SUBSET POLYNOMIAL SEMIRINGS AND SUBSET MATRIX SEMIRINGS

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Subset Polynomial Semirings and Subset Matrix Semirings

W. B. Vasantha Kandasamy
Florentin Smarandache

Educational Publisher Inc.
Ohio
2013

This book can be ordered from:

Education Publisher Inc.
1313 Chesapeake Ave.
Columbus, Ohio 43212,
USA
Toll Free: 1-866-880-5373

Peer reviewers:

Said Broumi, University Hassan II Mohammedia, Hay El Baraka Ben M'sik, Casablanca B.P. 7951, Morocco
Prof. Gabriel Tica, Bailesti College, Bailesti, Jud. Dolj, Romania.
Florentin Popescu, Facultatea de Mecanica, University of Craiova, Romania.
Dr. Sebastian Nicolaescu, 2 Terrace Ave., West Orange, NJ 07052, USA.

Many books can be downloaded from the following
Digital Library of Science:
http://www.gallup.unm.edu/eBooks-otherformats.htm

ISBN-13: 978-1-59973-217-6
EAN: 9781599732176

Printed in the United States of America

CONTENTS

PREFACE

In this book authors introduce the notion of subset polynomial semirings and subset matrix semirings. The study of algebraic structures using subsets were recently carried out by the authors. Here we define the notion of subset row matrices, subset column matrices and subset $m \times n$ matrices. Study of this kind is developed in chapter two of this book.

If we use subsets of a set X; say P(X), the power set of the set X; as the entries of the collection of subsets of $m \times n$ matrices say S; then we see $(S, \cup)$ is a semigroup (semilattice) and $(S, \cap)$ is a semigroup (semilattice). Thus $(S, \cup, \cap)$ is a semiring (a lattice). Hence if P(X) is replaced by a group or a semigroup we get the subset matrix to be only a subset matrix semigroup. If the semiring or a ring is used we can give the subset collection only the semiring structure.

The collection of subsets from the polynomial ring or a polynomial semiring can have only a semiring structure. Several types of subset polynomial semirings are defined described and developed in chapter three of this book.

Using the subset polynomials (subset matrices) we built subset semivector spaces. Study in this direction is interesting and innovative which forms the chapter four of this book. Every chapter is followed by a collection of problems.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this book authors for the first time study algebraic structures using subsets of a ring or a field or a semiring or a semifield. We define describe and develop mainly subset polynomial semiring and subset matrix semiring. It is important and interesting at this juncture to keep on record that the maximum algebraic structure we can give to these subset structures is a semifield (in some cases semiring).

Further the study of algebraic structures using subsets started in the 18^{th} century by Boole who made the collection of all subsets of a set into an algebra in a very special way was named after him. However $\{P(X), \cup, \cap\}$ is a Boolean algebra of order $2^{|X|}$ if $|X| < \infty$ and P(X) is the power set of X.

After this study we see there are not many well defined algebraic structures using subsets of an algebraic structure.

However we have the subsets of a set with '$\cup$' and '$\cap$' is made into a nice topological space. Thus we have topology developed on them. But we do not have any algebraic structure other than Boolean algebra developed using subsets of a set X.

Of course we have the concept of semilattices using subsets.

Here we develop a algebraic structure which is a semiring / semifield. In fact by this method we in the first place are in a position to generate finite non commutative semirings.

Further for every subset collection (subsets from algebraic structure) we can define two types of semirings. This is described in this book. Finally we in this book introduce the concept of polynomial subset semirings and subset polynomial semiring S[x] and P[x] respectively. We show how we can solve polynomial subset equations in P[x].

We have three cases to our surprise.

(i) Completely solvable subset polynomial equations for a subset solution.
(ii) Partially solvable subset polynomial equations and
(iii) Not solvable subset polynomial equations in P[x].

We leave it as an open conjecture.
If P[x] = {Collection of all subsets from $C(Z_p)[x]$, p a prime} be the polynomial subset semiring.

Can we say P[x] will be algebraically closed polynomial subset semifield?

For we see P[x] = {Collection of all subsets from C[x]} be the polynomial subset semifield then P[x] is an algebraically closed semifield for every pair A, B $\in$ P[x], A = B is completely solvable.

Chapter Two

SUBSET MATRICES

In this chapter we proceed onto study for the first time the notion of subset matrices; the subsets can be from a set or a semigroup or a ring or a field or a group or a semiring or a semifield. We give algebraic structure to these subset matrices.

Let X = $\{a_1, \ldots, a_n\}$ be a set, the power set of X denoted by P(X) where
P(X) = {Collection of all subsets of X including X and ϕ}.

DEFINITION 2.1: *Let X be a set $\{a_1, a_2, \ldots, a_n\}$ and P(X), the power set of X.*

Let $S_R = \{(p_1, p_2, \ldots, p_t)$ where $p_i \in P(X)\}$; then we define S_R to be the subset row matrix $(1 \leq i \leq t)$.

$$\text{Let } S_C = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix} \text{ where } p_j \in P(X),\ 1 \leq j \leq m;$$

we define S_C as the subset column matrix.

$$S_S = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \textit{ where } p_{ij} \in P(X);\ (1 \leq i, j \leq n)$$

is defined as the subset square matrix.

$$S_{Rect} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nm} \end{bmatrix} \textit{ where } p_{ij} \in P(X);\ 1 \leq i \leq n$$

and $1 \leq j \leq m$ *is defined as the subset rectangular matrix.*

We will illustrate this situation by some examples. Further even if we do not put S_C or S_R or S_S or S_{Rect}, the reader can follow by the very context.

Example 2.1: Let
P(X) = {Collection of all subsets of the set X = {1, 2, 3, 4}}, that is the power set of X.

S = ({ϕ}, {1, 2}, {4, 2}, {1, 2, 3}, {X}, {1,2}, ϕ, {2,3}) is a 1×8 subset row matrix of the power set P (X).

$$A = \begin{bmatrix} \{3\} \\ \{2\} \\ \{1,2\} \\ \phi \\ X \\ \{\phi\} \\ \{1,2,3\} \end{bmatrix}$$ is a 7×1 subset column matrix with entries

from P(X) where X = {0, 1, 2, 3}.

$$\text{Let } Y = \begin{bmatrix} \{\phi\} & \{1,2,3\} & \{1,2\} & \{3\} & \{4\} \\ \{X\} & \{2\} & \{3\} & \{4\} & \{\phi\} \\ \{2,1\} & \{3,2\} & \{3,4\} & \{4,2\} & X \\ \{\phi\} & \{2,4\} & \{2,3\} & \{\phi\} & \{1,2,3\} \end{bmatrix} \text{ be a } 4 \times 5$$

subset rectangular matrix with entries from the power set P(X) where X = {1, 2, 3, 4}.

$$\text{Let } B = \begin{bmatrix} \{\phi\} & X & \{1,2\} & \{1\} & \{2\} & \{3\} \\ X & \{\phi\} & \{3\} & \{3\} & \{4\} & \{1,2,3\} \\ \{2,3\} & \{1\} & \{\phi\} & \{\phi\} & \{3\} & \{1,2\} \\ \{3\} & \{4\} & \{4\} & \{1,2,3\} & \{\phi\} & X \\ \{4\} & \{3,2\} & \phi & \{1\} & \{2\} & \{4\} \\ \{4,3\} & \{3,2,4\} & & \{1,2,3\} & X & \{1,4,3\} \end{bmatrix}$$

be a 6 × 6 subset square matrix from P(X) where X = {1,2,3,4}.

Example 2.2: Let X = {a_1, a_2, a_3, a_4, a_5, a_6} be a set; P(X) the power set of X. Let

$$A = \begin{bmatrix} \{a_6, a_1\} \\ \{a_1\} \\ \{a_2, a_5\} \\ \{a_3, a_1, a_4\} \\ \{a_4\} \\ \{a_5, a_6, a_2\} \\ X \end{bmatrix} \text{ be the } 6 \times 1 \text{ subset column matrix}$$

of P(X).

Take B = (X, ϕ, {a_1, a_2, a_3, a_4, a_5}, {a_1}, {a_3, a_2, a_6}, {a_6, a_1}, {a_2, a_3, a_4}, {a_5, a_1, a_6}, ϕ), B is a 1 × 9 subset row matrix with entries from P(X) of X.

$$\text{Let } C = \begin{bmatrix} \{\phi\} & \{a_1,a_2,a_3\} & \{a_4\} & X \\ \{a_5\} & \{a_6\} & \{a_3,a_6\} & \{a_5\} \\ \{a_6\} & X & \{\phi\} & \{a_6,a_1,a_3\} \\ \{a_1,a_2,a_3\} & \{a_1,a_3\} & \{a_4\} & \{a_6,a_5\} \\ \{a_4,a_6,a_5\} & \{a_4,a_6\} & \{a_6,a_1\} & X \end{bmatrix}$$

be a 5 × 4 subset rectangular matrix of X or with entries from P(X).

$$D = \begin{bmatrix} \{a_1\} & \{a_2\} & \{a_3\} & \{a_4\} & \{a_5a_6\} \\ \{a_6a_1\} & \{a_1a_2\} & \{a_4\} & \phi & X \\ X & \phi & \{a_4a_6\} & \{a_4a_2\} & \{a_3a_1a_2\} \\ \{a_1a_2\} & \{a_4a_6\} & \{a_6\} & \{a_5\} & \{a_2a_3\} \\ \{a_1a_2a_3\} & X & \{a_3\} & \{a_4a_5a_6\} & \phi \end{bmatrix}$$

is a 5 × 5 subset square matrix of X or with entries from P(X).

Example 2.3: Let $X = Z^+ \cup \{0\}$,

P = {Collection of all subsets of X together with X and ϕ} be the power set of X. P is of infinite order.

Let $A = (p_1, p_2, \ldots, p_{14})$; $p_i \in P$, $1 \leq i \leq 14$ be the subset row matrix of X or with entries from P.

$$B = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{19} \end{bmatrix}; \; p_j \in P,\ 1 \leq j \leq 19 \text{ be a } 19 \times 1 \text{ subset column}$$

matrix of $X = Z^+ \cup \{0\}$.

Consider

$$C = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{19} \\ p_{21} & p_{22} & \cdots & p_{29} \\ \vdots & \vdots & & \vdots \\ p_{161} & p_{162} & \cdots & p_{169} \end{bmatrix};$$

C is a 16 × 9 subset matrix of X or with entries from P, $p_{ij} \in P$; $1 \leq i \leq 16$, $1 \leq j \leq 9$.

Let $D = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{151} \\ p_{21} & p_{22} & \cdots & p_{152} \\ p_{31} & p_{32} & \cdots & p_{153} \\ p_{41} & p_{42} & \cdots & p_{154} \end{bmatrix}$ be a 4 × 15 subset rectangular matrix of X or with entries from P.

Now we have seen examples of subset matrices of a set.

Next we show we can only define two operations on the subset of a set viz; ∪ and ∩ of subsets.

Thus when the set under consideration is just a set with no operations on it we on this subset matrices define operations '∪' and '∩'.

Let C_R = {Collection of all 1 × n subset row matrices with entries from a power set P(X) of X}.

We can define '∪' and '∩' on C_R. $\{C_R, \cup\}$ is a semilattice or a commutative semigroup.

$\{C_R, \cap\}$ is a semilattice or a commutative semigroup. $\{C_R, \cup, \cap\}$ is a lattice.

We will illustrate this situation by some examples.

Example 2.4: Let S = {Collection of all 1×5 subset matrices with entries from the power set P(X); where X = {1, 2, 3, 4, 5, 6, 7, 8}} be a semigroup under '$\cup$' i.e., {S, $\cup$} is a semilattice.

Let A = ({6,1}, {2,3,8}, {ϕ}, X, {5,6,1,7}) and

B = ({1,7}, {2,3,4}, {5,6,7}, {3,8,6}, X) be in S;

$A \cup B$ = ({6,1} $\cup$ {1,7}, {2,3,8} $\cup$ {2,3,4}, {ϕ} $\cup$ {5,6,7}, X $\cup$ {3,8,6}, {5,6,1,7} $\cup$ X)

= ({1,6,7}, {2,3,4,8}, {5,6,7}, X, X).

Clearly $A \cup B \in S$.

We see $B \cup A = A \cup B$.

(S, $\cup$) is a commutative subset matrix semigroup.

Example 2.5: Let S_C = {Collection of all 7×1 subset column matrices with entries from the power set of X, where X = {1, 2, …, 12}}.

(S_C, $\cap$) is a semilattice.

$$\text{Take } A = \begin{bmatrix} \{1,12,5\} \\ \{3,10,6\} \\ \{8,7,11\} \\ X \\ \{1,2,3,4,5\} \\ \phi \\ \{12,6,3,9\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{1,2,7\} \\ \{5,10\} \\ \{8,2,4\} \\ \{7,6,5,4,2,3\} \\ \{1,2,5,7,9,10\} \\ \{7,8,4,9,11\} \\ \{12,10,9,7\} \end{bmatrix} \text{ in } S_C.$$

$$\text{We find } A \cap B = \begin{bmatrix} \{1,12,5\} \cap \{1,2,7\} \\ \{3,10,6\} \cap \{5,10\} \\ \{8,7,11\} \cap \{8,2,4\} \\ X \cap \{7,6,5,4,2,3\} \\ \{1,2,3,4,5\} \cap \{1,2,5,7,9,10\} \\ \phi \cap \{7,8,4,9,11\} \\ \{12,6,3,9\} \cap \{12,10,9,7\} \end{bmatrix}$$

$$= \begin{bmatrix} \{1\} \\ \{10\} \\ \{8\} \\ \{7,6,5,4,2,3\} \\ \{1,2,5\} \\ \{\phi\} \\ \{12,9\} \end{bmatrix} \text{ is in } S_C.$$

It is easily verified (S, $\cap$) is a commutative subset matrix semigroup (or semilattice) of finite order.

Example 2.6: Let

$$S_R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in P(X) \right.$$

where X = {1, 2, …, 19}, $1 \leq i \leq 12$} be the collection of subset 3×4 rectangular matrices with entries from P(X).

We see {S_R, $\cup$} is a subset matrix semigroup which is commutative.

$$\text{Let } A = \begin{bmatrix} \{6,2,1\} & \{18,10\} & \{19,1\} & \{X\} \\ \{7,3,8\} & \{3,1,2,13,6\} & \{10,3,4\} & \{9,2,3\} \\ \{4,5,6,2,3\} & \{4,1,10,7\} & X & \phi \end{bmatrix}$$

and

$$B = \begin{bmatrix} \phi & X & \{1,19,12,9\} & \{9,7,18,6,3\} \\ \{3,9,11,14\} & \phi & \{3,7,9,2\} & \{10,6,4,9\} \\ \{6,7,9\} & \{10,8,1\} & \{16,9\} & X \end{bmatrix}$$

be in S_R.

$$A \cup B = \begin{bmatrix} \{6,2,1\} \cup \phi & \{18,10\} \cup X \\ \{7,3,8\} \cup \{3,9,11,14\} & \{3,1,2,13,6\} \cup \phi \\ \{4,5,6,2,3\} \cup \{6,7,9\} & \{4,1,10,7\} \cup \{10,8,1\} \end{bmatrix}$$

$$\begin{matrix} \{19,1\} \cup \{1,19,12,9\} & \{X\} \cup \{9,7,18,6,3\} \\ \{10,3,4\} \cup \{3,7,9,2\} & \{9,2,3\} \cup \{10,6,4,9\} \\ X \cup \{16,9\} & \phi \cup X \end{matrix} \Bigg]$$

$$= \begin{bmatrix} \{6,2,1\} & X & \{1,19,12,9\} & \{X\} \\ \{3,9,7,8,11,14\} & \{3,1,2,13,6\} & \{3,4,2,7,9,10\} & \{2,9,4,6,9,10\} \\ \{4,5,6,2,3,7,9\} & \{1,4,7,8,10\} & X & X \end{bmatrix}$$

is in S_R.

Thus $\{S_R, \cup\}$ is a subset matrix semigroup (or semilattice).

Example 2.7: Let $S_R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in P(X) \right.$

where $X = \{1, 2, 3, \ldots, 18\}$; $1 \leq i \leq 9\}$ be a 3×3 square subset matrix of the set X.

$\{S_R, \cap\}$ is a subset matrix semigroup / semilattice.

Take

$$A = \begin{bmatrix} \{6,18,5\} & \phi & \{2,3,4,5,6\} \\ \{7,8,9,10\} & \{11,12,13\} & X \\ \{14,15,16,17\} & \{18,1,2,5\} & \{3,8,13,18\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{6,1,5,11\} & \{3,2\} & \{4,5,6,8,10,11\} \\ \{3,8,5,10,11\} & X & \{5,6,7,8\} \\ \{16,17,1\} & \{1,5,3,2,11,13\} & \phi \end{bmatrix}$$

in S_R.

To find

$$A \cap B = \begin{bmatrix} \{6,18,5\} \cap \{6,1,5,11\} & \phi \cap \{3,2\} & \{2,3,4,5,6\} \cap \{4,5,6,8,10,11\} \\ \{7,8,9,10\} \cap \{3,8,5,10,11\} & \{11,12,13\} \cap X & X \cap \{5,6,7,8\} \\ \{14,15,16,17\} \cap \{16,17,1\} & \{18,1,2,5\} \cap \{15,3,2,11,13\} & \{3,8,13,18\} \cap \phi \end{bmatrix}$$

$$= \begin{bmatrix} \{6,5\} & \phi & \{4,6,5\} \\ \{8,10\} & \{11,12,13\} & \{5,6,7,8\} \\ \{16,17\} & \{1,2,5\} & \phi \end{bmatrix} \in S_R.$$

It is easily verified S_R is a subset matrix commutative semigroup and is of finite order.

$A \cap (\phi) = (\phi)$ where $(\phi) = \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix}$

$$A \cap (X) = (A) \quad \text{and } (X) = \begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \end{bmatrix} \text{ is in } S_R.$$

Example 2.8: Let S_M = {Collection of all subset matrices of the

form $\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$ with $p_i \in P(X)$ where

$X = \{1, 2, 3, \ldots, 10\}$, $1 \leq i \leq 12\}$. $\{S_m, \cup\}$ and $\{S_m, \cap\}$ are subset matrix semigroups of finite order which is commutative.

$$\text{Let } A = \begin{bmatrix} \{6,10\} & \{3,5\} & \{4\} & \phi & \{7,8,9\} & \{1\} \\ \{5,6,7\} & \phi & \{8,9,10\} & \{9,4\} & \{8,9\} & \phi \end{bmatrix}$$

and

$$B = \begin{bmatrix} \{9,7,3\} & \{3\} & \{8\} & \phi & \{7,6,4\} & \{1,2,3,4\} \\ X & \{8,6,2\} & \{7,9\} & \{10,7,8,9\} & X & \phi \end{bmatrix}$$

be in S_M. We find

$$A \cup B = \left[\begin{matrix} \{6,10\} \cup \{9,7,3\} & \{3,5\} \cup \{3\} & \{4\} \cup \{8\} \\ \{5,6,7\} \cup X & \phi \cup \{8,6,2\} & \{8,9,10\} \cup \{7,9\} \end{matrix}\right.$$

$$\left.\begin{matrix} \phi \cup \phi & \{7,8,9\} \cup \{7,6,4\} & \{1\} \cup \{1,2,3,4\} \\ \{9,4\} \cup \{10,7,8,9\} & \{8,9\} \cup X & \phi \cup \phi \end{matrix}\right]$$

$$= \left[\begin{matrix} \{10,6,9,7,3\} & \{3,5\} & \{8,4\} \\ X & \{8,6,2\} & \{7,8,9,10\} \end{matrix}\right.$$

$$\left.\begin{matrix} \phi & \{7,8,9,6,4\} & \{1,2,3,4\} \\ \{10,7,8,9,4\} & X & \phi \end{matrix}\right];$$

$A \cup B$ is in S_M.

$$A \cap B = \begin{bmatrix} \phi & \{3\} & \{\phi\} & \{\phi\} & \{7\} & \{1\} \\ \{5,6,7\} & \phi & \{9\} & \{9\} & \{8,9\} & \phi \end{bmatrix} \in (S, \cap).$$

Example 2.9: Let $S_T = \left\{ \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \\ p_5 & p_6 \\ p_7 & p_8 \\ p_9 & p_{10} \end{bmatrix} \middle| \; p_i \in P(X) \text{ where} \right.$

$X = \{1, 2, 3, 4, 5, 6\}$, $1 \leq i \leq 10\}$, $(S, \cap)$ be the subset column matrix semigroup of finite order.

$$\text{Let } A = \begin{bmatrix} \{1\} & \{3,6\} \\ \{2,4\} & \{1,2,3,4\} \\ \{5,1\} & \phi \\ X & \{4,5,1\} \\ \{2,3,4\} & \{1,6,5\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{1,2\} & \phi \\ X & \{1,2\} \\ \{3,5\} & \{1,2,3\} \\ \{6,3\} & \phi \\ \phi & X \end{bmatrix}$$

be in S_T.

$$A \cap B = \begin{bmatrix} \{1\} & \phi \\ \{2,4\} & \{1,2\} \\ \{5\} & \phi \\ \{6,3\} & \phi \\ \phi & \{16,5\} \end{bmatrix} \in S_T.$$

This is the way $\cap$ operation is performed.

All the operations done on them are natural product $\times_n$. We see for any matrix A,

$$I = \begin{bmatrix} X & X & \ldots & X \\ X & X & \ldots & X \\ \vdots & \vdots & & \vdots \\ X & X & \ldots & X \end{bmatrix}$$ acts as the identity under '$\cap$'.

For if $A = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix}$ and $I = \begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \end{bmatrix}$ are in S.

$A \cap I = A = I \cap A$.
However $A \cup I = I \cup A = I$ for all $A, I \in S$.

Further $(\phi) \cup A = A$ and $(\phi) \cap A = \phi$ where

$$(\phi) = \begin{bmatrix} \phi & \phi & \ldots & \phi \\ \phi & \phi & \ldots & \phi \\ \vdots & \vdots & & \vdots \\ \phi & \phi & \ldots & \phi \end{bmatrix}$$ for all $A \in S$.

