1) - [2j-1][2j]] a_{2j-1} x^{-[2j+1]} = -4 \sum_{i=0}^{\frac{n-1}{2}} i b_{2i} x^{-[2i+1]} - 4 \sum_{i=1}^{\frac{n-1}{2}} i b_{2i} x^{-[2i+1]} + 0$$

$$\Rightarrow [n(n+1) - [2i][2i-1]] a_{2i-1} = -4 i b_{2i} \qquad , \quad \left\{ i \in \mathbb{I} \mid 1 \le i \le \frac{n-1}{2} \right\}$$

Likewise, for: $\left(\lambda^2 - \frac{1}{4}\right) x^{-2} v_n - v_n'' = -2u_n'$:

$$\sum_{i=0}^{\frac{n-pol(n)}{2}} \left[\left(\lambda^2 - \frac{1}{4}\right) - [2i+1-pol(n)][2i+2-pol(n)] \right] b_{2i+1-pol(n)} x^{-[2i+3-pol(n)]} =$$

$$= 2 \sum_{i=0}^{\frac{n-pol(n)}{2}} [2i + pol(n)] a_{2i+pol(n)} x^{-[2i+1+pol(n)]}$$

Separating into evens and odds,
for even $n > 0$:

$$\sum_{i=0}^{\frac{n-2}{2}} \left[\left(\lambda^2 - \frac{1}{4} \right) - [2i+1][2i+2] \right] b_{2i+1} x^{-[2i+3]} = 2 \sum_{i=0}^{\frac{n}{2}} [2i] a_{2i} x^{-[2i+1]}$$

$$\Rightarrow \sum_{i=0}^{\frac{n}{2}-1} \left[\left(\lambda^2 - \frac{1}{4} \right) - [2i+1][2i+2] \right] b_{2i+1} x^{-[2i+3]} = 4 \sum_{i=1}^{\frac{n}{2}} i a_{2i} x^{-[2i+1]} + 0$$

$$\Rightarrow \sum_{j=1}^{\frac{n}{2}} \left[\left(\lambda^2 - \frac{1}{4} \right) - [2(j-1)+1][2(j-1)+2] \right] b_{2(j-1)+1} x^{-[2(j-1)+3]} = 4 \sum_{i=1}^{\frac{n}{2}} i a_{2i} x^{-[2i+1]}$$

$$\Rightarrow \sum_{j=1}^{\frac{n}{2}} [n(n+1) - [2j-1][2j]] b_{2j-1} x^{-[2j+1]} = 4 \sum_{i=1}^{\frac{n}{2}} i a_{2i} x^{-[2i+1]}$$

$$\Rightarrow [n(n+1) - [2i-1][2i]] b_{2i-1} = 4 i a_{2i} \qquad , \left\{ i \in \mathbb{I} \mid 1 \leq i \leq \frac{n}{2} \right\}$$

for odd $n > 0$:

$$\sum_{i=0}^{\frac{n-1}{2}} \left[\left(\lambda^2 - \frac{1}{4} \right) - [2i+1-1][2i+2-1] \right] b_{2i+1-1} x^{-[2i+3-1]} =$$

$$= 2 \sum_{i=0}^{\frac{n-1}{2}} [2i+1] a_{2i+1} x^{-[2i+1+1]}$$

$$\Rightarrow \sum_{i=0}^{\frac{n-1}{2}} [n(n+1) - [2i][2i+1]] b_{2i} x^{-[2i+2]} =$$

$$= 2 \sum_{i=0}^{\frac{n-1}{2}} [2i+1] a_{2i+1} x^{-[2i+2]}$$

$$\Rightarrow [n(n+1) - [2i][2i+1]] b_{2i} = 2[2i+1] a_{2i+1} \qquad , \left\{ i \in \mathbb{I} \mid 0 \leq i \leq \tfrac{n-1}{2} \right\}$$

Tabulating the results:

n even:

$[n(n+1) - [2i][2i-1]] b_{2i-1} = 4 i a_{2i}$	$\left\{ i \in \mathbb{I} \mid 1 \leq i \leq \frac{n}{2} \right\}$
$[n(n+1) - [2i](2i+1)] a_{2i} = -2[2i+1] b_{2i+1}$	$\left\{ i \in \mathbb{I} \mid 0 \leq i \leq \frac{n}{2} - 1 \right\}$

n odd:

$[n(n+1) - [2i][2i-1]] a_{2i-1} = -4 i b_{2i}$	$\left\{ i \in \mathbb{I} \mid 1 \leq i \leq \frac{n-1}{2} \right\}$
$[n(n+1) - [2i][2i+1]] b_{2i} = 2[2i+1] a_{2i+1}$	$\left\{ i \in \mathbb{I} \mid 0 \leq i \leq \frac{n-1}{2} \right\}$

We already have $J_{\pm \frac{2n+1}{2}} : n \in \{0, 1\}$, so as examples:

$n = 2$:

$b_1 = a_2$	$\{ i \in \mathbb{I} \mid 1 \leq i \leq 1 \}$
$-3a_0 = b_1$	$\{ i \in \mathbb{I} \mid 0 \leq i \leq 0 \}$

$$\left\{\begin{array}{|l|l|}\hline J_{\frac{6}{2}} \equiv (u_2)J_{\frac{1}{2}} - (v_2)J_{-\frac{1}{2}} & u_2 \equiv -a_0\left(\dfrac{3}{x^2} - 1\right) \\ \hline J_{-\frac{5}{2}} \equiv (v_2)J_{\frac{1}{2}} + (u_2)J_{-\frac{1}{2}} & v_2 \equiv -a_0\dfrac{3}{x} \\ \hline \end{array}\right.$$

(Clearly, the standard value is for: $a_0 = -1$)

$n = 3$:

$10a_1 = -4b_2$	$\{i \in \mathbb{I} \mid 1 \le i \le 1\}$
$12b_0 = 2a_1$	$\{i \in \mathbb{I} \mid 0 \le i \le 1\}$
$6b_2 = 6a_3$	

$$\left\{\begin{array}{|l|l|}\hline J_{\frac{6}{2}} \equiv (u_2)J_{\frac{1}{2}} - (v_2)J_{-\frac{1}{2}} & u_2 \equiv \dfrac{1}{15}a_3\left(\dfrac{15}{x^3} - \dfrac{6}{x}\right) \\ \hline J_{-\frac{5}{2}} \equiv (v_2)J_{\frac{1}{2}} + (u_2)J_{-\frac{1}{2}} & v_2 \equiv \dfrac{1}{15}a_3\left(\dfrac{15}{x^2} - 1\right) \\ \hline \end{array}\right.$$

(Here, clearly, the standard value is for: $a_3 = 15$)

Tabulating the results:

n even:

$b_{n-1} = a_n$	
$b_{2i+1} = -\dfrac{[n(n+1) - [2i](2i+1)][n(n+1) - [2i][2i-1]]}{8i[2i+1]}b_{2i-1}$	$\left\{i \in \mathbb{I} \mid 1 \le i \le \dfrac{n}{2} - 1\right\}$
$a_{2i} = \dfrac{n(n+1) - [2i][2i-1]}{4i}b_{2i-1}$	$\left\{i \in \mathbb{I} \mid 1 \le i \le \dfrac{n}{2} - 1\right\}$
$b_1 = -\dfrac{n(n+1)}{2}a_0$	

which may be summarized:

$$\left\{\begin{array}{|l|}\hline u_n \equiv -\dfrac{2}{n(n+1)}b_1 + \displaystyle\sum_{i=1}^{\frac{n}{2}-1}\dfrac{b_{2i-1}}{x^{2i}}\left[\dfrac{n(n+1) - [2i][2i-1]}{4i}\right] + \dfrac{b_{n-1}}{x^n} \\ \hline v_n \equiv \dfrac{b_1}{x} - \displaystyle\sum_{i=1}^{\frac{n}{2}-1}\dfrac{b_{2i-1}}{x^{2i+1}}\dfrac{[n(n+1) - [2i](2i+1)][n(n+1) - [2i][2i-1]]}{8i[2i+1]} \\ \hline \end{array}\right.$$

where:

$b_{2i+1} = -\dfrac{[n(n+1) - [2i](2i+1)][n(n+1) - [2i][2i-1]]}{8i[2i+1]}b_{2i-1}$	$\left\{i \in \mathbb{I} \mid 1 \le i \le \dfrac{n}{2} - 1\right\}$

n odd:

$-\dfrac{n(n+1)-[2i][2i-1]}{4i}a_{2i-1} = b_{2i}$	$\left\{i \in \mathbb{I} \mid 1 \le i \le \dfrac{n-1}{2}\right\}$
$\dfrac{[n(n+1)-[2i][2i+1]][n(n+1)-[2i][2i-1]]}{8i[2i+1]}a_{2i-1} = a_{2i+1}$	$\left\{i \in \mathbb{I} \mid 1 \le i \le \dfrac{n-1}{2}\right\}$
$n(n+1)b_0 = 2a_1$	

which may be summarized:

$$\left\{\begin{array}{l} u_n \equiv \dfrac{a_1}{x} + \sum_{i=1}^{\frac{n-1}{2}} \dfrac{a_{2i-1}}{x^{2i+1}}\left[\dfrac{[n(n+1)-[2i][2i+1]][n(n+1)-[2i][2i-1]]}{8i[2i+1]}\right] \\ v_n \equiv \dfrac{2}{n(n+1)}a_1 - \sum_{i=1}^{\frac{n-1}{2}} \dfrac{a_{2i-1}}{x^{2i}}\left[\dfrac{n(n+1)-[2i][2i-1]}{4i}\right] \end{array}\right.$$

where:

$a_{2i+1} = \dfrac{[n(n+1)-[2i][2i+1]][n(n+1)-[2i][2i-1]]}{8i[2i+1]}a_{2i-1}$	$\left\{i \in \mathbb{I} \mid 1 \le i \le \dfrac{n-1}{2}\right\}$

With these assignments:

$$\left\{\begin{array}{l} J_{\frac{2n+1}{2}} \equiv u_n J_{\frac{1}{2}} - v_n J_{-\frac{1}{2}} \\ J_{-\frac{2n+1}{2}} \equiv v_n J_{\frac{1}{2}} + u_n J_{-\frac{1}{2}} \end{array}\right.$$

establishes the solutions for the odd half-integer Bessel equation solutions.

Although any odd half integer Bessel equation solution may be obtained from the first, and the Recurrence formula: $J_{n-1} + J_{n+1} = \dfrac{2\pi}{x}J_n$ the above has reduced the problem to calculating and tabulating the recurrence relations for $\dfrac{n-1}{2}$ numerators over the powers of x .

The theory and confirmations above give confidence that this linear algebra technique provides a useful method of obtaining exact solutions to homogeneous linear ordinary differential equations.

The Bessel half integer solutions, being long well known, have been used here to verify with confidence the above linear independence technique for solving ordinary differential equations.

The Bessel second order LHODE has been used, here, as an example application of this vector space transformation technique. Clearly, it may be used on other non-general elementary second order LHODE, such as the Legendre, Laguerre, Hermite and other second order LHODE's. Using the two linearly independent solutions of a second order linear homogeneous ordinary differential equations insures that the two functions are linearly independent. However, any two linearly independent functions may be used, and the two resulting differential equations need not be the same, as was the case for the Bessel's above. In fact, clearly, the technique may also be used for higher order LHODE's, since there are N linearly independent solutions of an N-th order LHODE there would be an equal number of transformation equations.

So, the process proceeds analogously to the above.

$$Y_h = \sum_{j=1}^{n} u_h^j y_j \Leftrightarrow Y = Uy \Leftrightarrow y = U^{-1}Y$$

$$y'_h = \sum_{j=1}^{n} r_h^j y_j \Leftrightarrow y' = Ry$$

etc.