Example 2.10: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \right.$$ $a_i \in P(X)$; the power set of X;

where X = {1, 2, I, 2I, 3, 3I, 4, 4I, 5I, 5, 6, 6I}} that is a_i's are subsets from the power set P(X) where set
{1, 2, I, 2I, 3, 3I, 4, 4I, 5, 5I, 6, 6I} = X.

That is if $A = \begin{bmatrix} \phi & X & \{6, I, 6I\} \\ \{I\} & \{2, 2I\} & \phi \\ \{3, 3I, I\} & \phi & \{6I, 5I, I, 5, 2\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \{3,3I,6,6I,2,2I\} & \{3I\} & \phi \\ \{X\} & \{4,4I,2\} & \{3I,2\} \\ \{6,5I,5,4I\} & \phi & \{X\} \end{bmatrix} \text{ are in S.}$$

We find A ∪ B and A ∩ B.

$$A \cup B = \begin{bmatrix} \{3,3I,6,6I,2,2I\} & X & \{6,6I,I\} \\ X & \{2,2I,4,4I\} & \{3I,2\} \\ \{3,3I,I,5,5I,4I,6\} & \phi & X \end{bmatrix} \in S.$$

$$A \cap B = \begin{bmatrix} \phi & \{3I\} & \phi \\ \{I\} & \{2\} & \phi \\ \phi & \phi & \{6I,5I,I,5,2\} \end{bmatrix} \text{ is in S.}$$

This is the way ∪ and ∩ are defined for any arbitrary power set P(X) of the set X and the subset matrices take its entries from P(X).

Example 2.11: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \ a_i \in P(X) \right.$$

where $X = \{i_F, 3i_F, 8i_F, 2i_F, 2, 4, 6, 8, 9, 9i_F, 0, 1, 5i_F\} \subseteq C(Z_{10})$,

$$1 \leq i \leq 4\}$$

be a subset of square matrices.

{S, ∪} and {S, ∩} are subset matrix subsemigroups.

We just show how the operations are performed on them.

$$\text{Let } A = \begin{bmatrix} \{6,8,9,9i_F,2i_F\} & X & \{3,2i_F\} \\ \{8,0,1,i_F,5i_F\} & \{3\} & \{9,9i_F,1\} \\ \{i_F,2i_F,5i_F\} & \phi & X \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{8,8i_F,9,9i_F\} & X & \{2i_F,3,0,1\} \\ X & \{8\} & \phi \\ \{1,2,4,i_F,2i_F\} & X & \{0,1,2,4,6,8,9\} \end{bmatrix} \text{ be in S.}$$

$$A \cap B = \begin{bmatrix} \{8,9,9i_F\} & X & \{3,2i_F\} \\ \{8,0,1,i_F,5i_F\} & \phi & \phi \\ \{2i_F,i_F\} & \phi & \{0,1,2,4,6,8,9\} \end{bmatrix}.$$

$$A \cup B = \begin{bmatrix} \{6,8,9,9i_F,2i_F,8i_F\} & X & \{0,3,1,2i_F\} \\ X & \{8,3\} & \{9,1,9i_F\} \\ \{1,2,4,i_F,2i_F,5i_F\} & X & X \end{bmatrix}$$

$A \cup B$ and $A \cap B \in S$ for $A, B \in S$.

Example 2.12: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| \, a_i \in P(Z_5(g));\ g^2 = 0,\ 1 \le i \le 2 \right\}$$

be the collection of 6×2 subset matrices.

Let A = $\begin{bmatrix} \{0,2,2g,3+4g\} & \{1,2,3,g,2+3g,g+1\} \\ \phi & \{4g+4,2g,3g,1+2g\} \\ \{2,3,g\} & X \\ X & \{2,2g\} \\ \{1,g,2g\} & \{4,4g,3\} \\ \{g,1+2g,3g\} & \phi \end{bmatrix}$ and

B = $\begin{bmatrix} X & \{2g,3g,4,4g\} \\ \{3,4,1\} & \phi \\ \{2,2g,1,g\} & \{g,2g,3g,4g\} \\ \{\phi\} & X \\ \{1,2,3,4,0\} & \{1+g,2+g,3+g\} \\ \{1,g,2,2g,3,3g\} & \{4+g,4g,2g,3g,2\} \end{bmatrix}$ be in S.

We find $A \cap B$ and $A \cup B$.

$$A \cap B = \begin{bmatrix} \{0,2,2g,3+4g\} & \{\phi\} \\ \phi & \phi \\ \{2,g\} & \{2g,3g,4g,g\} \\ \phi & \{2,2g\} \\ \{1\} & \phi \\ \{g,3g\} & \phi \end{bmatrix}$$

$A \cup B =$

$$\begin{bmatrix} X & \{1,2,3,g,2+3g,g+1,2g,3g,4,4g\} \\ \{3,4,1\} & \{4g+4,2g,3g,1+2g\} \\ \{2,3,g,2g,1\} & X \\ X & X \\ \{1,2,3,4,g,2g\} & \{2,2g,1+g,2+g,3+g\} \\ \{g,1+2g,3g,1,2,2g,3\} & \{4+g,4g,2g,3g,2\} \end{bmatrix}.$$

Clearly $A \cup B$ and $A \cap B \in S$.

Now we see the subset matrix can be given two operations simultaneously say; $\cup$ and $\cap$.

If S is any subset matrix with entries from a power set P(X) of a set X, certainly $\{S, \cup, \cap\}$ is a distributive lattice. If X is of finite order so is S.

We will illustrate this situation by some examples.

Example 2.13: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in P(X); \right.$$

$$\text{where } X = \{1, 2, 3, 4, 5, 6, 7, 8\}; 1 \leq i \leq 5\}$$

be a subset 5×1 matrix; $(S, \cup, \cap)$ is a subset lattice of P(X).

Example 2.14: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_{13} \\ a_2 & a_{14} \\ \vdots & \vdots \\ a_{12} & a_{24} \end{bmatrix} \middle| \; \text{where } a_i \in P(X), \right.$$

$$X = \{8, I, 7I, 3+4I, -5, -8I+3, 9I+4, -3I, 5-4I, 9+12I, 20I\},$$

$$1 \leq i \leq 24\}$$

be the 12×2 subset column matrix of P(X). $(S, \cup, \cap)$ is a distributive lattice of finite order.

Example 2.15: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in P(Z_{12}(g)) \right.$$

where $X = \{Z_{12}(g) = \{a+bg\} \mid a, b \in Z_{12}, g^2 = 0\}$

be the set of dual modulo integers $1 \leq i \leq 6\}$ be the subset square matrices.

$\{S, \cap, \cup\}$ is a distributive lattice.

Infact $A \cup \{\phi\} = A$ where $\phi = \begin{bmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{bmatrix}$ for all $A \in S$.

$A \cap (\phi) = (\phi)$ for all $A \in S$.

$A \cap (X) = A$ where $X = \begin{bmatrix} X & X & X & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{bmatrix}$ and

$$A \cup (X) = X \text{ for all } A \in S.$$

Example 2.16: Let $S = \{6 \times 10$ subset matrices with entries from the power set. $P(X)$ where $X = \{1,2,3, \ldots,16\}\}$, $\{S, \cup, \cap\}$ is a lattice.

Now having seen examples of semigroups, lattices and semilattices of subset matrices with entries only from a power set $P(X)$ of a set X.

We now proceed onto define substructures of them.

DEFINITION 2.2: *Let S be a semigroup of subset matrices of a power set P(X)of a set X under the binary operation '$\cup$'. Let P $\subseteq$ S if (P, $\cup$) is itself a semigroup of subset matrices under $\cup$ we define (P, $\cup$) to be a subset matrix subsemigroup of S. (This is true if $\cup$ is replaced by the operation $\cap$).*

We define a subset matrix subsemigroup (P, $\cup$) to be a subset matrix ideal if for all p $\in$ P and s $\in$ S, p $\cap$ s is in P (p $\cup$ s $\in$ P in case (S, $\cap$) is the subset matrix semigroup taken for working).

We will first illustrate this situation by some examples.

Example 2.17: Let S ={Collection of all 3 × 3 subset matrices with entries from the power set P(X) where X = {0, 1, 2, …, 19}} be a subset 3 × 3 matrix semigroup under $\cup$. Take

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in P(Y) \right.$$

where Y = {0, 1, 2, …, 10} $\subseteq$ X; 1 $\leq$ i $\leq$ 9} $\subseteq$ S;

(P, $\cup$) is a subset 3 × 3 matrix subsemigroup of X.

Clearly (P, $\cup$) is not a subset matrix ideal of S.

$$\text{For take } A = \begin{pmatrix} \{0,1,2\} & \{3,4\} & \{5,6\} \\ \{10\} & \{9,6\} & \{3,6\} \\ \{1,2,3,4,5\} & \{9,10,1\} & \{7\} \end{pmatrix} \text{ in P.}$$

$$\text{Let } B = \begin{pmatrix} \{9,12\} & \{7,16\} & \{3,14\} \\ \{16\} & \{6,8,15\} & \{19,13,1\} \\ \{5,6,16\} & \{7,8\} & \{2\} \end{pmatrix} \in S.$$

We find A ∪ B

$$= \begin{bmatrix} \{0,1,2,9,12\} & \{3,4,7,16\} & \{3,5,6,14\} \\ \{10,16\} & \{6,8,9,15\} & \{3,6,1,13,19\} \\ \{1,2,3,4,5,6,16\} & \{7,8,9,10,1\} & \{2,7\} \end{bmatrix}.$$

Clearly A ∪ B ∉ P only A ∪ B ∈ S. So P is not an subset matrix ideal of S. Thus P is only subset matrix subsemigroup of S.

Example 2.18: Let S = {Collection of all 4 × 3 subset matrices with entries from the power set P(X), where X = {1, 2, …, 10}} be the subset matrix semigroup under the operation '∩'.

Take P = {Collection of all 4 × 3 subset matrices with entries from the power set P(Y) where Y = {2, 4, 6, 8, 10} ⊆ X}; {P, ∩} is a subset 4 × 3 matrix subsemigroup of S. We see {P ∩} is also a subset 4 × 3 matrix ideal of {S, ∩}.

Inview of these two examples we give the following theorems.

THEOREM 2.1: *Let S be a subset matrix semigroup under '∪'. Let {P, ∪} be a subset matrix subsemigroup of S. {P, ∪} is not an ideal of {S, ∪}.*

Proof is left as an exercise to the reader.

THEOREM 2.2: *Let {S, ∩} be a subset matrix semigroup under the operation ∩. Let (P, ∩) be a subset matrix subsemigroup of S (P, ∩) is a subset matrix ideal of (S, ∩).*

The proof of both the theorems are direct and hence left as an exercise to the reader.

Example 2.19: Let S = {Collection of all 1 × 7 row matrices with entries from P(X) where X = {g, 1+g, 2g, 5g, 3g, 6g, 4g, 2+3g, 5+6g, 1, 2, 3, 4, 5, 6, 3+2g, 6+6g, 5+5g, 4+4g}} be a subset 1×7 row matrix semigroup under $\cup$.

Take P = $\{(a_1\ \phi\ \ a_2\ \phi\ a_3\ \phi\ a_4) \mid a_i \in P(X), 1 \leq i \leq 4\} \subseteq S$ be the subset 1×7 row matrix subsemigroup of S under $\cup$. Clearly (P, $\cup$) is not a subset 1×7 row matrix ideal of S.

For if A = ({3, 2g, 3g}, {5+5g, 4g}, {1+g, 2g, 3g}, {4+4g}, {6+6g, 5+5g}, {6g, 4g, 5g, 3g}, {1,2,3,4}) $\in$ S and B = ({2g, 5+5g}, ϕ, {6+6g, 2+3g}, ϕ, {1,2,3,4,5}, ϕ, {1+g, 2g, 3g, 4g, 5g, 6g}) $\in$ P.

We see A $\cup$ B = ({3, 2g, 3g, 5+5g}, {4g, 5+5g}, {1+g, 2g, 3g, 6+6g, 2+3g}, {4+4g}, {6+6g, 5+5g, 1, 2, 3, 4, 5}, {6g, 4g, 5g, 3g}, {1, 2, 3, 4, 1+g, 2g, 3g, 4g, 5g 6g}) $\notin$ P.

Thus {P, $\cup$} is only a subset 1×7 row matrix subsemigroup of S and is not a subset 1×7 row matrix ideal of S.

Example 2.20: Let S = {Collection of all 5×3 matrices with entries from P(X) where X = $\{C(Z_{10})\}$} be the subset 5×3 matrix semigroup under $\cap$.

Let P = {Collection of all 5×3 subset matrices of the form

$$\begin{pmatrix} \{3g,1\} & \phi & \{2+2g,10,6,8\} \\ \phi & \{5,6,7g\} & \phi \\ \{8+3g,2g\} & \phi & \{1+2g,3g,4g\} \\ \phi & \{1,2,3,4,5,6,7\} & \phi \\ X & \{g,2g,3g,4g,5g\} & \{1+2g,1+3g,1+4g,1+5g\} \end{pmatrix}$$

entries of P are form P(X)} $\subseteq$ S; (P, $\cap$) is a subset 5×3 matrix subsemigroup as well as subset 5×3 matrix ideal of (S, $\cap$).

For take A $\in$ S and B $\in$ P, we see A $\cap$ B $\in$ P, hence the claim.

Thus under all conditions if (S, $\cap$) is a subset matrix semigroup. All subset matrix subsemigroup are subset matrix ideals of (S, $\cap$).

Example 2.21: Let S = {Collection of all subset 3×5 matrices with entries from the powerset P(X) where X = {C (Z_8) (g) = a + bg | a, b $\in$ C(Z_8), g^2 = g}} be the subset 3×5 matrix semigroup under $\cap$.

Let P = {Collection of all subset 3×5 matrices with entries from P(Z_8) $\subseteq$ P(X)} be the 3×5 matrix semigroup under $\cap$. P is a subset 3×5 matrix ideal of S.

If the operation '$\cap$' is replaced by $\cup$ certainly; P is not a subset matrix ideal of S.

Example 2.22: Let S = {Collection of all 8×8 matrices with entries from P(X) where X = Q} be the subset 8×8 matrix semigroup under $\cap$ (or $\cup$). Clearly S is of infinite order. S has subset 8×8 matrix subsemigroups and ideals under $\cap$. (However S has subset 8×8 matrix subsemigroups which are not subset 8×8 matrix ideals of S under $\cup$).

Now we proceed onto define in case of subset matrix lattices with entries from a power set P(X) the concept of sublattice of subset matrices, ideals of subset matrices and filters of subset matrices.

We just give an informal definition of these notions.

Consider S = {Collection of all $m \times n$ matrices with entries from a powerset P(X) of the set X}, (S, $\cup$, $\cap$) is a lattice of subset $m \times n$ matrices.

Let {P, $\cap$, $\cup$} $\subseteq$ {S, $\cup$, $\cap$}, if P by itself is a lattice of subset $m \times n$ matrices we define {P, $\cap$, $\cup$} to be the sublattice of subset of $m \times n$ matrices or subset $m \times n$ matrix sublattice of S.

A subset (P, $\cup$) is a subsemilattice of subset matrices (S, $\cup$) and for all p $\in$ P and s $\in$ S, p $\cap$ s $\in$ P then P is defined as the subset matrix ideal of the subset matrix lattice S.

If {F, $\cap$} is a subsemilattice of subset matrices and for all s $\in$ S and f $\in$ F; s $\cup$ f $\in$ F; then we define F to be a subset matrix of filter of the subset matrix lattice of S.

We will illustrate both the situations by some simple examples.

Example 2.23: Let

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \text{ a, b, c, d} \in P(X) \text{ where } X = \{1, 2, 3, 4, 5, 6\} \right\}$$

be a lattice of subset matrices.

$$\text{Take } P = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \text{ a, b, c, d} \in P(Y);\ Y = \{1, 3, 5\} \right\} \subseteq S,$$

{P, $\cup$} is a subset matrix ideal of S. {P, $\cap$} $\subseteq$ S is a subset matrix of sublattice of S.

Clearly P is not a subset matrix filter of S.

$$\text{For if } s = \begin{pmatrix} \{6,1\} & \{2,3\} \\ \{1,2,4\} & \{4,6,5\} \end{pmatrix} \text{ and } p = \begin{pmatrix} \{1,3\} & \phi \\ \{5\} & \{5,1\} \end{pmatrix} \in P.$$

$$\{P, \cap\} \text{ is also a semilattice so } s \cap p = \begin{pmatrix} \{1\} & \phi \\ \phi & \{5\} \end{pmatrix} \in P.$$

$$s \cup p = \begin{pmatrix} \{1,3,6\} & \{2,3\} \\ \{1,2,4,5\} & \{4,6,5,1\} \end{pmatrix} \notin P;$$

so P is not a filter of S.

We see P is not a filter but P is a ideal of subset matrix lattice.

Example 2.24: Let

$$S = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \right| a_i \in P(X); \right.$$

$$X = \{1, 2, \ldots, 18\},\ 1 \leq i \leq 8\}$$

be the subset matrix of lattice.

$$P = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \right| a_i \in P(Y); \right.$$

$$Y = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}; 1 \leq i \leq 8\} \subseteq S; \{P, \cup, \cap\}$$

and $\{P, \cap, \cup\}$ are just subset matrix sublattices of S.

$\{P, \cup\}$ is a subset matrix ideal of S; $\{P, \cap\}$ is not a subset matrix filter of S.

Interested reader can find examples of subset matrix ideals and subset matrix filters.

Now we proceed onto define subset matrix of a semigroup.

DEFINITION 2.3: *Let S = {Collection of all m × n matrices with entries from a semigroup P, under product ×}. If for A, B ∈ S we define in A × B; the product of subsets as product operation in the semigroup. Then (S, ×) is again a semigroup called the subset matrix semigroup of the semigroup (P, ×). (The × can be usual product of matrices or natural product $\times_n$ of matrices).*

We will illustrate this situation by some examples.

Example 2.25: Let S = {collection of all 1 × 5 row subset matrices from the subset of the semigroup P = $\{Z_4, \times\}$}.

Take A = ({0, 2}, {1}, {3,1}, {2,3}, {1,0})

and B = ({0,1,2}, {0,1}, {1}, {1,2},{3,1}) in S.

We see A × B = ({0,2}, {0,1}, {3,1}, {2,0,3}, {3,1,0}) ∈ S.

Clearly $\phi \notin S$.

(S, ×) is a commutative subset row matrix semigroup of the semigroup $\{Z_4, \times\}$.

Example 2.26: Let S = {Collection of all 7 × 1 subset matrices from the subsets of the semigroup $\{Z_6, \times\}$}, be the subset matrix semigroup of $\{Z_6, \times\}$.

$$\text{If } A = \begin{bmatrix} \{0,3\} \\ \{5,2\} \\ \{1,2,3\} \\ \{5,1,4\} \\ \{4,1\} \\ Z_6 \end{bmatrix} \text{ and } B = \begin{bmatrix} \{0,5,2\} \\ \{1,2\} \\ \{3,4\} \\ \{3,4,5\} \\ \{1,4,2,5\} \\ \{2,3,4,0\} \end{bmatrix} \text{ are in S.}$$

$$A \times B = \begin{bmatrix} \{0,3\} \\ \{5,2,4\} \\ \{3,4,2\} \\ \{3,2,1,4,5\} \\ \{1,4,2,5\} \\ Z_6 \end{bmatrix} \in S.$$

Example 2.27: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \text{ belongs to the subsets of} \right.$$

$$\left. \text{the semigroup } \{Z_8, \times\};\ 1 \le i \le 9 \right\}$$

be the subset 3 × 3 square matrix semigroup of the semigroup $\{Z_8, \times\}$.

Take

$$A = \begin{bmatrix} \{0,1,2,3\} & \{5,6,1\} & \{4,2,6,0\} \\ \{0,3,7\} & \{2,5\} & \{6,7,1\} \\ \{0,1,4,6\} & Z_8 & \{3,6,7,1\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{0,1,4\} & \{7,2\} & \{5,6\} \\ \{6,5,4\} & \{3,7\} & \{3,0\} \\ Z_8 & \{3,1,2\} & \{0,6,2\} \end{bmatrix} \text{ in S,}$$

$$A \times B = \begin{bmatrix} \{0,1,2,3,4\} & \{7,2,1,4,3\} & \{2,4,0,6\} \\ \{0,2,7,5,4,3\} & \{6,2,3,7\} & \{0,2,5,3\} \\ Z_8 & Z_8 & \{0,6,2,4\} \end{bmatrix} \in S.$$

Example 2.28: Let S = {Collection of all 2 × 7 subset matrices with entries from subset of $\{Z_3, \times\}$; the semigroup} be the subset 2 × 7 matrix semigroup of the semigroup $\{Z_3, \times\}$.

Let A = ({0}, {1}, {1 2}, {1, 0}, {2, 0}, {1}, {2}) and B = ({1}, {0,1}, {2},{0},{1},{2,1}, {2,0}) be in S.

We find A × B = ({0}, {0,1},{2,1},{0},{2,0},{2,1}, {1,0}) is in S. Thus S is a subset 2 × 7 matrix semigroup of the semigroup $\{Z_3, \times\}$.

Example 2.29: Let S = {Collection of all subset 4 × 2 matrices with subsets from the semigroup $\{Z_{10}, \times\}$} the subset 4 × 2 matrix semigroup of the semigroup $\{Z_{10}, \times\}$.

$$\text{Let } A = \begin{bmatrix} \{5\} & \{1,2,3,4\} \\ \{3,4,5,6\} & \{2,7\} \\ \{1,2,3,4,5\} & \{7,8,6,1,9,4\} \\ \{4,2,0\} & \{1\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{0,2,6,4,8\} & \{1,3,5,7,9\} \\ \{2\} & \{3\} \\ \{5,1,0\} & \{9,1,2,6\} \\ \{7\} & \{1,4,5\} \end{bmatrix} \text{ be in S.}$$

$$A \times B = \begin{bmatrix} \{0\} & \{13,4,5,7,9,2,6,8\} \\ \{6,8,2\} & \{6,1\} \\ \{5,1,0,2,3,4\} & \{7,8,6,1,9,4,3,2\} \\ \{8,4,0\} & \{1,4,5\} \end{bmatrix} \text{ is in S.}$$

Example 2.30: Let S = {Collection of all 5 × 2 subset matrices from the subsets of the semigroup P = $\{Z_{15}, \times\}$} be the subset matrix semigroup of the semigroup $\{Z_{15}, \times\}$.

$$\text{Take two subsets A, B} \in \text{S where } A = \begin{bmatrix} \{7,8\} & \{9,10,11\} \\ \{12,13\} & \{14,5,0\} \\ \{1,2\} & \{3,4,5\} \\ \{6,7\} & \{8,9,10\} \\ \{11,12\} & \{13,14,0\} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \{1,2,3,4\} & \{2,3\} \\ \{5,6,7,8\} & \{6,7\} \\ \{9,10,11,12\} & \{10,11\} \\ \{13,14,0,1\} & \{14,0\} \\ \{2,3,4,5\} & \{3,4\} \end{bmatrix} \in S.$$

$$A \times B = \begin{bmatrix} \{7,14,6,13,8,1,9,2\} & \{3,12,5,0,7\} \\ \{0,12,9,6,5,3,1,14\} & \{9,0,5,8\} \\ \{9,10,11,12,3,5,7,9\} & \{3,10,14,5\} \\ \{6,0,3,9,7,18\} & \{0,14,5,6\} \\ \{7,3,14,10,9,6\} & \{0,9,12,7,11\} \end{bmatrix} \in S.$$

Thus {S, ×} is a subset 5 × 2 matrix semigroup of the semigroup $\{Z_{15}, \times\}$.

Example 2.31: Let S = {Collection of all subset 2 × 5 matrices with entries from the subsets of the semigroup $\{Z_{12}, \times\}$} be the subset matrix semigroup of the semigroup $\{Z_{12}, \times\}$.

$$\text{Let } A = \begin{pmatrix} \{0,4,6\} & \{4,6,7,8,1\} & \{3\} & \{1,3\} & \{9\} \\ \{1,2\} & \{1\} & \{5,0\} & \{7,2\} & \{0\} \end{pmatrix}$$

$$\text{and } B = \begin{pmatrix} \{6,1\} & \{8,1,5,9\} & \{5,2\} & \{4\} & \{0\} \\ \{8,9\} & \{10,1\} & \{3\} & \{6\} & \{7,9\} \end{pmatrix} \text{ be in S.}$$

$$A \times B = \begin{pmatrix} \{0,4,6\} & \{8,4,6,9,5,1\} & \{6,15\} & \{4\} & \{9\} \\ \{8,9,4,6\} & \{10,1\} & \{0\} & \{6\} & \{0\} \end{pmatrix} \in S.$$

Now we have seen several subset matrix semigroup of the semigroup.

We now study the structure using the group.

Example 2.32: Let S = {Collection of all subset 3 × 5 matrices from the subset of the group {Z_7, +}} be the subset matrix of the group {Z_7, +}.

$$\text{Let } A = \begin{bmatrix} \{0\} & \{1,2\} & \{3,4\} & \{1\} & \{6\} \\ \{2,1\} & \{0\} & \{5\} & \{0\} & \{2\} \\ \{1,3,2\} & \{1\} & \{3,5,1\} & \{6,1\} & \{3,2\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{1,2\} & \{0\} & \{1,3,5\} & \{6,2,3\} & \{3\} \\ \{3\} & \{1,2\} & \{6,1\} & \{5\} & \{1,2,3,4\} \\ \{1,2,3\} & \{4,5,6\} & \{1,2\} & \{4,6,0\} & \{1,0,3,5\} \end{bmatrix} \text{ be in S.}$$

Now

A + B =

$$\begin{bmatrix} \{1,2\} & \{1,2\} & \{4,6,1,5,0,2\} & \{0,3,4\} & \{2\} \\ \{5,4\} & \{1,2\} & \{6,4\} & \{5\} & \{3,4,5,6\} \\ \{2,3,4,5,6\} & \{5,6,0\} & \{4,6,2,5,0,3\} & \{3,5,6,1,0\} & \{4,3,6,1,2,0,5\} \end{bmatrix}$$

$\in$ S.

(S, +) is a subset matrix semigroup of the group.

Example 2.33: Let S = {Collection of all subsets 3 × 3 matrix with entries from the subsets of the group {Z_{12}, +}} be the subset 3 × 3 matrix semigroup of the group order +; i.e., {S, +} is a semigroup.

$$\text{Let } A = \begin{bmatrix} \{3,7\} & \{0\} & \{6,5\} \\ \{1,2,3\} & \{0,6,8\} & \{5,6,7,8,9\} \\ \{10,11,0\} & \{1,2,3,4,5\} & \{6,7,8,9\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{1,2\} & \{3,4,5,6,7\} & \{8,9,10\} \\ \{11\} & \{0,1,2,3,4\} & \{5,6\} \\ \{7,8,9\} & \{10,11\} & \{0,1,2\} \end{bmatrix} \text{ be in S.}$$

We find A + B =

$$\begin{bmatrix} \{4,8,9,5\} & \{3,4,5,6,7\} & \{2,3,4,1\} \\ \{0,1,2\} & \{0,1,2,3,4,7,6,8,9,11,10\} & \{11,0,10,1,2,3\} \\ \{7,8,9,5,6\} & \{11,0,1,2,3,4\} & \{6,7,8,9,10,11\} \end{bmatrix} \in S.$$

(S, +) is a subset 3 × 3 matrix semigroup of the group G.

Example 2.34: Let S = {collection of all subset 2 × 4 matrices with subsets from the group G = {1, g, g^2, …, g^5 where $g^6 = 1$}; under ×} be the subset 2 × 4 matrix semigroup of the group {G, ×}. Let us take A and B in S where

$$A = \begin{pmatrix} \{1\} & \{g,g^2\} & \{g,g^3,1\} & \{1,g,g^5\} \\ \{1,g^2,g^4\} & \{g^3\} & \{g\} & \{1\} \end{pmatrix}$$

and

$$B = \begin{pmatrix} G & \{1,g\} & \{g^2,g^3,g^5,1,g\} & \{1,g^5,g^4\} \\ \{g,g^2,g^3\} & \{g^4\} & \{1\} & \{1,g,g^2,g^3\} \end{pmatrix} \in S.$$

A × B =

$$\begin{bmatrix} G & \{g,g^2,g^3\} & \{1,g,g^2,g^3,g^5,g^4\} & \{1,g^4,g^5,g^3\} \\ \{g,g^2,g^3,g^4,g^5,1\} & \{g\} & \{g\} & \{1,g,g^2,g^3\} \end{bmatrix}$$

is in (S, ×).

(S, ×) is a subset 2 × 4 matrix semigroup of the group G = {g | $g^6 = 1$}.

Example 2.35: Let S = {Collection of all subset 8 × 1 matrices from the subsets of the group (Z,+)} be the subset 8 × 1 matrix semigroup of the group Z under '+'.

$$\text{Let } A = \begin{bmatrix} \{0,1,-1\} \\ \{2,-3,6\} \\ \{-8,-11,-5\} \\ \{1\} \\ \{-1\} \\ \{0\} \\ \{-7,8,9\} \\ \{-10,12\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{1,2\} \\ \{3,4\} \\ \{5,6\} \\ \{7,8\} \\ \{9,10\} \\ \{11,12\} \\ \{13,14\} \\ \{15,16\} \end{bmatrix} \text{ be in S.}$$

$$\text{Now we find } A + B = \begin{bmatrix} \{0,2,3,1\} \\ \{5,6,0,1,9,10\} \\ \{-3,-2,-6,-5,0,-1\} \\ \{8,9\} \\ \{8,9\} \\ \{11,12\} \\ \{6,7,21,22,23\} \\ \{5,6,27,28\} \end{bmatrix} \in \text{S.}$$

Thus (S, +) is a subset 8 × 1 matrix semigroup of the group. Clearly this semigroup subset matrix is of infinite order.

Now we proceed onto give illustration of substructures of subset matrix semigroups.

Example 2.36: Let S = {Collection of all subset 2 × 2 matrices with subsets from the group, $\{Q^+ \setminus \{0\}, \times\}$ be the subset 2 × 2 matrix semigroup of the group $\{Q^+, \times\}$.

Let A = $\begin{bmatrix} \{1,2,3,4,5,7\} & \{\sqrt{7},\sqrt{3}\} \\ \{5,8,11/2,7\} & \{3\sqrt{7},5\sqrt{3},10,11\} \end{bmatrix}$ and

B = $\begin{bmatrix} \{4,3\} & \{7\sqrt{7},3\sqrt{3},5\} \\ \{7,8,1\} & \{1,2\} \end{bmatrix}$ be in S.

$$A \times B = \begin{bmatrix} \begin{matrix}\{4,8,12,16,20,28, \\ 3,6,9,15,21\}\end{matrix} & \begin{matrix}\{49,9,7\sqrt{21},3\sqrt{21}, \\ 5\sqrt{7},5\sqrt{3}\}\end{matrix} \\ \begin{matrix}\{35,40,5,56,7,64,8,77/2, \\ 4,11/2,49\}\end{matrix} & \begin{matrix}\{3\sqrt{7},5\sqrt{3},10,11,22, \\ 20,6\sqrt{7},10\sqrt{3}\}\end{matrix} \end{bmatrix} \in S.$$

Example 2.37: Let S = {Collectiin of all subset 3 × 2 matrices with entries as subsets from the semigroup {Z_{16}, ×}} be the subset 3 × 2 matrix semigroup of the semigroup {Z_{16}, ×}.

Take P = {Collection of all subset 3 × 2 matrixces with subsets from the subsemigroup {0, 2, 4, 6, 8, 10, 12, 14} ⊆ Z_{16}} ⊆ S} be the subset 3 × 2 matrix subsemigroup of S.

Let B = $\begin{bmatrix} \{8,4\} & \{0,2\} \\ \{2,4,6\} & \{10,12\} \\ \{14,0\} & \{0\} \end{bmatrix}$ and

A = $\begin{bmatrix} \{6,2\} & \{4,6,8\} \\ \{0,4\} & \{0,4,6,10,12\} \\ \{0,6,2\} & \{8,2,0\} \end{bmatrix}$ ∈ P.

A × B = $\begin{bmatrix} \{0,8\} & \{0,8,12\} \\ \{0,8\} & \{0,8,12,4\} \\ \{0,4,12\} & \{0\} \end{bmatrix}$ ∈ P.

Infact it can be easily observed; P is an ideal of S. For take the same $A \in P$ and

$$M = \begin{bmatrix} \{3,5,2\} & \{0,11,7,5\} \\ \{2,4,11,3\} & \{0,13,15\} \\ \{1,2,3,5\} & \{7,11,9,13\} \end{bmatrix} \text{ be in S.}$$

We find

$$M \times A = \begin{bmatrix} \{6,10,4,2,14,12\} & \{0,12,4,8,2,10,14\} \\ \{0,8,12\} & \{0,4,12,14,10,2,6\} \\ \{0,2,4,6,10,12,14\} & \{0,14,6,2,10,8\} \end{bmatrix} \in P.$$

Hence P is a subset matrix ideal of the semigroup we can have subsemigroup which are not ideals in this case also.

Example 2.38: Let S = {Collection of all subset 6×1 matrices with entries from the subset of the semigroup $\{Z_{20}, \times\}$} be the subset matrix semigroup of the semigroup $\{Z_{20}, \times\}$.

Take M = {Collection of all subset 6×1 matrices from the subsets of the set $\{0, 5, 10, 15\} \subseteq Z_{20}$} be the subset matrix subsemigroup of S.

$$\text{For take } A = \begin{bmatrix} \{0\} \\ \{5\} \\ \{10\} \\ \{5,10\} \\ \{5,10,15\} \\ \{0,5\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{5,10\} \\ \{10,5\} \\ \{5,10,15\} \\ \{0\} \\ \{10,5\} \\ \{5,10,15\} \end{bmatrix} \in S.$$

$$A \times B = \begin{bmatrix} \{0\} \\ \{10,5\} \\ \{10,0\} \\ \{0\} \\ \{5,10,0\} \\ \{0,5,10,15\} \end{bmatrix} \in M.$$

We see M is also a subset matrix ideal of the subset matrix semigroup of the semigroup S.

Example 2.39: Let S = {Collection of all subset 1 × 5 matrices from the subsets of the group; G = {Z_{12}, +}} be the subset matrix semigroup of the group G.

Take M = {Collection of all subset 1 × 5 matrices from the subsets of the set {0, 2, 4, 6, 8, 10} ⊆ G} be the subset matrix subset of S under '+'.

M is only a subset matrix subsemigroup and is not an ideal.

For if

X = ({3, 4, 7}, {1, 5, 11}, {9}, {10, 11, 3}, {5, 3, 2, 1, 11}) ∈ S and A = ({2, 4, 6}, {8, 10, 0}, {2, 4}, {6, 2}, {8, 10}) ∈ M.

We see

X + A = ({5, 7, 9, 6, 8, 10, 11, 1}, {9, 11, 1, 3, 5, 11, 7}, {11, 1}, {4, 0, 5, 1, 9, 5}, {1, 11, 10, 9, 7, 3, 0}) ∉ M.

So M is not a matrix subset ideal of the semigroup only a subsemigroup.

Example 2.40: Let S = {Collection of all 3 × 1 subset matrices of the subset of the group, (Z, +)} be the subset 1 × 3 matrix semigroup under '+'. Take M = {Collection of all 3 × 1 subset matrices of the set {2Z, +} ⊆ (Z, +)} ⊆ S. M is only a subset matrix subsemigroup of S. Clearly M is not a subset matrix ideal of S.

$$X = \begin{bmatrix} \{0,-3,9\} \\ \{5,-7,11\} \\ \{13,15,49\} \end{bmatrix} \in S \text{ and } A = \begin{bmatrix} \{0,2,4,8\} \\ \{10,16,64\} \\ \{120,-144,80\} \end{bmatrix} \in M.$$

$$X + A = \begin{bmatrix} \{0,-1,1,5,9,11,13,17\} \\ \{15,21,69,3,9,57,21,27,75\} \\ \{133,-131,93,135,-129,95,169,-95,31\} \end{bmatrix} \notin M;$$

so M is not a subset matrix ideal of the subset matrix semigroup S.

Example 2.41: Let S = {Collection of all 6 × 1 subset matrices with entries from the subsets of the group S_3} be the subset 6 × 1 matrix semigroup of the group. Clearly S is a non commutative subset matrix semigroup of finite order.

Let P = $\{(a_1, a_2, a_3, a_4, \{e\}, \{e\}) \mid a_i \in$ {subsets of the set e, $p_1\}$ $1 \leq i \leq 4\} \subseteq S$ be the subset matrix subsemigroup of the group S_3.

Take A and B in P where A = $(\{e, p_1\}, \{p_1\}, \{p_1, e\}, \{e\}, \{e\}, \{e\})$ and B = $\{\{p_1\}, \{p_1\}, \{e\}, \{e\}, \{e\}, \{e\})$ be in S. We see A × B = $(\{e, p_1\}, \{e\}, \{p_1, e\}, \{e\}, \{e\}, \{e\}) \in S$.

Thus P is a subset matrix subsemigroup of the group S_3. Clearly P is a commutative subset matrix subsemigroup of the non commutative subset matrix subsemigroup. Further P is not a subset matrix ideal of the subset matrix semigroup S of the group S_3. For take
M = $\{\{e, p_4\}, \{p_5\}, \{p_4\}, \{e, p_4, p_5\}, \{p_4, p_5\}, \{p_3\}) \in S$ and A = $(\{p_1\}, \{1, p_1\}, \{p_1\}, \{p_1\}, \{e\}, \{e\}) \in P$. AM = $(\{p_1 p_3\}, \{p_5 p_2\}, \{p_3\}, \{p_1, p_2, p_3\}, \{p_4, p_5\}, \{p_3\}) \notin P$.

$$MA = (\{p_1 p_2\}, \{p_5 p_3\}, \{p_2\}, \{p_1, p_2, p_3\}, \{p_4, p_5\}, \{p_3\}) \notin P.$$

We see MA ≠ AM and P is not a subset matrix ideal of the subset matrix semigroup S of the group S_3.

Take M = {Collection of all subset 6×1 matrices with entries from the subsets of $\{e, p_4, p_5\}$} be the subset matrix subsemigroup of S.

Example 2.42: Let S = {Collection of all subset 1×5 matrices with entries from the subsets of the semigroup S(5)} be the subset matrix semigroup of the symmetric semigroup S(5).

We see S is also non commutative subset matrix semigroup of the symmetric semigroup.

Example 2.43: Let S = {Collection of all subset 3×3 matrices with subsets taken from the group $D_{2,7}$} be the subset 3×3 matrix semigroup of the group $D_{2,7}$. Clearly S is a non commutative subset matrix semigroup of the group $D_{2,7}$.

Example 2.44: Let S = {Collection of all subset 7×1 matrices with entries from the subsets of the group S_{20}} be the subset 7×1 matrix semigroup of the group S_{20} which is clearly non commutative.

Example 2.45: Let S = {Collection of all subset 3×7 matrices from the subsets of the group $G = S_8 \times D_{2,7}$} be the subset 3×7 matrix semigroup of the group G. Clearly S is non commutative.

Thus we can say in general a subset $m \times n$ matrix semigroup can have zero divisors if and only if the semigroup over which it is built has zero and the semigroup is also assumed to be under product.

We will first illustrate this by some examples.

Example 2.46: Let S = {Collection of all subset 5×1 matrices with subsets from the semigroup $\{Z_{12}, \times\}$} be the subset 5×1 matrix semigroup of the semigroup $\{Z_{12}, \times\}$. This subset matrix semigroup has zero divisors and idempotents.

Take A = $\begin{bmatrix} \{0\} \\ \{2,4,9\} \\ \{4\} \\ \{6\} \\ \{4,8\} \end{bmatrix}$ and B = $\begin{bmatrix} \{1,2,3,4,5,6\} \\ \{0\} \\ \{6,3\} \\ \{4,8,2\} \\ \{6,3,9\} \end{bmatrix}$ in S

we see A × B = $\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$ $\in$ S is a zero divisor in S.

Consider A = $\begin{bmatrix} \{4\} \\ \{1\} \\ \{9\} \\ \{0\} \\ \{4\} \end{bmatrix}$ $\in$ S.

We see A^2 = $\begin{bmatrix} \{4\} \\ \{1\} \\ \{9\} \\ \{0\} \\ \{4\} \end{bmatrix}$ = A is an idempotent in S.

Thus S has idempotents and zero divisors.

Also take X = $\begin{bmatrix} \{6\} \\ \{0,6\} \\ \{0\} \\ \{6\} \\ \{0,6\} \end{bmatrix}$ $\in$ S we see X^2 = $\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$.

Thus we have seen S has idempotents, nilpotents and zero divisors.

However now we are going to show S has elements which can contribute to dual like special dual like number and special quasi dual numbers.

$$\text{Take } X = \begin{bmatrix} \{6\} \\ \{0,6\} \\ \{6\} \\ \{0,6\} \\ \{6\} \end{bmatrix} \in S. \text{ Clearly } X^2 = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \text{ so,}$$

this X can act for the generation of dual number a + bX as $X^2 = (\{0\})$.

$$\text{Now take } Y = \begin{bmatrix} \{9\} \\ \{4\} \\ \{4\} \\ \{9\} \\ \{9\} \end{bmatrix} \in S \text{ we see } Y^2 = \begin{bmatrix} \{9\} \\ \{4\} \\ \{4\} \\ \{9\} \\ \{9\} \end{bmatrix} = Y.$$

Now this Y can be used to get the special dual like numbers of the form a + bY with $Y^2 = Y$.

$$\text{Finally consider } Z = \begin{bmatrix} \{8\} \\ \{3\} \\ \{8\} \\ \{3\} \\ \{8\} \end{bmatrix} \in S. \text{ We see } Z^2 = \begin{bmatrix} \{4\} \\ \{9\} \\ \{4\} \\ \{9\} \\ \{9\} \end{bmatrix} = -Z,$$

hence Z can be used to get special quasi dual numbers.

So using subset matrix semigroup S we can have dual numbers, special dual like numbers and special quasi dual numbers.

Example 2.47: Let S = {Collection of all subset 1 × 6 matrices with entries from the subsets of the semigroup $\{Z_{36}, \times\}$} be the subset 1 × 6 matrix semigroup of the semigroup $\{Z_{36}, \times\}$.

Take
AA = ({0, 6}, {0,12,6}, {0,12},{0,18}, {0,6,18}, {0,6,18}) ∈ S.
We see A^2 = ({0}, {0}, {0},{0},{0},{0}) that is a + b A is the dual number collection for varying a and b reals.

Infact S has zero divisors for

X = ({6}, {0}, {6, 12}, {12}, {18}, {0}) and

Y = ({12, 18}, {3, 5, 7}, {18}, {6}, {6, 12}, {1, 2, 3, 4, 5, 6, 7}) are in S it is easily verified

X × Y = ({0}, {0}, {0}, {0}, {0}, {0}), so S has zero divisors.

S has idempotents also, for take

X = ({9, 0}, {0}, {0}, {9}, {9, 0}, {9}) ∈ S is such that

X^2 = ({0,9}, {0}, {0}, {9}, {0,9}, {9}) = X so this X can serve as the special dual like number of S.

Take P = ({8}, {27}, {8}, {27}, {8}, {27}) ∈ S is such that P^2 = ({28}, {9}, {28}, {9}, {28}, {9})

= ({–8}, {–27}, {–8}, {–27}, {–8}, {–27}) ∈ S; so P can serve as the special quasi dual element.

It is left as an exercise to find such elements in subset matrix semigroup of a semigroup.

Example 2.48: Let S = {Collection of all 7 × 1 subset matrices with entries from the subsets of the semigroup $\{Z_{15}, \times\}$} be the subset 7 × 1 matrix semigroup of the semigroup $\{Z_{15}, \times\}$.

Take $A = \begin{bmatrix} \{6,12\} \\ \{0,3\} \\ \{9,6,12\} \\ \{3,12,6\} \\ \{12\} \\ \{6,0\} \\ \{0,9,12\} \end{bmatrix} \in S$; we have $B = \begin{bmatrix} \{5,10\} \\ \{5,10,0\} \\ \{5\} \\ \{10\} \\ \{5,10\} \\ \{0,10\} \\ \{0,10,5\} \end{bmatrix}$ in S

such that

$$A \times B = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}.$$

Thus S has zero divisors.

$M = \begin{bmatrix} \{0,6\} \\ \{10\} \\ \{0,6\} \\ \{10\} \\ \{6\} \\ \{0\} \\ \{0,6\} \end{bmatrix} \in S$ we see $M^2 = \begin{bmatrix} \{0,6\} \\ \{10\} \\ \{0,6\} \\ \{10\} \\ \{6\} \\ \{0\} \\ \{0,6\} \end{bmatrix} = M \in S$.

So S has nontrivial idempotent.

Example 2.49: Let S = {Collection of all subset 3×3 matrices with entries from the subsets of the symmetric semigroup S(8)} be the subset 3×3 matrix semigroup of the symmetric semigroup S(8).

Clearly S has no zero divisors.

Example 2.50: Let S = {Collection of all subset 5×3 matrices with entries from the subsets of the group S_9} be the subset 5×3 matrix semigroup of the group S_9.

Clearly S has no zero divisors or idempotents.

Now having seen examples of zero divisors subset matrix structure we now proceed onto give examples of subset $n \times m$ matrix semirings over a semifield or a semiring.

Example 2.51: Let S = {Collection of all subset 3×1 matrices with entries from the subsets of the semifield $Z^+ \cup \{0\}$} be the subset 8×1 matrix semiring of the semifield.

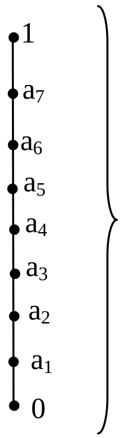
Let A = ({0}, {2, 4, 8, 9, 1}, {8, 5}) and
B = ({4, 9, 8, 21, 28, 103, 148, 1}, {0}, {0}) be in S we see $A \times B$ = ({0}, {0}, {0}) but $A \neq (0)$ and $B \neq (0)$.

So A, B is a zero divisor of S that is why we can say S is only a semiring of subset matrices.

Clearly S is not a semifield. Further S is of infinite order. Infact S is a commutative semiring.

Now we give examples of subset matrices from semirings or semifields which are algebraic structures with two binary operations.

Example 2.52: Let S = {Collection of all 2×5 subset matrices with entries from the semifield



be the subset matrix semiring of the semifield.

Let $A = \begin{pmatrix} \{0,1,a_1\} & \{a_3\} & \{a_2,a_5\} & \{1\} & \{a_7\} \\ \{a_7\} & \{a_3,a_2\} & \{a_1,a_4\} & \{a_5\} & \{1,a_2,a_7\} \end{pmatrix}$ and

$$B = \begin{pmatrix} \{0,a_3,a_4\} & \{a_4\} & \{a_2,a_4,a_7\} & \{a_6,0,a_1\} & \{a_7,0,1\} \\ \{a_4,a_5,a_2\} & \{a_2,a_3\} & \{a_1,0,1\} & \{a_7,0,a_5\} & \{1\} \end{pmatrix}$$

be in S. '+' is the lattice union and × is the lattice intersection.

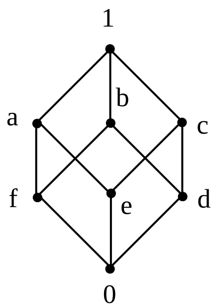
$$A + B = \begin{pmatrix} \{0,1,a_3,a_4,a_1\} & \{a_4\} & \{a_2,a_4,a_5,a_7\} & \{1\} & \{a_7,1\} \\ \{a_7\} & \{a_2,a_3\} & \{1,a_1,a_4\} & \{a_7,a_5\} & \{1\} \end{pmatrix}$$

and

$$A \times B = \begin{pmatrix} \{0,a_3,a_4,a_1\} & \{a_3\} & \{a_2,a_4,a_5\} & \{0,a_6,a_1\} & \{0,a_7\} \\ \{a_4,a_2,a_5\} & \{a_2,a_3\} & \{0,a_1,a_4\} & \{0,a_5\} & \{1,a_2,a_7\} \end{pmatrix}.$$

Clearly A + B and A × B are in S.

Example 2.53: Let S = {Collection of all 5 × 11 subset matrices with entries from the subsets of the semiring



be the subset 5 × 1 matrix semiring of the semiring.

$$\text{Let } A = \begin{bmatrix} \{0,a\} \\ \{0,b,1\} \\ \{a,b,d\} \\ \{c,d,0\} \\ \{d,a,e\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{a,b,c\} \\ \{1,0,d\} \\ \{f,0,1\} \\ \{b,e,1\} \\ \{0,1\} \end{bmatrix} \text{ be in S.}$$

$$A \cup B = \begin{bmatrix} \{a,1\} \\ \{1,c,b,d\} \\ \{1,a,b,d\} \\ \{1,e,b,c\} \\ \{1,d,a,e\} \end{bmatrix} \in S.$$

$$A \cap B = \begin{bmatrix} \{0,a,f,e\} \\ \{0,b,c,d,1\} \\ \{a,b,d,1,0,f\} \\ \{0,c,d\} \\ \{d,a,e,0\} \end{bmatrix} \in S.$$

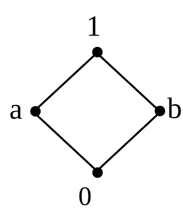
Thus S is only a semiring.

For if $X = \begin{bmatrix} \{f\} \\ \{0\} \\ \{d\} \\ \{b\} \\ \{e\} \end{bmatrix}$ and $Y = \begin{bmatrix} \{e\} \\ \{a\} \\ \{e\} \\ \{0\} \\ \{0\} \end{bmatrix} \in S$.

$$\text{We see } X \cap Y = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix};$$

thus S has zero divisors which proves S cannot be a semifield.

Example 2.54: Let S = {Collection of all 1 × 4 subset row matrices with entries from the subsets of the semiring



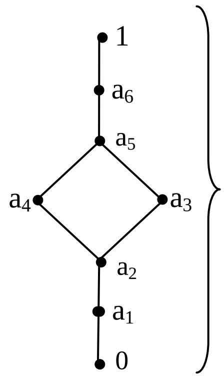
}

be the subset 1 × 4 row matrix semiring of the semiring.

Let $X = \begin{bmatrix} \{0\} \\ \{b\} \\ \{a\} \\ \{1\} \end{bmatrix}$ and $Y = \begin{bmatrix} \{a\} \\ \{a\} \\ \{b\} \\ \{0\} \end{bmatrix} \in S$,

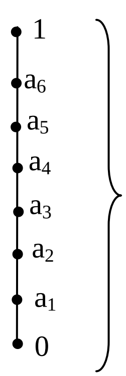
we see $X \times Y = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$ is a zero divisor so is not a semifield.

Example 2.55: Let S = {Collection of all subsets of 1×7 matrices where the subsets are from the semiring



be the subset 1×7 matrix semiring. For S has zero divisors. Take A = ({0}, $\{a_1\}$, $\{b_1\}$, {0}, $\{a_6\}$, $\{a_5\}$, {0}) and B = ($\{a_6\}$, {0}, {0}, $\{a_1\}$, {0}, {0}, {1}) $\in$ S is such that A $\cap$ B = ({0},{0},{0},{0}, {0},{0},{0}). So S is only a semiring. It is important to record at this juncture that even if the entries of the subset matrix are from the subset of a semfield still the resultant need not in general be a semifield in most cases it is a semiring.

Example 2.56: Let S = {Collection of all subset 5×1 matrices with subsets from the semifield



be the subset matrix semiring. Clearly S has zero divisors S is only a semiring and not a semifield though the subsets are from the semifield.

Next we proceed onto give examples of subset m × n matrices with subsets from a ring or a field.

Example 2.57: Let S = {Collection of all subset 3 × 2 matrices with subsets from the ring Z_{12}} be the subset 3 × 2 matrix semiring of the ring.

We see S in general is not a ring. S can only be a semiring. For subsets under any inherited operations never form a group only a semiring. But by this method we get many semirings.

We find for A, B ∈ S.

$$\text{Let } A = \begin{bmatrix} \{3,4\} & \{6,2\} \\ \{4,0,10\} & \{8,9,1\} \\ \{11,1\} & \{3,2,5,4\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{5,2,7\} & \{8,0,5,3\} \\ \{1,2,3\} & \{4,5,6,7\} \\ \{8,9,10\} & \{11,0,3\} \end{bmatrix} \in S.$$

$$A + B = \begin{bmatrix} \{8,5,10,9,6,11\} & \{2,6,11,9,10,2,7,5\} \\ \{5,6,7,1,2,3,11\} & \{5,6,7,8,0,1,2,3,4\} \\ \{7,8,9,10,11\} & \{3,2,5,4,1,6,8,7\} \end{bmatrix}$$

$$A \times B = \begin{bmatrix} \{6,3,9,8,4\} & \{0,6,4,10\} \\ \{4,8,0,10,6\} & \{8,4,0,9,5,7,6,3\} \\ \{8,9,10,4,3,2\} & \{9,0,6,10,3,7,8\} \end{bmatrix} \in S.$$

S is only a semiring for S has zero divisors.

Let

$$X = \begin{bmatrix} \{3\} & \{6\} \\ \{3,6\} & \{0,9\} \\ \{9,6\} & \{4,8\} \end{bmatrix} \text{ and } Y = \begin{bmatrix} \{4,8,0\} & \{2,4,6\} \\ \{4\} & \{4,8,0\} \\ \{8,4,0\} & \{3,6,9,0\} \end{bmatrix} \text{ be in S.}$$

$$X \times Y = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}.$$

Thus S is only a semiring and not a semifield.

Now we give some more examples of subset m × n matrix semirings using fields.

Example 2.58: Let S = {Collection of all subset 2 × 7 matrices with subsets from the ring Z_{10}} be the subset 2 × 7 matrix semiring of the ring Z_{10}.

S is only a semiring. S has zero divisors and idempotents; S is not a ring or a field or a semifield.

Example 2.59: Let S = {Collection of all subset 4 × 2 matrices with entries from the subsets of the ring Z} be the subset 4 × 2 matrix semiring of the ring Z.

Clearly S is of infinite order.

Let

$$A = \begin{bmatrix} \{3\} & \{5,2\} \\ \{1,7\} & \{9,4\} \\ \{0,5\} & \{-1,3\} \\ \{-3,2\} & \{-5,1\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{5,-7\} & \{3,-2\} \\ \{-1,2\} & \{-7,8\} \\ \{1,0,-1\} & \{8,2\} \\ \{0,-1,-2\} & \{0,1,2,3\} \end{bmatrix} \text{ be in S.}$$

$$A + B = \begin{bmatrix} \{8,-4\} & \{8,0,3,5\} \\ \{0,6,3,9\} & \{2,-3,12,17\} \\ \{1,0,-1,6,5,4\} & \{7,1,5,11\} \\ \{-3,2,-4,-5,1,0\} & \{-5,1,-4,2,-3,3,-2,4\} \end{bmatrix} \text{ is in S.}$$

$$A \times B = \begin{bmatrix} \{15,-21\} & \{15,-10,6,-4\} \\ \{-1,2,-7,14\} & \{-63,72,-28,32\} \\ \{0,5,-5\} & \{-8,-2,6,24\} \\ \{0,3,-2,6,-4\} & \{-5,1,-4,2,-3,3,-2,4\} \end{bmatrix} \text{ is in S.}$$

$$\text{Let } A = \begin{bmatrix} \{0\} & \{7,8,3\} \\ \{5,2\} & \{0\} \\ \{0\} & \{3,-9\} \\ \{7,8\} & \{0\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{6,3,-2\} & \{0\} \\ \{0\} & \{7,3,4,-5\} \\ \{7,3,-5\} & \{0\} \\ \{0\} & \{8,0,-40,-59\} \end{bmatrix} \text{ be in S.}$$

$$A \times B = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}.$$

Thus S is only a semiring and not a semifield as S has zero divisors.

Example 2.60: Let S = {Collection of all 5 × 2 subset matrices with entries from the subsets of the field Z_5} be the subset 5 × 2 matrix semiring of the field Z_5.

Let A = $\begin{bmatrix} \{0,3\} & \{0,1,4\} \\ \{2,3\} & \{4,0,1\} \\ \{4,1\} & \{2,1\} \\ \{0\} & \{3,1,4\} \\ \{1,2\} & \{3,4\} \end{bmatrix}$ and B = $\begin{bmatrix} \{2,1\} & \{4,1,2\} \\ \{3\} & \{4,3\} \\ \{2\} & \{4\} \\ \{0,3\} & \{2,0\} \\ \{1\} & \{2\} \end{bmatrix}$ be in S.

$$A + B = \begin{bmatrix} \{2,1,0,4\} & \{4,1,2,3,0\} \\ \{0,1\} & \{4,3,2,0\} \\ \{3,1\} & \{0,1\} \\ \{0,3\} & \{0,3,1,4\} \\ \{2,3\} & \{0,1\} \end{bmatrix} \in S.$$

$$A \times B = \begin{bmatrix} \{0,1,3\} & \{0,4,1,2,3\} \\ \{1,4\} & \{0,4,1,3,2\} \\ \{3,2\} & \{3,4\} \\ \{0\} & \{0,2,3,1\} \\ \{1,2\} & \{1,3\} \end{bmatrix} \in S.$$

Thus S is a semiring for S has zero divisors.

A = $\begin{bmatrix} \{0\} & \{1\} \\ \{2\} & \{0\} \\ \{0\} & \{1,2,3\} \\ \{0\} & \{4,3\} \\ \{1,2\} & \{0\} \end{bmatrix}$ and B = $\begin{bmatrix} \{1,2,3\} & \{0\} \\ \{0\} & \{3,4\} \\ \{4,2\} & \{0\} \\ \{1,2,3\} & \{0\} \\ \{0\} & \{1,4,2\} \end{bmatrix}$ be in S.

It is easily verified A × B = $\begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}$.

Example 2.61: Let S = {Collection of all subset 4 × 3 matrices with entries as subsets from the field Q} be the subset 4 × 3 matrix semiring over the field Q. S is of infinite order S has zero divisors so S is not a semifield.

Example 2.62: Let S = {collection of all subset 5 × 5 matrix with entries from the subsets of the ring Z_{42}} be the subset 5 × 5 matrix semiring of the ring Z_{42}. S is a semiring with zero divisors and idempotents.

Example 2.63: Let S = {Collection of all subset 2 × 6 matrices with entries from the subsets of the ring $C(Z_{12})$} be the subset 2 × 6 matrix semiring of the complex modulo integer ring $C(Z_{12})$. S has zero divisors and idempotents.

Infact using S we can get dual number g ∈ S with $g^2 = 0$ and special dual like numbers $g_1 \in S$ with $g_1^2 = g_1$ and special quasi dual numbers g_2 with $g_2^2 = -g_2$.

Thus S is a rich structure in getting dual numbers, special dual like numbers and special quasi dual numbers.

Example 2.64: Let S = {Collection of all subsets 3 × 2 matrices with entries from the subsets of the ring $C(Z_7)$ (g) with $g^2 = 0$} be the subset 3 × 2 matrix semiring of the ring $C(Z_7)$ (g).

$$\text{Let } A = \begin{bmatrix} \{g, 2g, 4g\} & \{0\} \\ \{5, 2\} & \{g, 3, 2\} \\ \{4, 3, 2g\} & \{2, 3g\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{g, 6g\} & \{2+2g, 4+4g\} \\ \{2+3g, 2g\} & \{3+2g, 5+g\} \\ \{4g, g\} & \{2+g, 5g+1\} \end{bmatrix} \text{ be in S.}$$

To find A + B;

A + B =

$$\begin{bmatrix} \{2g, 0, 3g, g5g\} & \{2+2g, 4+4g\} \\ \{3g, 2+2g, 5+2g, 4+3g\} & \{3+3g, 5+2g, 6+2g, g+1, 5+2g, g\} \\ \{4+4g, 3+4g, 6g, 4+g, 3+g, 3g\} & \{4+g, 2+4g, 5g+3, g+1\} \end{bmatrix}$$

is in S.

A × B =

$$\begin{bmatrix} \{0\} & \{0\} \\ \{3+g, 3g, 4+6g, 4g\} & \{3g, 5g, 2+6g, 1+3g, 6g+4g, 3+2g\} \\ \{2g, 5g, 4g, 3g\} & \{4+2g, 3g+2, 6g, 3g\} \end{bmatrix} \in S.$$

This is the way operation are performed on S.

It is easily verified S is not a ring or a field or a semifield only a semiring.

Infact the notion of subset matrices have paved way to construction of infinite number of finite semirings.

Except for this we would not be having finite semirings barring distributive lattices. We also get non commutative semirings of finite order.

Example 2.65: Let S = {Collection of all subset 2 × 7 matrices with entries from the complex modulo integer dual ring $C(Z_{30})$ (g, g_1) where $g_1^2 = g^2 = g_1g = gg_1 = 0$} be the subset matrix semiring of the ring $C(Z_{30})$ (g, g_1).

S has zero divisors units, idempotents, dual elements, special dual like numbers and special quasi dual numbers.

Now having seen examples of semirings of subset matrices we now proceed onto define / recall some properties enjoyed by these rings.

Some of these subset matrix semiring contain subsets which are semifields or rings. Such structure study is interesting and innovative.

Example 2.66: Let S = {Collection of subset 3×1 matrix with entries from the semifield $Q^+ \cup \{0\}$} be the subset matrix semiring of the semifield $Q^+ \cup \{0\}$.

S has zero divisors and no idempotents. S has substructures like subsemirings and ideals. We can think of idempotents in subset matrix semirings only when it is defined over Z_n, $C(Z_n)$, Z or Z_n (g_1, g_2, g_3) or $C(Z_n)$ (g_1, g_2, g_3) or Boolean algebra or chain lattices or other distributive lattices. We have elaborately discussed about these.

Now we can not give the set theoretic '$\cup$' or '$\cap$' when S is built over semirings or rings or fields as the collection will not contain the empty set. However by adjoing the empty set we can give the set theoretic operations on them so that S becomes a semiring.

To this end we will illustrate by some examples.

Example 2.67: Let S = {Collection of all subset 1×5 matrices with entries from the subset ring $Z_6(g)$ with $g^2 = g$} be the subset 1×8 matrix semiring of the ring $Z_6(g)$.

Take $S_1 = \langle S \cup \phi \rangle$, now on S_1 give the two set theoretic operation $\cup$ and $\cap$ then $\{S_1, \cup, \cap\}$ is semiring.

Let A = ({2 + 2g, 3g, 0}, {4g}, {3g+5, 2+2g, g}, {g+5, 3g+1}, {g+1, 2g+2, 3g+3}) and B = ({g, g+3}, {3g + 2, 2+4g}, {3+2g, 4+4g, g}, {5g, 3g + 4}, {g, 4g+2, 2g + 3}) $\in S_1$.

$A \cup B$ = ({2 + 2g, 3g, 0, g, g+3}, {4g, 3g + 2, 2 + 4g}, {3g + 5, 2+2g, g, 3+2g, 4+4g, g}, {5g, 3g+4, g+5, 3g+1}, {g, 4g+2, 2g+3, g+1, 2+2g, 3g + 3}) $\in S_1$.

We can find $A \cap B$ = ({ϕ},{ϕ}, {g}, {ϕ}, {ϕ}) $\in S_1$.

Thus (S_1, $\cup$, $\cap$) is a semiring.

Example 2.68: Let S = {collection of all subset 2 × 5 matrices with entries from the subsets of the ring $Z_{15}(g)$ where $g^2 = 0$} be the subset 2 × 5 matrix semiring of the ring Z_{15} (g), $S_1 = \langle S \cup \phi \rangle$.

Clearly this semiring has dual numbers, special dual like numbers and special quasi dual numbers. S_1 has zero divisors, idempotents and also S_1 is not a semifield.

$$A = \begin{bmatrix} \phi & \{10, g, 3g\} & \{0, 7g, 1\} & \{2g+4, 5g\} & \{3g, 1\} \\ \{2g\} & \{4+g\} & \{3, 5g\} & \{1\} & \phi \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{3\} & \{2+g, 4\} & \{3g+2\} & \{5g+1, 3g\} & \{1, 2g\} \\ \{4g\} & \{2+3g\} & \{g\} & \{g, 1\} & \{3+4g\} \end{bmatrix} \in S_1$$

we find $A \cup B$

$$= \begin{bmatrix} \{3\} & \{10, g, 3g, 2+g, 4\} & \{3g+2, 1, 7g, 0\} \\ \{2g, 4g\} & \{2+3g 4+g\} & \{3, g, 5g\} \end{bmatrix.$$
$$\begin{matrix} \{2g+4, 5g, 5g+1, 3g\} & \{1, 2g, 3g\} \\ \{1, g\} & \{3+4g\} \end{matrix}\Bigg]$$

$$A \cap B = \begin{bmatrix} \phi & \phi & \phi & \phi & \{1\} \\ \phi & \phi & \phi & \{1\} & \phi \end{bmatrix} \in S_1.$$

A + B =

$$\begin{bmatrix} \{3\} & \begin{matrix}\{14, 4+g, 4+3g, 12+g, \\ 2+2g, 2+4g\}\end{matrix} & \{0, 3g+3, 10g+2\} & \begin{matrix}\{7g+5, 10g+1, \\ 5g+4, 8g\}\end{matrix} & \begin{matrix}\{2, 3g+1, \\ 1+2g, 5g\}\end{matrix} \\ \{6g\} & \{6+4g\} & \{g+3, 6g\} & \{1+g, 2\} & \{3+4g\} \end{bmatrix}$$

We see A + B ≠ A ∪ B.

Now we find

A × B =

$$\begin{pmatrix} \{\phi\} & \{5+10g, 2g, 6g, 10, 4g, 12g & \{3g+2\} & \{14g, 0\} & \{3g, 1, 2g\} \\ \{0\} & \{8+14g\} & \{3g\} & \{g\} & \{\phi\} \end{pmatrix}$$

and A × B = A ∩ B.

Example 2.69: Let S = {Collection of all subset 3 × 1 matrices with entries from the subsets of the ring $C(Z_4)$ (g)} be the subset matrix semiring of the ring $C(Z_4)$ (g).

We see $S_1 = \langle S \cup \phi \rangle$; we get entirely a very different subset matrix semiring using S_1.

Now having seen such examples of these new structures we describe how a topology can be build using them.

We will take S = {Collection of all subset m × n matrices with subsets from the semigroup or group or semiring or ring or a field}.

S has a semigroup structure if S we take a group or a semigroup. S has a semiring structure if we take a semiring or a ring or a semifield or a field.

Suppose S is a subset matrix semigroup of a semigroup. Let P $\subseteq$ S (P a proper subset of S).

Let M $\subseteq$ S be a subset matrix subsemigroup of S. If for all p $\in$ P and m $\in$ M, mp and pm $\in$ P then we define P to be a set subset matrix ideal of S. The same is true if the subset matrix semigroup is built using the group.

We will first illustrate this situation by some examples.

Example 2.70: Let S = {Collection of all subset 1 × 4 matrices with subsets from the semigroup (Z_{12}, ×)} be the subset 1 × 4 matrix semigroup of the semigroup (Z_{12}, ×).

Take P = (({0,1,2}, {1,1}, {2,2}, {0,4,5}), ({0}, {0}, {0}, {0}), ({9, 2}, {5, 4, 2}, {3, 0, 1}, {1, 2, 3, 4, 5}), ({7, 6, 8, 9, 10}, {11, 0, 1, 3, 5}, {7, 9, 11, 0} {2, 4, 6, 8})} $\subseteq$ S.

Consider the subset matrix subsemigroup
M = {({0}, {0}, {0}, {0}), ({1}, {1}, {1}, {1})} $\subseteq$ S.

Clearly P is a set subset 1 × 4 matrix ideal of the subset 1 × 4 matrix semigroup over the subset 1 × 4 matrix subsemigroup M of S.

Example 2.71: Let S = {Collection of all subset 2 × 3 matrices with subset entries from the semigroup $C(Z_4)$} be the subset 7 × 3 matrix semigroup of the semigroup $C(Z_4)$.

$$\text{Let } M = \left\{ \begin{pmatrix} \{0\} & \{0,2\} & \{2\} \\ \{2\} & \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{\} & \{0\} & \{0\} \end{pmatrix}, \right.$$

$$\left.\begin{pmatrix} \{2\} & \{0\} & \{0,2\} \\ \{2\} & \{2\} & \{2\} \end{pmatrix}, \begin{pmatrix} \{2\} & \{2\} & \{0,2\} \\ \{0,2\} & \{0,2\} & \{2\} \end{pmatrix}\right\} \subseteq S$$

be a subset 2 × 3 matrix subset of S.

Take

$$P = \left\{\begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0\} \end{pmatrix}, \left\{\begin{pmatrix} \{1\} & \{1\} & \{1\} \\ \{1\} & \{1\} & \{1\} \end{pmatrix},\right.\right.$$

$$\left\{\begin{pmatrix} \{2\} & \{2\} & \{2\} \\ \{2\} & \{2\} & \{2\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0,2\} & \{2\} \\ \{2\} & \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{2\} & \{0\} & \{0,2\} \\ \{2\} & \{2\} & \{2\} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \{2\} & \{2\} & \{0,2\} \\ \{0,2\} & \{0,2\} & \{2\} \end{pmatrix}\right\} \subseteq S;$$

be a subset 2 × 3 matrix subset of S. Clearly P is a subset 2 × 3 matrix set ideal of S over the subset 2 × 3 matrix subsemiring M of S.

Example 2.72: Let S = {Collection of all subset 2 × 4 matrices with entries from the subsets of the semigroup Z_6 (g) with $g^2 = 0$ under product} be the subset 2 × 4 matrix semigroup of the semigroup Z_6 (g). Take $S_1 = Z_6$ a subsemigroup of Z_6 (g).

T = {Collection of all subset 2 × 4 matrix set ideals of S over the subsemigroup Z_6 of Z_6 (g)} is the subset matrix set ideal topological space of S over Z_6.

Suppose S_2 = {0, g, 2g, 3g, 4g, 5g} ⊆ Z_6(g) be the subsemigroup. T_2 = {Collection of all subset matrix set ideals of the semigroup S} be the set ideal topological subset semigroup of S.

We see both of them are distinct and are of finite order.

We can as in case of usual semigroup define in case of subset matrix semigroups also construct S-prime set ideal, S-strong quasi set ideal and S-set ideal topological subset matrix semigroup spaces [17-8].

We will illustrate all these situations by some examples.

Example 2.73: Let S = {Collection of all subset 8×1 matrices with subsets from the semigroup Z_{24} (g) where $g^2 = 0$} be the subset 8×1 matrix semigroup of the semigroup {Z_{24} (g), $\times$}.

Let M = {Collection of all subset 8×1 matrices with subsets from P = {0, g, 2g, …., 23g}$\subseteq$ Z_{24} (g)} be the set ideal subset matrix subsemigroup of the semigroup S over the semigroup $S_1 = Z_{24}$.

Clearly every set ideal in M is Smarandache quasi set ideal subset matrix of S relative to S_1 as N = {collection of all subset 8×1 matrices with subsets from S_2 = {0, 3g, 6g, …, 21g} $\subseteq$ P} is a subset 8×1 matrix subsemigroup of M. Thus M is a S-quasi set subset 8×1 matrix topological space of S relative to the subsemigroup S_1 of S.

Example 2.74: Let S = {Collection of all subset 3×3 matrices with subsets from the semigroup Z(g), $\times$} be the subset 3×3 matrix semigroup of the semigroup {Z(g), $\times$}.

Take P = {all subset 3×3 matrices with subsets from {3Z(g), $\times$} $\subseteq$ {Z(g), $\times$}}, P is a Smarandache quasi set ideal of S relative to the subsemigroup S_1 {Z, $\times$} $\subseteq$ {Z(g), $\times$}} for P_1 = {all subset 3×3 matrices with subsets from {3Z, $\times$}} is a subsemigroup of P.

Next we proceed onto give examples of Smarandache perfect quasi set ideal of a subset matrix semigroup.

Example 2.75: Let S = {Collection of all subset 2×3 matrices with subsets from Z_{30} (g)} be the subset 2×3 matrix semigroup over the semigroup Z_{30} (g).

Take $P_1 = \{0, 1, 2, \ldots, 29\}$, $P_2 = \{0, g, 2g, \ldots, 29g\}$; $P_3 = \{0, 15\}$, $P_4 = \{0, 15g\}$, $P_5 = \{0, 10, 20\}$ and $P_6 = \{0, 10g, 20g\}$ subsemigroups of Z_{30} (g).

Take M = {Collection of all subset 2×3 matrices with subsets from $\{a + bg \mid a, b \in 2Z_{30}\} \subseteq Z_{30}(g)$} be the subset 2×3 subsemigroup of S over $P_1, P_2, \ldots, P_6$.

M is the Smarandache perfect quasi set ideal subset 2×3 matrix of S over each P_i, $1 \leq i \leq 6$.

Example 2.76: Let S = {Collection of all subset 3×6 matrices with subsets from $\{Z(g), \times\}$, $g^2 = 0$} be the subset 3×6 matrix semigroup of the semigroup $\{Z(g), \times\}$.

P = {Collection of all subset 3×6 matrices with subsets from $\{3Z(g), \times\}$} be the set ideal of S with respect to the group $G = \{1, -1\} \subseteq Z(g)$.

P is the strong set subset 3×6 matrix ideal of the semigroup S over the group G of S.

Clearly this subset matrix semigroup has only one group $G = \{1, -1\} \subseteq Z(g)$ so if T_G = {Collection of all strong set ideal subset matrix semigroup of S over G} is the strong set ideal subset matrix semigroup topological space of S over G.

Interested reader can construct several such examples of strong set ideal topological spaces of a subset matrix semigroup S.

Now we just discuss the advantage of defining subset $m \times n$ matrix set ideals of semigroups (semirings) over subsemigroups (or subsemirings or fields or groups or semifields).

In the first case given a semigroup of subset $m \times n$ matrix set ideals of a semigroup we can have only one topology defined on it that is the usual topology with '$\cup$' and '$\cap$', however the other one with the new topology $\cup_N$ and $\cap_N$ cannot be defined using subset semigroup.

Both the type of topologies can be defined on semirings; we have several of them depending on the number of substructures and with the appropriate algebraic structure on them.

Further for every one of such collection we can have two topologies usual topology and the new topology only in case of semirings.

Thus this is one of the advantages of using set ideal subset matrix topological semirings.

We will illustrate these situations by some examples.

Example 2.77: Let S = {Collection of subset 1×5 matrices with subsets from the semigroup Z_6} be the subset 1×5 matrix semigroup of the semigroup Z_6.

Consider P = {Collection of all subset 1×5 matrices from the subsets $\{0, 3\} \subseteq Z_6$} be the subset 1×5 matrix subsemigroup of S.

Let

M = {Collection of all subset matrix set ideals of S over P}; $\{M, \cup, \cap\}$ is a set ideal subset 1×5 matrix topological subsemigroup of S over P.

We can give on M a new topology so that $\{M, \cup_N, \cap_N\}$ is the set ideal subset 1×5 matrix new topological space of S over P.

Suppose we have

$$X = (\{0,2\}, \{0, 2, 4\}, \{0\}, \{0\}, \{0, 4\})$$
and $Y = (\{2\}, \{4\}, \{2, 4\}, \{0\}, \{4\})$

then $X \cap Y = (\{0\}, \{4\}, \{\phi\}, \{0\}, \{4\})$ and

$X \cup Y = (\{0,2\}, \{0, 2, 4\}, \{0,4,2\}, \{0\}, \{0, 4\})$.

$X \cap_N Y = (\{0\}, \{0, 2\}, \{0\}, \{0\}, \{0\})$ and
$X \cup_N Y$; has no meaning, because our underlying structure is only a semigroup so has only one operation can be defined on it.

Thus for subset $m \times n$ matrix semigroups we cannot define the concept of the new topology T_N.

We have only one topology using ideals of a subset matrix semigroups and several topologies using set ideals of subset matrix semigroups over subsemigroups.

We now show by an example or two the new topology on the subset matrix semirings.

Example 2.78: Let S = {Collection of all subset 3×1 matrices with entries from the subsets of the ring Z_{12}} be the subset 3×1 matrix semiring of the ring.

Let T = {Collection of all subset matrix semiideals of S including ϕ, the emply set}; $\{T, \cup, \cap\}$ is the subset matrix ideal topological space semiring of S.

$\{T, \cup_N, \cap_N\}$ is the subset matrix ideal new topological space semiring of S.

$$\text{Let } x = \begin{bmatrix} \{0,2,4\} \\ \{0,4\} \\ \{2\} \end{bmatrix} \text{ and } y = \begin{bmatrix} \{0,2\} \\ \{6\} \\ \{4,6\} \end{bmatrix}$$

$$x \cap_N y = \begin{bmatrix} \{0,4,8\} \\ \{0\} \\ \{8,0\} \end{bmatrix} \text{ and } x \cup_N y = \begin{bmatrix} \{0,4,6,2\} \\ \{6,10\} \\ \{6,8\} \end{bmatrix}.$$

This is the way operation is performed.

$$x \cap_N x = \begin{bmatrix} \{0,4,8\} \\ \{0,4\} \\ \{4\} \end{bmatrix} \neq x. \quad x \cup_N x = \begin{bmatrix} \{0,2,4,6,8\} \\ \{0,4,8\} \\ \{4\} \end{bmatrix} \neq x.$$

This new topology on subset 3 × 1 matrices of the semiring is different from the usual topology for $x \cap_N x \neq x$ and $x \cup_N x \neq x$ in general.

Now we proceed onto illustrate by an example that this sort of new topology can be defined as set ideal subset matrix new topological space of semirings.

Example 2.79: Let S = {Collection of all subset 2 × 2 matrices with subsets from the ring Z_{10}} be the subset 2 × 2 matrix semiring of the ring Z_{10}.

T = {Collection of all set ideals of the subset 2 × 2 matrix semiring over the subsemiring

$$P = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \{0, 2, 4, 6, 8\} \subseteq Z_{10} \right\}\}.$$

$\{T, \cup_N, \cap_N\}$ is the new topological space of subset 2 × 2 matrix semiring over the subsemiring P.

$$\text{Let } X = \begin{pmatrix} \{0,2,6\} & \{4,2\} \\ \{0,4,8,6\} & \{0,6\} \end{pmatrix} \text{ and}$$

$$Y = \begin{pmatrix} \{0,4,8\} & \{0,2,6,8\} \\ \{0,6\} & \{0,8\} \end{pmatrix} \in T.$$

$$X \cap_N Y = \begin{pmatrix} \{0,8,6,4\} & \{0,4,2,6,8\} \\ \{0,4,8,6\} & \{0,8\} \end{pmatrix} \in T.$$

$$X \cup_N Y = \begin{pmatrix} \{0,2,6,4,8\} & \{0,2,6,4,8\} \\ \{0,4,6,8,2\} & \{0,6,8,4\} \end{pmatrix} \in T.$$

$(T, \cup_N, \cap_N) = T_N$ is a new topological subset matrix ideal topological semiring space.

Take $S_1 = \{0, 5\} \subseteq Z_{10}$ to be a subring of S.

P_1 = {Collection of all subset 2 × 2 matrices with entries from the subsemiring $S_1 = \{0, 5\}$}.

P_1 is a subset 2 × 2 matrix subsemiring of S.

Take T = {Collection of all set ideals of subset 2 × 2 matrix of S over the subset matrix subsemiring S_1}

$$= \left\{ \left\{ \begin{pmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{2\} & \{2\} \\ \{2\} & \{2\} \end{pmatrix} \right\}, \left\{ \begin{pmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{2\} & \{0\} \\ \{0\} & \{0\} \end{pmatrix} \right\}, \right.$$

$$\left\{ \begin{pmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{2\} \\ \{0\} & \{0\} \end{pmatrix} \right\}, \left\{ \begin{pmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0\} \\ \{2\} & \{0\} \end{pmatrix} \right\},$$

$$\left. \left\{ \begin{pmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0\} \\ \{0\} & \{2\} \end{pmatrix} \right\} \text{ and so on} \right\}.$$

We see $\{T, \cup_N, \cap_N\}$ is a set ideal subset 2 × 2 matrix new topological semiring space of S over the subset 2 × 2 matrix subsemiring S_1.

We can have several such to new topological spaces by varying the subset matrix subsemirings over which these are defined.

Now it is left for the reader to define different types of new topological set ideals using strong set ideal of subset semiring collection, special strong set ideal new topological subset matrix semiring space and other types of set ideal new topological subset matrix semirings. It can be done as a matter of routine with some appropriate changes.

Now we proceed onto suggest some problems for the reader.

Problems

1. Let X = {1, 2, 3, 4, 5}, and P(X) the power set of X.

   Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \end{bmatrix} \middle| \; a_i \in P(X);\ 1 \leq i \leq 12 \right\}$ be the subset 2×6 matrix.

   (i) Find the number of elements in S.
   (ii) Show $(S, \cup)$ is a commutative semigroup.
   (iii) Show $(S, \cap)$ is a commutative semigroup.

2. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{11} \\ a_{12} & a_{13} & \dots & a_{22} \\ a_{23} & a_{24} & \dots & a_{33} \end{bmatrix} \middle| \; a_i \in P(C(Z_{12})),\ 1 \leq i \leq 33, \right.$

   $X = \{C(Z_{12}) = a + bi_F \mid a, b \in Z_{12},\ i_F^2 = 11\}\}$ be a collection of all 3×11 subset matrices.

   (i) Find o(M).
   (ii) Show $(M, \cup)$ is a commutative semigroup.

(iii) Show $(M, \cap)$ is a commutative semigroup.

3. Obtain some interesting properties associated with subset matrices of a power set.

4. What can be benefits of studying such algebraic structure?

5. Find some nice applications of these new structures.

6. Let S = {Collection of all subset 3×2 matrices with subsets from the semigroup $\{Z_{40}, \times\}$} be the subset 3×2 matrix semigroup of the semigroup $\{Z_{40}, \times\}$.

   (i) Find the order of S.
   (ii) Can S have subset matrix subsemigroups which are not subset matrix ideals?
   (iii) Find atleast two subset matrix ideals of S.
   (iv) Can S have zero divisors?
   (v) Can S have S-idempotents?
   (vi) Can S have S-units or units?
   (vii) Find any other interesting property associated with S.

7. Let S = {Collection of all subset 3×6 matrices with entries from the subsets of the semigroup $\{C(Z_5), \times\}$} be the semigroup.

   (i) Find order of S.
   (ii) Find subset 3×6 matrix subsemigroups of S which are not subset matrix ideals.
   (iii) Find subset 3×6 matrix ideals of S.
   (iv) Can S have zero divisors?
   (v) Find idempotents and zero divisors of S.
   (vi) Can S have S-zero divisors and S-units?
   (vii) Is S a S-semigroup?

8. Let S = {collection of all subset 7×2 matrices with entries from the subsets of the semigroup $\{C(Z_{20}), \times\}$} be the subset 7×2 matrix semigroup with entries from the semigroup $\{C(Z_{20}), \times\}$.

(i) Find o(S).
(ii) Can S have zero divisors?
(iii) Show S has idempotents.
(iv) Give an example of a subset matrix subsemigroup which is not a subset matrix ideal.
(v) Can S have S-units?

9. Find some special features enjoyed by subset 1×9 matrix semigroup built using $C(Z_n)$.

10. Find the order of S = {Collection of all $m \times n$ subset matrices with entries from the subsets of the semigroup Z_p}, the subset $m \times n$ matrix semigroup (p prime).

(i) If p a replaced by t, t a composite number find o(S).
(ii) Find the order of S if Z_p is replaced by $C(Z_p)$.
(iii) Find the order of S if Z_p is replaced by $Z_p(g)$ where $g^2 = 0$.
(iv) Find the order of S if Z_p is replaced by $C(Z_p)$ (g) such that $g^2 = 0$.

11. Let S = {Collection of all subset 3×5 matrices with entries from the semigroup $\{Z, \times\}$} be the subset 3×5 matrix semigroup of the semigroup $\{Z, \times\}$.

(i) Prove S is of infinite order.
(ii) Prove S has zero divisors.
(iii) Can S have idempotents?
(iv) Can S have S-zero divisors?

12. Let S = {Collection of all subset 5×1 matrices with entries from the subsets of the semigroup $\{Q^+ \cup \{0\}, \times\}$} be the subset 5×1 matrix semigroup with entries from subsets of the semigroup $\{Q^+ \cup \{0\}, \times\}$.

(i) Study questions (i) to (iv) of problem 11.
(ii) Can S have units?
(iii) Can S have S-units?

13. Let S = {Collection of all 2×2 subset matrices with entries from the subsets of the semigroup $\{(Q^+ \cup \{0\})\,(g)\}$} be the subset 2×2 matrix semigroup of the semigroup $\{(Q^+ \cup \{0\})\,(g), \times\}$.

    (i) Prove S has zero divisors.
    (ii) Prove S has a zero square subsemigroups.
    (iii) Can S have S-units?
    (iv) Is it possible to have subset matrix ideals of S?
    (v) Can S have subset matrix subsemigroups which are not subset matrix ideals?

14. Obtain some special features enjoyed by subset matrix semigroups of the semigroup $\{Q, \times\}$ or $\{R, \times\}$ or $\{C, \times\}$ or $\{Q(g), \times\}$ or $\{R(g), \times\}$, $\{C(g), \times\}$ where $g^2 = 0$.

15. Specify some special features enjoyed by infinite subset matrix semigroup of $\{Z[x], \times\}$ (or $\{Q[x], \times\}$ or $\{R[x], \times\}$ or $\{C[x], \times\}$ or $\{Z^+[x] \cup \{0\}\}$ or $\{Q^+[x] \cup \{0\}\}$).

16. Let S = {Collection of all 7×3 matrices whose entries are subsets from the semigroup of $\{Z_n[x], \times\}$

    (i) Will the subset matrix semigroups of the semigroup $\{Z_n[x], \times\}$ be finite? Justify.
    (ii) Can S have zero divisors?
    (iii) Can S have units?
    (iv) Can S have S-idempotents?
    (v) Can S have subset matrix subsemigroups which are not subset matrix ideals?

17. Let S = {Collection of all subset 3×3 matrix semigroup with subsets from the symmetric semigroup S(5)} be the subset semigroup matrix of symmetric semigroup S(5).

    (i) Find order of S.
    (ii) Prove S is a non commutative semigroup.
    (iii) Can S have subset matrix ideals?
    (iv) Is it possible for S to contain zero divisors?
    (v) Can S have S-units?

18. Let S = {Collection of all 3×3 subset matrices with subsets from S(n), the symmetric semigroup} be the subset matrix semigroup of the semigroup S(n).

    (i) Find o(S).
    (ii) Prove S has no zero divisors.
    (iii) Prove S is non commutative.
    (iv) Can S have idempotents?
    (v) Is S a S-subset matrix semigroup?

19. If in the problem 18 the 3×3 matrix is replaced by 5×3 matrix study questions (i) to (iv).

20. Let S = {Collection of all subset 2×5 matrices with entries from the symmetric semigroup S(7)} be the subset matrix semigroup of the symmetric semigroup S(7).

    (i) Find o(S).
    (ii) Prove S is non commutative.
    (iii) Can S have zero divisors?
    (iv) Prove S can have units where

    $$I = \begin{bmatrix} \{e\} & \{e\} & \{e\} & \{e\} & \{e\} \\ \{e\} & \{e\} & \{e\} & \{e\} & \{e\} \end{bmatrix} \text{ is the unit in S.}$$

    {e} identity element of S(7). Clearly $A \times I = I \times A = A$ for all $A \in S$.
    (v) Can S have idempotents?
    (vi) Can S have S-units?
    (vii) Can S have subset matrix subsemigroup which is commutative?
    (viii) Is S a S-semigroup?

21. Let S = {Collection of all subset 7×2 matrices with subsets from the group {Z, +}} be the subset 7×2 matrix semigroup of the group G.

    (i) Find ideals of S.

(ii) Is S a S-semigroup?
(iii) Can S have zero divisors?
(iv) Can S have S-units?
(v) Can S have S-idempotents?
(vi) Does S contain subset matrix subsemigroups which are not subset matrix ideals?

22. Let S = {Collection of all 3×7 subset matrices with entries from the subsets of the group G = $\{Z_{10}, +\}$} be the subset matrix semigroup of the group G.

(i) Find o(S).
(ii) Is S commutative?
(iii) Can S have subset matrix ideals?
(iv) If G is replaced by the semigroup P = $\{Z_{10}, \times\}$, enumerate all the special features enjoyed by S.
(iv) Differentiate both structure like units, S units, zero divisors, S-zero divisors, idempotents and S-idempotents when G is replaced by P = $\{Z_{10}, \times\}$.

23. Let S = {Collection of all subset 3×7 matrices with entries from the subsets of the group G = $\{Z_{17}, +\}$} be the subset matrix semigroup of the group G.

(i) Study questions (i) to (iii) (given in problem 22).
(ii) If G is replaced by P = $\{Z_{17}, \times\}$ study question (iv).

24. Obtain some special features enjoyed by subset matrix semigroups over a group.

25. Let S = {Collection of all subset 4×3 matrices with subsets from the semigroup P = $Z_{11} \times Z_6 \times C(Z_7) \times Z_4(g)$} be the subset matrix semigroup of the semigroup P.

(i) Find o(S).
(ii) Find atleast three subset matrix subsemigroups of S.
(iii) Find atleast three subset matrix ideals of S.
(iv) Can S have S-zero divisors?

26. Let S = {Collection of all subset 5×2 matrices whose subsets are from the group G = $D_{2.7}$} be the subset matrix semigroup over the group G.

    (i) Find o(S).
    (ii) Can S have subset matrix subsemigroups which are not a subset ideals?
    (iii) Can S have S-zero divisors?
    (iv) Can S have idempotents?
    (v) Give some special properties about S.
    (vi) Prove S is non commutative.
    (vii) Can S be a S-semigroup?

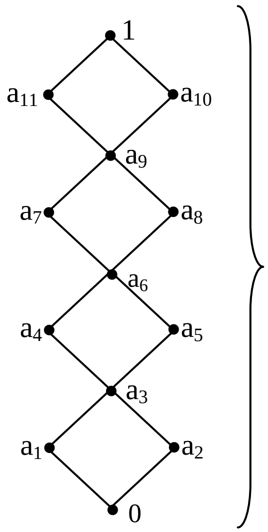
27. Let S = {Collection of all subset 2×4 matrices with subsets from the group G = $S_7 \times D_{2.7} \times (Z_{11} \setminus \{0\}, \times)$}} be the subset 2×4 matrix semigroup over the group G.

    (i) Find order of S.
    (ii) Can S have zero divisors?
    (iii) Is S a S-semigroup?
    (iv) Can S have S-subsemigroups?
    (v) Is it possible for S to have S-ideals?
    (vi) Study (i) to (v) using H = S_7 alone.

28. Let S = {Collection of all subset 3×3 matrices with subsets from $Q^+ \cup \{0\}$} be the subset 3×3 matrix semiring.

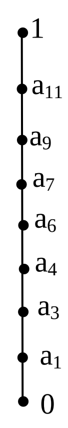
    (i) Is S a commutative semiring?
    (ii) Is S a S-semiring?
    (iii) Can S have zero divisors?
    (iv) Can S have idempotents?
    (v) Is A = $\begin{pmatrix} \{1\} & \{0\} & \{0,1\} \\ \{1\} & \{0,1\} & \{0\} \\ \{0\} & \{1\} & \{1\} \end{pmatrix} \in$ S an idempotent of S?

29. Let S = {Collection of all 2 × 4 subset matrices with subsets from the semiring



be the subset 2 × 4 matrix semiring over the semiring P.

(i) Find o(S).
(ii) Is S a S-semiring?
(iii) Can S have zero divisors?
(iv) If P is replaced by the semifield



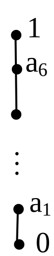
and S is defined over P study the above questions (i) to (iii)

(v) Can S have S-units?
(vi) Can S have S-idempotents?
(vii) Can S have idempotents which are not S-idempotents?

30. Let S = {Collection of all subset 7×1 matrices with subsets from the Boolean algebra B of order 32} be the subset 7×1 matrix semiring over B.

(i) Find o(S).
(ii) Can S have S-zero divisors?
(iii) Can S have S-idempotents?
(iv) Can S have S-ideals?
(v) Is S a S-semiring?
(vi) Can S have S-units?

31. Let in problem 30 the Boolean algebra B be replaced by a chain lattice C_8 =



Study question (i) to (vi) of problem 30 for this S over C_8.

32. Let S = {Collection of all subset 2×2 matrices with subsets from the ring Z_{40}} be the special subset 2×2 matrix semiring over the ring.

(i) Find order of S.
(ii) Is S a commutative structure?
(iii) Can S have zero divisors?

(iv) Can S be a S-ring?
(v) Can S have S-ideals?

33. Let S = {Collection of all subset 8 × 1 matrices where subsets are taken from the field Z_7} be the subset 8 × 1 matrix semiring of the field.

(i) Find o(S).
(ii) Find S-subsemirings.
(iii) Can S have S-idempotents?
(iv) Prove S has zero divisors.

34. Let S = {Collection of all subset 1 × 9 matrix subsets from the field Q} be the subset 1 × 9 matrix semiring of the field Q.

(i) Can S have zero divisors?
(ii) Can S have S-units?
(iii) Can S have S-ideals?

(iv) Can S have subset matrix subsemirings which are not ideals?
(v) Is S a S-semiring?

35. Let S = {Collection of all subset 2 × 4 matrix, subsets from the ring Q[x]} be the subset matrix semiring over of the S-ring Q[x].

(i) Prove S is of infinite order.
(ii) Can S have S-subset matrix subsemiring which is not a subset matrix ideal?
(iii) Can S have S-ideals?
(iv) Can S have zero divisors?
(v) Can S have S-idempotents?
(vi) Can S have S-units?
(vii) Obtain some stricking features about S.

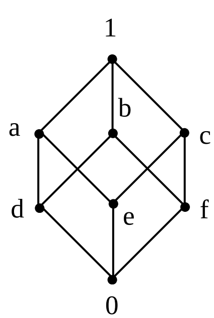
36. Let S = {Collection of all subsets of 3×5 matrices with subsets from $Z_3[x]$} be the subset 3×5 matrix semiring of the ring.

    (i) Prove S is of infinite order.
    (ii) Can S have zero divisors?
    (iii) Is it possible for idempotents to be in S?
    (iv) Can S have units?
    (v) Give two examples of subset matrix S-ideals in S.
    (vi) Give two examples subset matrix S-subsemirings which are not subset matrix ideals.
    (vii) Give an example of a subset matrix ideal which is not a subset matrix S-ideal?

37. Let S = {Collection of all subsets of 2×6 matrices with subsets from the quasi dual like ring $Z_n(g)$ with $g^2 = -g$} be the subset 2×6 matrix semiring of the ring $Z_n(g)$.

    (i) Find the order of S.
    (ii) Can S have zero divisors?
    (iii) Can S have S-idempotents?
    (iv) Find a subset matrix S-subsemiring of S.
    (v) Find subset matrix S-ideals of S.
    (vi) Find a subset matrix subsemiring which is not a subset matrix S-ideal.

38. Let S = {Collection of all subset of 3×1 matrices with subsets from the semiring



}

be the subset 3×1 matrix semiring.

(i) Find o(S).
(ii) Find zero divisors in S.
(iii) Find units of S.
(iv) Find idempotents of S.
(v) Find subset matrix S-ideals of S.
(vi) Find subset matrix S-subsemiring of S.

39. Let S = {Collection of all subset 2 × 5 matrices with subsets from the ring Z_{20} $(g_1, g_2) = g_1^2 = g_2^2 = 0 = g_1g_2 = g_2g_1$} be the subset matrix semiring of the ring Z_{20} (g_1,g_2).

(i) Find zero divisors of S.
(ii) Find o(S).
(iii) Can S have S-units?
(iv) Can S have subset matrix ideals which are not subset matrix S-ideals?
(v) Can S have subset matrix S-subsemiring which are not subset matrix ideals?

40. Let S = {Collection of all subsets of 1 × 8 matrices with subsets from the ring $C(Z_9)$} be the subset 1 × 8 matrix semiring of the ring $C(Z_9)$.

(i) Can S have zero divisors?
(ii) Find o(S).
(iii) Is S a subset matrix S-semiring?
(iv) Can S have subset matrix S-ideals?
(v) Can S have S-units?
(vi) Can S have units which are not S-units?
(vii) Can S have zero divisors which are not S-zero divisors?

41. Let S = {Collection of all subset 5 × 5 matrices with subsets from the field Z_{19}} be the subset 5 × 5 matrix semiring of the field Z_{19}.

(i) Find order of S.
(ii) Prove S has zero divisors.

(iii) Prove S has units.
(iv) Can S have S-units and S-zero divisors?
(v) Can S have subset matrix S-ideals?

42. Let S = {Collection of all subset 3×2 matrices with entries from subsets from the semigroup Z_{15}} be the subset 3×2 matrix semigroup of the semigroup Z_{15}.

(i) Find all the set ideals of the subset 3×2 matrix semigroup S.
(ii) If P = {0,1} be the subsemigroup of Z_{15}. Find collection of set subset matrix ideals T of S over P.
(iii) If P_1 = {0, 14, 1} be the subsemigroup of the semigroup Z_{15}, find collection of all subset matrix set ideals T_1 of S over P_1.
(iv) Compare T with T_1
(v) If $P_2 = \{0, 3, 9, 12, 6\} \subseteq Z_{15}$ be a subsemigroup of the semigroup Z_{15} find the collection of all subset matrix set ideals T_2 of S over P_2.
(vi) If $P_3 = \{0, 5, 10\} \subseteq Z_{15}$ be the subsemigroup of Z_{15} find the collection of all subset matrix set ideals T_3 of S over P_3.
(vii) Compare T_3, T_1 and T_2.
(viii) Using the '$\cup$' and '$\cap$' of subsets of these subset matrix set ideals define topologies on them.

43. Let S = {Collection of all subset 3×7 matrices with subsets from the semigroup $Z_{12}(g)$} be the subset 3×7 matrix semigroup of the semigroup Z_{12} (g).

(i) Take $P_1 = \{0, 6, 6g\} \subseteq Z_{12}$ (g) a subsemigroup of the semigroup Z_{12} (g). T_1 = {Collection of all set ideals of S over P_1} be the subset matrix set ideal topological space of S over P_1.

(a) Find a basic set of T_1.
(b) Find $o(T_1)$.

Study question (i) of a and b for $P_2 = \{6, 0\}$, $P_3 = \{0, 6g\}$, $P_4 = \{1, 0\}$, $P_5 = \{0, 11, 1\}$, $P_6 = \{g, 0\}$, $P_7 = \{g, 11g, 0\}$, $P_8 = \{0, 2, 4, 6, 8, 10\}$, $P_9 = \{0, 3, 6, 9\}$, $P_{11} = \{0, 4g, 8g\}$ and $P_{12} = \{0, 3g, 6g, 9g\}$.

44. Let S = {Collection of all subset 5×2 matrices with subsets from the semigroup $S = \{C(Z_6)(g); \times\}$ $g^2 = 0\}\}$ be the subset 5×2 matrix semigroup of the semigroup S.

    (a) Find the total number of set ideal subset 5×2 matrix topological spaces of S over the appropriate subsemigroup of S.

45. Let S = {Collection of all subset 1×7 matrices with subsets from the semigroup $S_1 = \{Z_4 \times Z_6, \times\}\}$ be the subset 1×7 matrix semigroup of S_1.

    Study question (a) in problem 44 for this S.

46. Let S = {Collection of all subset 2×5 matrices, entries of the matrices are taken from subsets of S(5)} be the subset 2×5 matrix semigroup of the semigroup S(5).

    (i) Find all set ideal subset matrix topological spaces of S over subsemigroup of S.
    (ii) How many are identical for the distinct subsemigroups?
    (iii) Find subset matrix set ideals of S over the group S_5 and find the related subset matrix set ideal topological space.
    (iv) Can S have a Smarandache quasi set ideal subset matrix relative to a subsemigroup S_1 of S?

47. Obtain some special features enjoyed by Smarandache perfect quasi set ideal subset $m \times n$ matrix semigroup S; a subset $m \times n$ matrix subset semigroup with subsets from the semigroup P.

48. Study Smarandache perfect quasi set ideal $m \times n$ matrix topological semigroup from subset matrix semigroup of a semigroup.

49. Distinguish this (48 problem) from the set ideal topological space of a semigroup.

50. Let S = {Collection of all subset 2×7 matrices with subsets from the ring Z_{18}} be the subset 2×7 matrix semiring of the ring Z_{18}.

    (i) How many subset 2×7 matrix subsemirings exist?
    (ii) How many set ideal subset 2×7 matrix over subsemirings of S exist?
    (iii) Construct the related topology on them.
    (iv) Distinguish between set ideals of all subset matrix subsemigroup topological space over S and that of the ideals of the subset matrix subsemigroup topological space over S.

51. Let S be the collection of all subset 3×2 matrix semiring with entries from the subsets from the field Z_{19}. Study questions (i) to (iv) of the problem 50.

52. Let S = {Collection of all subset 3×7 matrices with subsets from the semifield $Z^+ \cup \{0\}$} be the subset 3×7 matrix semiring of the semifield.

    (i) Find the ideal subset topological 3×7 matrix semiring of the semiring S.
    (ii) Find all set ideal subset topological 3×7 matrix of S over subsets of S.
    (iii) Prove S has zero divisors.
    (iv) Can S have S-zero divisors?

53. Let S = {Collection of all subset 3×1 matrices with subsets from the ring Z_{42} (g)} be the subset 3×1 matrix semiring of the ring.

(i) Find o(S).
(ii) Can S have zero divisors which are not S-zero divisors?
(iii) Find the subset matrix ideal topological space of the subset matrix ideals of S.
(iv) Find all the set ideal topological subset 3×1 matrix ideals space of S over proper subsemirings of S.

54. Let S = {Collection of all subset 5×2 matrices with subsets from the symmetric semigroup S(3)} be the subset 5×2 matrix semigroup of the symmetric semigroup S(3).

(i) Find o(S).
(ii) Prove S is non commutative under the natural product $\times_n$ of matrices.
(iii) Prove S cannot have zero divisors.
(iv) Prove S is not a semifield.
(v) Find a strong subset matrix set ideal of the subset matrix semigroup S.
(vi) Find a special strong subset matrix set ideal of the subset matrix semigroup S.
(vii) Find the strong subset matrix set ideal topological space of the subset matrix semigroup space T of S.
(viii) Find the special strong set ideal topological subset matrix semigroup space T_1 of S.
(ix) Compare the T and T_1.
(x) Can S have minimal set ideal topological subset matrix semigroup space?
(xi) Can S have maximal set ideal topological subset matrix space semigroup T_4? (Justify).
(xii) Find the ideal topological subset matrix semigroup space T_2 of S.
(xiii) Compare T, T_1 and T_2.
(xiv) Find the prime set ideal topological subset matrix semigroup space T_3 of S.
(xv) Compare T_4 with T_3; can they be identical?
(xvi) Find the Smarandache set ideal topological subset matrix subsemigroup space T_5 of S.

(xvii) Find a Smaradache quasi set ideal topological subset matrix subsemigroup space T_6 of S.

(xviii) Find the Smarandache strongly quasi set ideal topological subset matrix subsemigroup space of S.

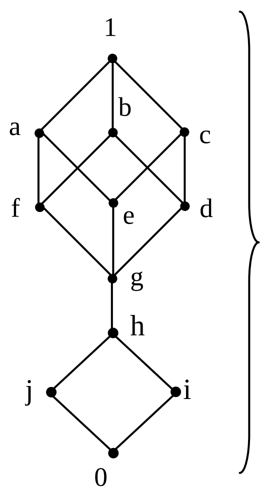
55. Let S_1 = {Collection of all subset 3×1 matrices with entries from the subsets of the semigroup Z_{60}} be the subset 3×1 matrix semigroup of the semigroup Z_{60}.

    Study questions (i) to (xviii) of problem (54). (For Question (iii) prove S_1 has zero divisors).

56. Let S_2 = {Collection of all subset 1×3 matrices with entries from the subsets of the group G = $D_{2,7}$} be the subset matrix semiring of the group G = $D_{2,7}$.

    Study question (i) to (xviii) in problem 54 for this S_2.

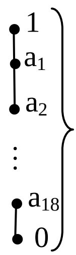
57. Let S = {Collection of all subset 3×2 matrices with subsets from the semiring



be the subset 3×2 matrix semiring of the semiring L.

(i) Study questions (i) to (xviii) of problem 54 for this S.
(ii) Construct T, T_1, T_2, …, T_5 in problem 54 the topological space of subset matrix semirings.
(iii) Build on T, T_1, …, T_5 the new topologies with $\cup_N$ and $\cap_N$ and distinguish the two topological subset matrix spaces of the subset matrix semiring.

58. Let S = {Collection of all subset 2 × 5 matrices with subset from the chain lattice C_{20} =



be the subset 2 × 5 matrix semiring of the chain lattice C_{20}.

(i) Study all the questions mentioned in problem (54) by replacing L by C_{20}.
(ii) Prove S has zero divisors.

59. Let S_1 = {Collection of all subset 2 × 4 matrices with subsets from the ring Z_{20}} be the subset 2 × 4 matrix semiring of the ring Z_{20}. Study all the questions mentioned in problem (54) for this S_1.
Does S_1 enjoy any stricking properties as ring is used? Justify.

60. Study problem (54) if Z_{20} is replaced by Z.

61. Let S = {Collection of all subset 7 × 1 matrices with subsets from the field Z_{43}} be the subset 7 × 1 matrix semiring of the field Z_{43}.
Study questions mentioned in problems (i) to (xviii) mentioned in problem 54.

62. Let B = {Collection of all subset 3 × 3 matrices with entries from the subsets of the ring Z_{24} (g); $g^2 = -g$} be the subset 3 × 3 matrix semiring of the ring Z_{24} (g).
Study all the questions (i) to (xviii) of problem 54 for this B.

63. Let M = {Collection of all subset 6 × 3 matrices with subset entries from the mixed dual number ring Z_{10} (g, g_1) where $g_1^2 = g_1$, $g_2 = 0$ $g_1 g_2 = g_2 g_1 = 0$} be the subset 6 × 3 matrix semiring of the ring.
Study all questions (i) to (xviii) of problem 54 for this M.

64. Let N = {Collection of all subset 9 × 1 matrices with subset entries from the ring $C(Z_9)$ (g), $g^2 = 0$} be the subset 9 × 1 matrix semiring of the ring $C(Z_9)$ (g).
(i) Study questions (i) to (xviii) in problem 54 for this N.
(ii) Does N enjoy any other special properties?

65. For problems 60, 61, 62 and 63 find all the possible new topologies that can be built on these subset matrix semirings mentioned in those problems.

Chapter Three

POLYNOMIAL SUBSETS

In this chapter for the first time the authors introduce the concept of polynomials with subset coefficients which will be known as subset polynomials or polynomials subsets. Here we define, describe and develop these new concepts.

DEFINITION 3.1: *Let X be a set with say n-elements {a_1, a_2, …, a_n}. S= P(X) = {All subsets of X including X and ϕ}. Take S[x] =* $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P(X) = S \right\}$. *S[x] is defined as the subset polynomial or polynomial in the variable x with coefficients from S or polynomial subsets.*

As polynomials play a major role in almost all the fields of science and more so in mathematics, we are forced to develop the concept of subset polynomials as we have already developed algebraic structures using subsets of a set or a semigroup or a group or a ring or a field or a semiring or a semifield.

We will first describe the subset polynomials in the variable x with coefficients from the subsets of a power set P(X) of the set X.

Example 3.1: Let X = {1, 2, 3, 4} be the set.

S = P(X) = {power set of X}. S[x] = $\left\{ \sum_{i=0}^{\infty} s_i x^i \,\middle|\, s_i \in S \right\}$ is the subset polynomial in the variable x.

We show how the operations '$\cup$' and '$\cap$' are performed on S[x]. Let p(x) = $\{1,2\}x^5 + \{3, 1, 2\} x^3 + \{1, 4\} x + \{4\}$ and q (x) = $\{1, 2, 3\} x^2 + \{4, 2\} x + \{3\}$ be two subset polynomials in S[x].

p(x) $\cup$ q(x) = $\{1, 2\} x^5 + \{3, 1, 2\} x^3 + \{1, 2, 3\} x^2 + (\{1,4\} \cup \{4, 2\}) x + \{4\} \cup \{3\}$

= $\{1,2\} x^5 + \{3, 1, 2\} x^3 + \{1, 2, 3\} x^2 + \{1, 4, 2\} x + \{4, 3\} \in S[x]$.

Thus {S, $\cup$} is a commutative semigroup.

Now p(x) $\cap$ q(x) = $(\{1, 2\} \cap \{1,2,3\}) x^5 \times x^2 + (\{1,2,3\} \cap \{1,2,3\}) x^3 \times x^2 + \{1,4\} \cap \{1,2,3\} x \times x^2 + \{4\} \cap \{1, 2, 3\} x^2 + \{1,2\} \cap \{4, 2\}x^5 \times x + \{1,2,3\} \cap \{4,2\} x^3 \times x + \{1,4\} \cap \{4,2\} x \times x + \{4\} \cap \{4, 2\} x + \{1,2\} \cap \{3\} x^5 + \{3,1,2\} \cap \{3\} x^3 + \{1,4\} \cap \{3\} x + \{4\} \cap \{3\}$

= $\{1,2\} x^7 + \{1,2,3\} x^5 + \{1\} x^3 + \phi x^2 + \{2\} x^6 + \{2\} x^4 + \{4\}x^2 + \{4\}x + \{\phi\}x^5 + \{3\}x^3 + \{\phi)x + \phi$

= $\{1,2\}x^7 + \{2\}x^6 + \{1,2,3\}x^5 + \{2\}x^4 + (\{1\} \cup \{3\}) x^3 + \{4\}x^2 + \{4\} x$.

= $\{1,2\}x^7 + \{2\}x^6 + \{1,2,3\}x^5 + \{2\}x^4 + \{1,3\}x^3 + \{4\}x^2 + \{4\}x$ as $\phi \cup \{2\} = \{2\}$ and $\phi\, x^n$ = empty set ϕ.

Clearly p(x) $\cap$ q(x) $\in$ S[x]. Thus (S[x] $\cap$) is a commutative semigroup.

Further $\{S[x], \cup, \cap\}$ is a distributive lattice of subset polynomials of infinite order.

We see many differences

(i) Clearly S[x] can never become a ring.
(ii) S[x] can never be a group
(iii) Maximum S[x] can be a semigroup under a single binary operation, $\cup$ or $\cap$.
(iv) S[x] with two binary operations can be a semiring or a semifield.
(v) Unless restrictions are made on degree of the polynomials always S[x] is of infinite order.

Example 3.2: Let X = {1, 2, 3, 4, 5, 6} be a set; S = P(X) the power set of X.

$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ be the subset polynomials.

$\{S[x], \cup, \cap\}$ is a semiring / lattice of infinite order.

Let $p(x) = \{1\} x^3 + \{3,1\}x + \{3\}$
and $q(x) = \{2\} x + \{2\} \in S[x]$

$p(x) \cap q(x) = \{1\} \cap \{2\} (x^3 \times x) + (\{3,1\} \cap \{2\}) (x \times x) + (\{3\} \cap \{2\}) (x) + (\{1\} \cap \{2\})x^3 + \{3,1\} \cap \{2\}x + \{3\} \cap \{2\} = \phi$.

Thus we see in case of subset polynomials we can have intersection of two non empty polynomials to be empty.

(We say two subset polynomials are subset zero divisors are product empty if $p(x) \cap q(x) = \phi$).

This never occurs in usual polynomials unless the elements are from $Z_n[x]$; i.e., rings with zero divisors. Here even if the rings have no zero divisors we still arrive at this conclusion.

Instead of saying zero polynomial when we use subsets from powerset of a set X we say empty polynomial.

In an analogous way we have an empty polynomial of degree n which is as follows:

$$\phi = \phi x^n + \phi x^{n-1} + \ldots + \phi x + \phi.$$

We see p(x) $\cup$ ϕ = p(x) and p(x) $\cap$ ϕ = ϕ

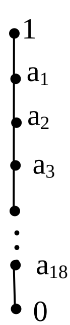
Also p(x) q(x) = ϕ can occur without p(x) = ϕ or q(x) = ϕ; this we call as subset zero divisors.

All these concepts happen to be true only in case of subset polynomials whose coefficients are from the powerset P(X) of a set X.

Example 3.3: Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} be the set; S = P(X) the power set of X. S[x] the subset polynomial semiring under $\cup$ and $\cap$.

[S[x], $\cup$, $\cap$] has non empty divisors and is of infinite order.

Example 3.4: Let S = {Collection of all subsets of the lattice C_{20} given by C_{20} =



and the empty set ϕ}.

$S[x]$ be the subset polynomial $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S\right\}$; $\{S[x], \cup, \cap\}$ is a subset polynomial semiring.

For take $p(x) = \{0,1,a_3\}\, x^3 + \{1,a_4, a_{10}, a_6\}\, x^2 + \{a_1, a_2, a_3, a_7\}$ and $q(x) = \{1, a_6, a_7, a_9\}x^2 + \{a_3, a_2, a_{10}, a_{11}, a_{17}\}$.

We find $p(x) \cup q(x)$ and $p(x) \cap q(x)$.

$p(x) \cup q(x) = \{0,1,a_3\}\, x^3 + (\{1,a_4,a_{10},a_6\} \cup \{1,a_6, a_7, a_9\})x^2 + (\{a_1, a_2, a_3, a_7\} \cup \{a_3, a_2, a_{10}, a_{11}, a_{17}\})$

$= \{0, 1, a_3\}\, x^3 + \{1, a_4, a_6, a_7, a_9, a_{10}\}\, x^2 + \{a_1, a_2, a_3, a_7, a_{10}, a_{11}, a_{17}\} \in S[x]$.

$p(x) \cap q(x) = (\{1, a_3, 0\} \cap \{1, a_6, a_7, a_9\})\, (x^5) + (\{1, a_4, a_{10}, a_6\} \cap \{1, a_6, a_7, a_9\})\, (x^4) + (\{a_1, a_2, a_3, a_7\} \cap \{1, a_6, a_7, a_9\})\, x^2 + \{1, a_3, 0\} \cap \{a_2, a_3, a_{10}, a_{11}, a_{17}\}\, x^3 + (\{1, a_4, a_6, a_{10}\} \cap \{a_2, a_3, a_{10}, a_{11}, a_{17}\})\, x^2 + (\{a_1, a_2, a_3, a_7\} \cap \{a_3, a_2, a_{10}, a_{11}, a_{17}\})$

$= \{1\}x^5 + \{1, a_6\}\, x^4 + \{\phi\}x^2 + \{a_3\}\, x^3 + \{a_{10}\}\, x^2 + \{a_2, a_3\}$
$= \{1\}\, x^5 + \{1, a_6\}\, x^4 + \{a_3\}\, x^3 + \{a_{10}\}\, x^2 + \{a_2, a_3\} \in S[x]$.

It is pertinent to keep on record that we used only set theoretic union and not the intersection and union of the lattice. Suppose we use the union and intersection of the lattice C_{20} we get the corresponding

$p(x) \cap_L q(x) = \{1, a_6, a_7, a_9, a_3, 0\}x^5 + \{1, a_6, a_7, a_9,a_{10}, a_4\}x^4 + \{a_1, a_2, a_3, a_7, a_6, a_9\}\, x^2 + \{a_2, a_3, a_{10}, a_{11}, a_{17}, 0\}\, x^3 + \{1, a_2, a_3, a_{10}, a_{11}, a_{17}, a_4, a_6\}\, x^2 + \{a_3, a_2, a_{10}, a_{11}, a_{17}, a_7\}$

$= \{1, a_6, a_7, a_9, a_3, 0\}\, x^5 + \{1, a_6, a_7, a_4, a_9, a_{10}\}\, x^4 + \{a_1, a_2, a_3, a_6, a_7, a_9\}\, x^2 + \{a_2, 0, a_3, a_{10}, a_{11}, a_{17}\}\, x^3 + \{a_2, a_3, a_7, a_{10}, a_{11}, a_{17}\} \in S[x]$

We see $p(x) \cap_L q(x) \neq p(x) \cap q(x)$. Thus we can have two polynomial subset semirings.

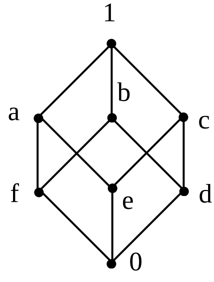
Now consider p(x) $\cup_L$ q(x) = {0, 1, a_3} x^3 + {1, a_4, a_9, a_6, a_7} x^2 + {a_1, a_2, a_3, a_7} $\in$ S[x].

Clearly p(x) $\cup_L$ q(x) $\neq$ p(x) $\cup$ q(x).

We see if we use other than powersets as coefficients for the same set S[x] we get two different semirings of infinite order.

However the other sets must be any algebraic structure with two binary operations.

Example 3.5: Let S = {Collection of all subsets of the Boolean algebra B =



together with ϕ}.

S[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ be the subset polynomial semiring [S[x], $\cup$, $\cap$] is a subset polynomial semiring with usual set theoretic union and intersection.

{S[x], $\cup_L$, $\cap_L$} is a subset polynomial ring with operations of the Boolean algebra B.

Take p(x) = {f}x^3 + {d}x^2 + {0,d} and
q(x) = {e, 0}x^2 + {e} $\in$ S[x].

p(x) $\cup$ q(x) = {f} x^3 + {d} $\cup$ {e, 0} x^2 + {0, d} $\cup$ {e}
= {f} x^3 + {0, d, e} x^2 + {0, d, e}.

$p(x) \cap q(x) = \{f\} \cap \{e, 0\}\ x^3 \times x^2 + \{d\} \cap \{e, 0\}\ x^2 \times x^2 + \{0, d\} \cap \{e\} + \{f\} \cap \{e\}\ x^3 + \{d\} \cap \{e\}\ x^2 + \{0, d\} \cap \{e\}$

$= \{\phi\}\ x^5 + \{\phi\}\ x^4 + \{0\}\ x^2 + \phi\ x^4 + \{\phi\}\ x^2 + \phi\ = \{0\}\ x^2.$

Now we find
$p(x) \cup_L q(x) = \{f\}\ x^3 + (\{d\} \cup_L \{e, 0\})x^2 + \{0, d\} \cup_L \{e\}$
$= \{f\}\ x^3 + \{d\ c\}\ x^2 + \{e, c\}.$

We see $p(x) \cup_L q\ (x) \neq p(x) \cup q(x)$.

Consider $p(x) \cap_L q(x) = \{f\} \cap_L \{e, 0\}\ x^5 + \{d\} \cap_L \{e, 0\}\ x^4 + \{d, 0\} \cap \{e, 0\}\ x^2 + \{f\} \cap_L \{e\}\ x^3 + \{d\} \cap_L \{e\}\ x^2 + \{0, d\} \cap_L \{e\}$

$= \{0\}x^5 + \{0\}\ x^4 + \{0\}x^2 + \{0\}x^3 + \{0\}x^2 + \{0\}.$

This is the zero subset polynomial. Thus a zero subset polynomial will be of the form

$p(x) = \{0\}x^n + \{0\}x^{n-1} + \ldots + \{0\}\ x + \{0\}.$

We call if $p(x) \cap_L q(x)$ = zero subset polynomial that is $\{0\}$, then $p(x)$ is a zero divisor in $S[x]$. However the concept of empty polynomial has no meaning in $\{S[x], \cup_L, \cap_L\}$. Likewise the concept of empty subset polynomial is present in $\{S[x], \cup, \cap\}$ and this $\{S[x], \cup, \cap\}$ has no relevance to the zero subset polynomial. For if $p(x)$ is a zero subset polynomial and $q(x)$ subset polynomial in $S[x]$.

$p(x) \cap q(x)$ need not be a zero subset polynomial it can be a subset polynomial with coefficient $\{\phi\}$ and $\{0\}$ or only $\{\phi\}$ or only $\{0\}$.

We for any subset polynomial $S[x]$ ($S = P(X)$ or S a semiring) we define degree of the polynomial $p(x) \in S[x]$ to be the highest degree of x with non zero (non empty) subset coefficient.

Let $p(X) = a_0 + a_1x + \ldots + a_tx^t$

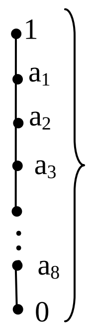
$a_i \in S$; $0 \leq i \leq t$ with $a_t \neq \{\phi\}$ (or $a_t = \{0\}$) then the degree of the subset polynomial p(x) is 't'.

We will give some examples.

Let $p(x) = \{6, 1, 2\}\, x^7 + \{\phi\}x^6 + \{5, 7, 6, 1\}\, x^5 + \{2, 3, 1\}$ here $\{6,1,2\}$, ϕ, $\{5, 7, 6, 1\}$ and $\{2, 3, 1\} \in P(X) = S$; where $X = \{1, 2, \ldots, 10\}$ be a subset polynomial in S(X).

The degree of the subset polynomial p(x) is x^7 as the subset coefficient is $\{6, 1, 2\}$.

Let $p(x) = \{0, a_1\}\, x^8 + \{0\}\, x^6 + \{0, a_1, a_2, a_3\}\, x^3 + \{a, a_8\} \in$ S[x] where S = {Collection of all subsets of the lattice C_{10} =



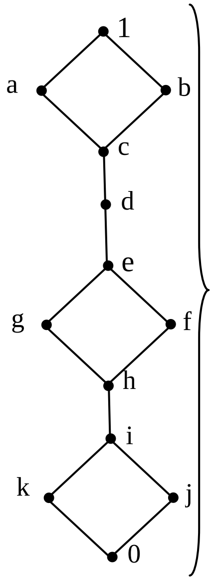
under $\cap_L$ and $\cup_L$ operation.

However if $p_1(x) = \{0\}x^5 + \{a_1, a_2, 0\}\, x^4 + \{a_1, a_2, a_3\} \in \{S[x], \cup, \cap\}$ the degree of p(x) is 5 and degree of $p_1(x)$ in $\{S[x], \cup_L, \cap_L\}$ is four.

However $p_1(x) \in \{S[x], \cap_L, \cup_L\}$ and degree of $p_1(x)$ is four. The concept of degree of the subset polynomial can be easily got depending on the context. Thus this task is simple. We do not accept zero {0} as the coefficient in case of $\{S[x], \cup_L, \cap_L\}$.

Now we proceed onto give examples of subset polynomial semirings.

Example 3.6: Let S = {Collection of all subsets of the lattice L =

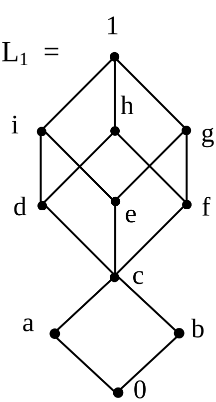


;

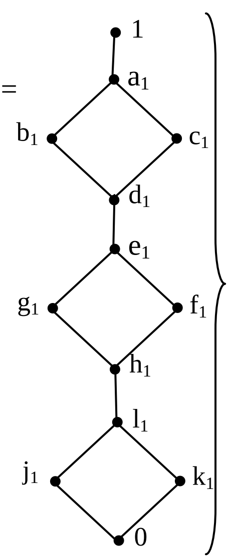
and S[x] the subset polynomial semiring. {S′[x], ∪, ∩} where S′[x] means ‘ϕ’ set is included in S and {S[x], $\cap_L$, $\cup_L$} be another subset polynomial semiring. The reader can easily find degrees of any subset polynomial in S[x] or S′[x].

Example 3.7: Let S = {Collection of all subset of the lattice L = $L_1 \times L_2$ where

L_1 =



and L_2 =



;

{S[x], $\cup_L$, $\cap_L$} and {S′[x], $\cup$, $\cap$} be the subset polynomial semirings.

Now we can define subset polynomial subsemirings of a subset polynomial semiring. We can also define subset polynomial ideal of S[x].

DEFINITION 3.2: *Let S[x] be a subset polynomial semiring. Let $P \subseteq S[x]$; if P is also a subset polynomial semiring under the operations of S[x] we define P to be the subset polynomial subsemiring of S[x]. If P in S[x] is such that for all $a(x) \in S[x]$ and $p(x) \in P$. p(x) a(x) and a(x) p(x) are in P then we define P to be the subset polynomial ideal of S[x].*

We will first illustrate both the situations by some examples.

Example 3.8: Let S = {Collection of all power sets from the set X = {1, 2, 3}}. S[x] the subset polynomial semiring

$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{\text{power set of } \{1, 2\}\} \right. \subseteq S[x]$ and P is a polynomial subset subsemiring. Clearly P is also a polynomial subset ideal of S[x].

Example 3.9: Let S = {Collection of all subsets of X = {1, 2, …, 10} together with X and ϕ} = P(X).

S[x] the polynomial subset semiring.

Consider $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{\text{Collection of all subsets of Y} \right.$ where Y = {2, 4, 6, 5, 8{ $\subseteq$ X} = P(Y)} $\subseteq$ S[x]. P is a subset polynomial subsemiring of S[x]. Infact P is a subset polynomial ideal of S[x].

Inview of this we have the following theorem.

THEOREM 3.1: *Let*

S = {P(X); X = {1, 2, …, n}} = {Powerset of X}. S[x] the subset polynomial semiring. $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P(Y) = \{\text{powerset of } Y;\ Y \text{ is subset of } X\} \right\} \subseteq S[x]$ *is a subset polynomial ideal of S[x].*

Proof is direct hence left as an exercise to the reader.

Example 3.10: Let S[x] be the subset polynomial semiring with coefficients from S = P(X) where X = {1, 2, …, 18}. Take P = {All polynomials of degree less than or equal to 19 in S[x]} $\subseteq$ S[x].

P is a subset polynomial subsemiring of S[x]. However P is not a subset polynomial ideal of S[x].

For if p(x) = {9, 2, 8, 1} x^{19} + {3, 10, 17} x + {4, 5} $\in$ P.
Let a(x) = {5, 2} x^{10} + {7, 8, 4} $\in$ S[x].

We see a(x) $\cap$ p(x)

= ({9, 2, 8, 1} $\cap$ {5, 2}) x^{29} + {3, 10, 17} $\cap$ {5, 2} x^{11} + {4, 5} $\cap$ {5, 2} x^{10} + {9, 2, 8, 1} $\cap$ {7, 8, 4} x^{19} + {3, 10, 19} $\cap$ {7, 8, 4} x + {4, 5} $\cap$ {7, 8, 4}

= {2} x^{29} + {5} x^{10} + {8} x^{19} + {4} $\notin$ P. So P is not a subset polynomial ideal of S[x].

Inview of this we have the following theorem.

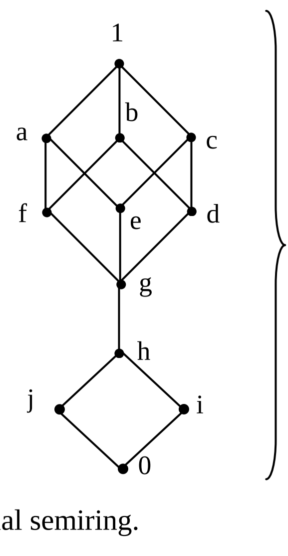
THEOREM 3.2: *Let S[x] be a subset polynomial semiring with coefficients from S = P(X) where X = {1, 2, …, n}. A subset polynomial subsemiring in general is not a subset polynomial ideal of S[x].*

Proof follows from the fact that if P[x] = {Collection of all subset polynomials of degree less than or equal to n, n a fixed positive integer} $\subseteq$ S[x] it is easily verified that P[x] is only a subset polynomial subsemiring and not a subset polynomial ideal of S[x].

Inview of this we see we have a class of subset polynomial subsemirings which are not subset polynomial ideals of S[x].

Now having seen the concept of subset polynomial subsemirings and ideals in powerset coefficient polynomials we proceed onto study this concept in case of powersets replaced by semirings.

Example 3.11: Let S [x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in$ S = {Collection of all subsets of the semiring L =



be the subset polynomial semiring.

Take P[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in$ {to the collection of all subsets of the subsemiring {j, h, I, 0} $\subseteq$ L} $\subseteq$ S[x]; it is easily verified P[x] is a subset polynomial subsemiring as well as subset polynomial ideal of S[x].

Take $P_1[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{1, a, b, c, d, e, f, g\} \subseteq L\right\}$ be the subset polynomial subsemiring of S[x].

Clearly $P_1[x]$ is not a subset polynomial ideal of S[x] for take $s(x) = \{0, h, j\} x^3 + \{i, j\}$ in S[x] and
$p(x) = \{g, 1, a\} x + \{b, c, d\}$ in $P_1[x]$, we see $s(x) \cap_L p(x)$

$= \{0, h, j\} \cap_L \{g, 1, a\} x^4.x + \{i, j\} \cap_L \{g, 1, a\} x + \{0, h, j\} \cap_L \{b, c, d\} x^3 + \{I, j\} \cap_L \{b, c, d\}$

$= \{0, h, j\} x^5 + \{i, j\} x + \{0, h, j\} x^3 + \{I, j\} \notin P_1[x]$.

Hence the claim.

However if we take $\cap$ and $\cup$ of subsets we see $p(x)\, s(x) \in P_1[x]$ as the product $p(x)\, s(x)$ is the empty subset polynomial of $P_1[x]$.

So we see in $\{S_1 [x], \cap, \cup\}$, P[x] is a subset polynomial ideal of $S_1[x]$; however in $\{S[x], \cap_L, \cup_L\}$; P[x] is not a subset polynomial ideal of S[x] only a subset polynomial subsemiring.

We see by using the operation $\cap_L$ and $\cup_L$ the subset polynomial subsemiring cannot be made into a subset polynomial ideal and however using the operation $\{S[x], \cup, \cap\}$ we get P[x] is also a subset polynomial ideal of S[x]. This is one of the marked difference between these two subset polynomial semirings.

Let us now study subset polynomial semirings over the semifields of characteristic zero.

Example 3.12: Let $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all}\right.$ subsets of the semifield $Z^+ \cup \{0\}\}$ be the subset polynomial semiring with coefficients from the subsets of semifield $Z^+ \cup \{0\}$.

Let $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in P$ = {Collection of all subsets of the semiring $5Z^+ \cup \{0\}\}\} \subseteq S[x]$.

P[x] is a subset polynomial subsemiring as well as a subset polynomial ideal.

Infact S[x] has infinite number of subset polynomial subsemirings which are ideals. Also S[x] has infinite number of subset polynomial subsemirings which are not ideals.

For take $P_n[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in P$ = {collection of all subsets of the semiring $nZ^+ \cup \{0\}$; n a positive integer} $\subseteq$ S} $\subseteq$ S[x] is a subset polynomial subsemiring which is a subset polynomial ideal.

Take $P^n[x] = \left\{ \sum_{i=0}^{n} a_i x^i \right| a_i \in S$ or any collection of subsets from any subsemiring $mZ^+ \cup \{0\}$ (m an integer)}. $n < \infty$ and $P^n[x]$ contains only subset polynomials of degree less than or equal to n. Clearly $P^n[x]$ is a subset polynomial subsemiring of S[x] which is not a subset polynomial ideal of S[x].

Example 3.13: Let S[x] = {Collection of all subsets coefficient polynomials in the variable x from the subsets of the semifield $Q^+ \cup \{0\}$} be the subset polynomial semiring. S[x] has infinite number of subset polynomial subsemirings which are not subset polynomial ideals.

For take $P[x] = \left\{ \sum a_i x^i \right| a_i \in$ {Collection of all subsets of the semifield $nZ^+ \cup \{0\}\}\} \subseteq S[x]$. P[x] is only a subset polynomial subsemiring and not an ideal; n a positive integer.

We can vary n in Z^+ and we get infinite collection of subset polynomial subsemirings which are not subset polynomial ideals.

However it is pertinent to keep on record that if we replace $\{S[x], \cup_L, \cap_L\}$ by $\{S[x], \cup, \cap\}$ then under $\cup$ and $\cap$ they are ideals. Having seen these properties we extend to give some more examples.

Example 3.14: Let
S = {Collection of all subsets of the semiring $R^+ \cup \{0\}$}.
S[x] be the subset polynomial ring.

Take $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \right.$ {Collection of all subsets of $Q^+ \cup \{0\}\} \subseteq S\}\} \subseteq S[x]$; $\{P[x], \cup_L, \cap_L\}$ is only a subset polynomial subsemiring of S[x], however P[x] is not a subset polynomial ideal of S[x]. Suppose ϕ is taken in S and $\{S'[x], \cup, \cap\}$ (i.e., $\phi \in S'$) be the subset polynomial semiring.

$\{P'[x], \cup, \cap\}$ be the subset polynomial subsemiring of S′[x]; then clearly P′[x] is the subset polynomial ideal of S′[x].

We have infinitely many subset polynomial subsemirings in $\{S[x], \cup_L, \cap_L\}$ which are not subset polynomial ideals of $\{S[x], \cup_L, \cap_L\}$. Also $\{S'[x], \cup, \cap\}$ has subset polynomial ideals.

Interested reader can study these structures.

Now having seen examples of both infinite subset polynomial semirings using powerset of a set X or a semiring we now proceed onto develop techniques of solving subset polynomials.

We call $a_1 + a_2x$ as linear subset polynomials where $a_1, a_2 \in P(X)$ as we cannot have the concept of inverse or unit in case of powersets; for $X \cap A = A$ for all $A \in P(X)$ and $A \cap \phi = \phi$ for all $A \in P(X)$. $A \cup \phi = A$, $X \cup A = X$; we cannot have inverse of a subset in P(X). Suppose

$p(x) = \{1, 2, 3\} x + \{2, 5, 1\}$ and $q\{x\} = \{2, 4, 6\} x + \{6, 7, 8\}$ where $\{1,2, 3\}, \{2, 5, 1\}, \{2, 4, 6\}, \{8, 6, 7\} \in P(X)$ where $X = \{1, 2, 3, \ldots, 10\}$.

$p(x) \times q(x) = (\{1, 2, 3\} x + \{2, 5, 1\}) \times (\{2, 4, 6\} x + \{6, 7, 8\})$

$= \{2\} x^2 + \{2\} x + \{\phi\}$.

To solve the equation we can have several ways of reducing this as two linear subset polynomials.

We give few of the solutions.

$(\{2\}x + \{4, 8, 6\})\ (\{2\}x + \{2, 7\})$
$= \{2\}x^2 + \{2\}x + \phi$.

So $\{2\}x + \{4, 8, 6\} = p_1 (x)$ and
$\{2\}x + \{2, 7\} = q_1 (x)$.

$(\{2,4,8\} x + \{8\}) (\{2, 6, 9\} x + \{2, 6\})$
$= \{2\} x^2 + \phi x + \{2\}x + \phi$.

So that $p_2 (x) = \{2, 4, 8\} x + \{8\}$
and $q_2 (x) = \{2, 6, 9\} x + \{2, 6\}$.

Take $(\{2, 4\}x + \{5\}) (\{2\} x + \{6, 2\})$
$= \{2\}x^2 + \{2\}x + \phi$
thus $p_3 (x) = \{2, 4\} x + \{5\}$
and $q_3 (x) = \{2\}x + \{6, 2\}$

and so on.

We have many ways of reducing a given second degree subset polynomial with subset coefficients from a power set P(X) of the set X.

This is the marked difference between usual polynomials and the subset polynomials. Thus the breaking of a nth degree subset polynomial linearly is not unique we have several ways of doing it. So we can think of any analogous concept like for fundamental theorem of algebra which says every nth degree polynomials has n and only n roots which can be equivalently termed as every nth degree polynomial can be written uniquely except for the order as the product of n linear polynomials of course repetition is accepted.

Let us consider two polynomials
$(\{3\}x + \{6\})\ \ (\{3\}x + \{6\})$
$\{3\}x^2 + \{6\}$
$= (\{3, 2\}x + \{6\})\ (\{2, 5\}x + \{6, 9\})$
$= \{2\}x^2 + \{6\}$
$= (\{8, 2, 1\}x + \{6, 4, 5\}) \times (\{9, 3, 2\}\ x + \{6, 7\})$
$= \{2\}\ x^2 + \{6\}$ and so on.

So we see even a second degree subset polynomial equation of this simple form has several ways of decomposition. Infact the larger the X we take the powerset P(X) give more and more decompositions the smaller the set, the lesser number of linear decomposition as the product of linear subset polynomials.

Consider $p(x) = (\{1, 2, 3\}x + \{4, 5\})\ (\{1, 2, 5\}x + \{9, 10\})$ $(\{1, 2, 7\}x + \{8, 7\})$ where $\{1, 2, 3\}$, $\{4, 5\}$, $\{1, 2, 5\}$, $\{9, 10\}$, $\{1, 2, 7\}$ and $\{8, 7\}$ are in $S = P(X)$, where $X = \{1, 2, 3, \ldots, 15\}$.

$p(x) = (\{1, 2, 3\}x + \{4, 5\})\ (\{1, 2\}x^2 + \phi x + \phi x + \phi)$
$= \{1,2\}x^3 + \phi x^2 + \phi x + \phi$
$= \{1,2\}\ x^3$.

Thus we see polynomials of the form $p(x) = Ax^n$; with $A \in P(X)$ can be factored into subset linear polynomials.

Let $p(x) = \{1, 4, 6, 8\}\ x^5$ then $p(x) = (\{1, 4, 6, 8\}x + \{2\})$ $(\{1, 4, 6, 8, 7\}x + \{3\})\ (\{1, 4, 6, 8, 5\}x + \{10\})\ (\{1, 4, 6, 8, 5, 12\}x + \{11\})\ (\{1, 4, 6, 8, 13\}x + \{13\})$; where $\{10\}$, $\{11\}$, $\{3\}$, {2}, {13}, {1, 4, 6, 8}, {1, 4, 6, 8, 5}, {1, 4, 6, 8, 7}, {1, 4, 6, 8, 5, 12}, {1, 4, 6, 8, 13} $\in$ P(X).

Thus this is the marked difference between usual polynomials and subset polynomials. For $p(x) = x^n$, we write it clearly as $p(x) = x.x.x. \ldots, x = x^n$, but if $p(x) = \{A\}x^n$ where $A \in P(X)$ we can break it linearly into n terms and n of them distinct.

A natural question would be can we have linear decomposition for $p(x) = Ax^n$ for any n or does it depend on X of P(X). The answer is the decomposition of $p(x) = Ax^n$ is highly dependent on the cardinality of X and the n.

For if we take X = {1, 2, 3} and $p(x) = \{2, 1\}\, x^4$ then we cannot break it as 4 distinct linear subset polynomials.

For $p(x) = \{2,1\}\, x^4$

$= (\{1,2\}x + \phi)\, (\{1, 2, 3\}\, x + \{3\})\, (\{1, 2\}x + \{\phi)\}\, (\{1,2, 3\}\, x + \phi)$.

We see only two of them are distinct that also in a very different way so $p(x) = Ax^n$ may not be decomposable as distinct linear factors.

It is left as open problems.

Problem 3.1: If $X = \{a_1, a_2, \ldots, a_n\}$ and $Ax^m = p(x)$; $m > n$, $A \in P(X)$ then can we decompose p(x) into linear subset polynomials which are distinct?

If $m < n$, is it always possible to decompose p(x) into distinct m linear subset polynomials?

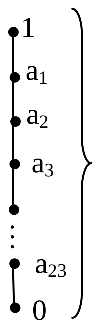
Now we proceed onto state the problem.

Problem 3.2: Let P(X) be a power set of X where $|X| = n$.

Let $p(x) = Ax^m + B$, $A, B \in P(X)$ $m < n$. Can $p(x)$ be decomposed into distinct linear subset polynomials in $S[x]$ where $S = P(X)$?

We leave these two open problems for any researcher.

Example 3.15: Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \right.$ {Collection of all subsets of the semiring $C_{25} = L =$



be the subset polynomial semiring.

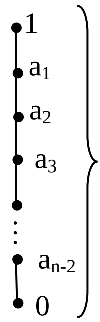
Take $p(x) = \{a_6\}x^2 \in S[x]$; if possible.

$p(x) = (\{a_6\}x + \{a_1\})\ (\{a_6, a_2\}\ x + \{a_3\})$
$= \{a_6\}x^2 + \{a_6, a_2\}x + \{a_6\}x + \{a_3\}$ so we cannot present in this way.

We are at a problem of finding a possible linear distinct decomposition of subset polynomials.

This is also left as an open problem.

Problem 3.3: Let $S[x] = \left\{ \sum a_i x^i \,\middle|\, a_i \in \right.$ {Collection of all subsets of a chain lattice $C_n = L =$



be a subset polynomial semiring.

(i) Can $p(X) = Ax^n + B$ where $A, B \in S$ = {Collection of all subsets of the chain lattice C_n=L} be linearly decomposed into distinct subset polynomials with coefficient from S?

(ii) For what values of n and $A \in S$ we can solve $p(x) = Ax^n$ into distinct linear subset polynomials in S[x]?

Suppose in problem 3.3, L is replaced by $Z^+ \cup \{0\}$ can we solve $p(x) = Ax^n$ and $p_1(x) = Ax^n + B$ where $A, B \in S$ as distinct linear decomposition of subset polynomials?

Example 3.16: Let $S[x] = \left\{ \sum a_i x^i \,\middle|\, a_i \in \right.$ {Collection of all subsets from the semiring $Z^+ \cup \{0\}$}} be the subset polynomial semiring. Let $p(x) = \{2, 4, 3\}\, x^4 + \{8, 9, 10, 11\} \in S[x]$. $p(x) = (\{2, 4, 3\}x + \{8, 9, 10, 11\})\, (\{2, 4, 3, 19\}\, x + \{8, 9, 10, 11, 18\})\, (\{2, 4, 3, 25\}x + \{8, 9, 10, 11, 27\})\, (\{2, 4,3,4,0)x + \{8,9,10,11,90\})$; that is p(x) is linearly decomposable into four distinct linear subset polynomials.

Inview of this we see $p(x) = Ax^n + B$; $A, B \in S$ can be decomposed linearly into n linear subset polynomials. This is possible as the cardinality of the semiring is infinite, that is $Z^+ \cup \{0\}$.

Example 3.17: Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{\text{Collection of all subsets of the semiring } Q^+ \cup \{0\}\}\right\}$ be the subset polynomial semiring.

All subset polynomials of the form $Ax^n + B = p(x)$ and $q(x) = Ax^m$ are linearly decomposable as distinct linear subset polynomials.

However if $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ is replaced by P(X), power set of a finite set X or by a chain lattice L of finite order or by a any distributive lattice of finite order both the subset polynomials $p(x) = Ax^n + b$ and $q(x) = Ax^m$ are not linearly decomposable for all m and n.

Problem 3.4: It is left as an open problem to find the highest values of m and n in the subset polynomials $p(x) = Ax^n + b$ and $q(x) = Ax^m$ A, B $\in$ S = {Collection of all subsets of a semiring of order t} so that the subset polynomials are not linearly decomposible for all m and n when

(i) $t > m$ and $t > n$
(ii) $t < m$ and $t < n$
(iii) $t = m = n$
(iv) $t > m$ and $t < n$ and
(v) $t < m$ and $t > n$.

Now one of the interesting features about these subset polynomials is for a given nth degree subset polynomial we have two possibilities.

(i) The nth degree subset polynomial can be linearly decomposable into n distinct subset polynomials in many ways.
(ii) Sometimes of nth degree polynomial may not be linearly decomposable into distinct subset polynomials only as repeated linear subset polynomials.

Thus we see subset polynomials behave in a very different way from the usual polynomials. Further we cannot write a value for x, however we can only say with some subset coefficient we have a related subset associated with it.

At this stage the usual way of solving polynomials is not possible.

We proceed onto construct examples of subset polynomial rings over fields and rings.

In the definition of a subset polynomial semiring if the semiring or the powerset is replaced by a ring still we get only a subset polynomial semiring (The same is true if the semiring is replaced by a field).

Example 3.18: Let

$S = \{\text{Collection of all subsets of the ring } Z_{12}\}$.

$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ be the subset polynomial semiring and $\cap_N$ and $\cup_N$ i.e., the operations on sets inbibe the operations from Z_{12}.

Consider $p(x) = \{2\}x + \{6\}$ we see
$p(\{3\}) = \{2\}\ \{3\} + \{6\} = \{6\} + \{6\} = \{12\} = \{0\}$.

Thus $\{3\}$ is the subset root of the polynomial $p\{x\}$.
Take $x = \{3, 9\} \in S$ we see $p(\{3, 9\}) = \{2\}\ \{3, 9\} + \{6\}$
$= \{6, 6\} + 6$
$= \{6\} + \{6\} = \{0\}$.

Thus $\{3, 9\} \in S$ is also a root of $p(x)$.

Now consider the subset polynomial

$$p(x) = \{4\}x + \{8\} \in S[x].$$

Put x = {4}
p ({4}) = {4} {4} + {8}
= {4} + {8} = {0}.

So {4} is a subset root of p(x).

Take x = {7}, p(x) = p ({7}) = {4} {7} + {8}
= {4} + {8} = {0}
So {7} is also a subset root of p(x).
Let x = {10}, p({10}) = {4} × {10} + {8}
= {4} + {8} = {0}.

{10} is also a root of p(x). Thus the linear subset polynomial has three distinct subset roots.

Suppose p(x) = {4} x^3 ∈ S[x] to find roots of p(x).
p(x) = {4} x^3 take x = {3}
p({3}) = {4} $(\{3\})^3$ = {4} ({3} × {3} × {3})
= {4} ({9} × {3})
= {4} {3} = {0}.

So {3} is a root.
{0} is a trivial root. {6} is a root, {9} is also a root for p({9}) = {4} $(\{9\})^3$ = {0}.

{3,6} is also a subset root of p(x) for

p({3, 6}) = {4} $\{(3, 6)\}^3$ = {0}.

{3, 6, 0} is also a subset root p({3, 6, 0}) = {0}.
{0, a} is also a root of p(x),
{3, 9} is also a root of p(x),
{0, 3, 9} is also a root of p(x),
{0, 3, 9, 6} is also a root of p(x) and
{3, 9, 6} is also a root of p(x).

Thus we see the subset polynomial p(x) = {4}x^3 has several subset roots.

This is a marked difference and an interesting feature of the subset polynomials.

The main advantage of such subset polynomial equations are if we are given a usual polynomial which is linear we have one and only one root; so if someone wants to apply it in some situation and has no choice but forced to take the solution whether feacible or not but incase of subset polynomial equations we have several such roots and we can choose the feacible one.

But however by using the subset linear polynomial we can have several solutions for this linear subset polynomial equation. So depending on the nature of the problem one can make an appropriate or a feacible solutions from the set of subset solutions. Thus without any doubt this method has an advantage over the other methods.

Example 3.19: Let

S = {Collection of all subsets from the ring Z_{24}}. S[x] be the subset polynomial semiring in the variable x.

Let $p(x) = \{2, 4, 6\}x^2 + \{7, 8, 9\}$ and
$q(x) = \{6, 8, 12\}\, x^3 + \{3, 2, 0\}x^2 + \{1, 2\}$ be in S(x).

We show how we perform the operation of '+' and $\times$ on S[x].

$p(x) + q(x) = \{6, 8, 12\}\, x^3 + (\{2, 4, 6\} + \{3, 2, 0\})\, x^2 + (\{7, 8, 9\} + \{1, 2\})$

$= \{6, 8, 12\}\, x^3 + \{5, 4, 2, 4, 7, 9, 8, 6\}\, x^2 + \{8, 9, 10, 11\} \in$ S[x].

This is the way '+' operation in S[x] is performed.

Consider $p(x) \times q(x) =$

$\{2, 4, 6\} \times \{6, 8, 12\}x^5 + \{7, 8, 9\} \{6, 8, 12\}x^3 + \{2, 4, 6\} \times \{3, 2, 0\} x^4 + \{7, 8, 9\} \{3, 2, 0\} x^2 + \{2, 4, 6\} \times \{1, 2\} x^2 + \{7, 8, 9\} \times \{1, 2\}$

$= \{12, 0, 16, 8\} x^5 + \{18, 8, 12, 16, 0, 6\}x^3 + \{0, 6, 4, 12, 8, 18\} x^4 + \{21, 14, 0, 16, 3, 18\} x^2 + \{2, 4, 6, 8, 12\} x^2 + \{7, 8, 9, 14, 16, 18\}$

$= \{0, 8, 12, 16\} x^5 + \{0, 4, 6, 8, 12, 18\} x^4 + \{0, 6, 8, 12, 16, 18\} x^3 + \{2, 4, 6, 8, 12, 23, 16, 18, 5, 20, 1, 7, 22, 3, 9, 0, 11, 9, 15\} x^2 + \{7, 8, 9, 14, 16, 18\}$.

This is the way product in S[x] is performed.

We can also speak of the degree of the polynomials in S[x] after performing addition and multiplication.

If the coefficients which are subsets are from the chain lattice or fields then deg (p(x) × q(x)) = deg (p(x) + deg (q(x)).

If the subsets are from a Boolean algebra of order greater than two or from rings with zero divisors

deg (p(x) × q(x)) ≤ deg p(x) + deg q(x).

We will illustrate this situation by some examples.

Let us take a polynomial q(x), p(x) from S[x] where S = {Collection of all subsets from Z}. Let $p(x) = m_t x^t + m_{t-1} x^{t-1} + \ldots + m_0$ and $q_0 = n_s x^s + n_{s-1} x^{s-1} + \ldots + n_0$ where $m_t \neq 0$ and $n_s \neq 0$; m_i and $n_j \in S$; $0 \leq i \leq t$ and $0 \leq j \leq s$.

Clearly deg (p(x) × q(x)) = deg p(x) + deg q(x).

deg (p(x) + q(x)) = deg p(x) (or deg q(x)) according as deg (p(x)) > deg q(x) or deg q(x) > deg p(x).

Let $p(x) = \{0, 3, 2, 7\} x^7 + \{13, 5\} x^2 + \{-7, -4, 9, 91\}x + \{120, 14, -20, 11\}$ and $q(x) = \{8, 1, 9\}x^{12} + \{-3, -2, -1\}x^8 + \{6, 7\} x^7 + \{8\} x^3 + \{20, -14\} x + \{3, 1, 2\} \in S[x]$.

Clearly deg p(x) = 7 and deg q(x) = 12.

Now deg (p(x) + deg q(x)) = deg ({8, 1, 9}x^{12} + {–3, –2, –1}x^8 + {0, 3, 2, 7, 6} x^7 + {8} x^3 +{13, 5} x^2 + {–7, –4, 9, 91, 20, –14} x + {3, 1, 2, 120, 14, –20, 11})

= 12 (= deg q(x)).

Now p(x) × q(x) = {8, 1, 9} {0, 3, 7, 2} × x^{12} × x^7 + {–3, –2, –1} × {0, 3, 2, 7} x^8 × x^7 {6, 7} {0, 3, 2, 7} x^7 × x^7 + {120, 14, –20, 11} × {3, 1, 2}.

Thus deg (p(x) × q(x)) =
deg {0, 24, 3, 27, 7, 56, 63, 16, 2, 18} x^{19} + {0, –9, –6, –21, –6, –4, –14, –7, –3, –2} x^{15} + … + {360, 42, –60, 33, 120, 14, –20, +11, 240, 28, –40, 22}) = 19.

Clearly in S = {Collection of all subsets of the ring Z}. So if A, B ∈ S. Clearly A × B ≠ {0}.

Now A + B = {0} can occur for take A = {7} and B = {–7} then A + B = {0}. So when we take two subset polynomials p(x) and q(x) of same degree say n, then see

deg (p(x) + q(x)) ≤ deg (p(x)) (or deg q(x)).

Let p(x) = {–10}x^8 + {8, 9, 4, 3}x^5 + {2, 3} x^2 + {7, 8, 10, –150, 4}

and q(x) = {10} x^8 + {3, 4, 5} x^4 + {–40, 80, 129, –59} x^3 + {8} x^2 + {7, 4, –10} in S[x].

We see p(x) + q(x) = ({–10} +{10}) x^8 + {8, 9, 4, 3} x^5 + {3, 4, 5} x^4 + {–40, 80, 129, –59} x^3 + ({2, 3} + {8}) x^2 + ({7, 8, 10, –150, 4} + {7, 4, –10})

= {0}x^8 + {8, 9, 4, 3}x^5 + {3, 4, 5} x^4 + {–40, 80, 129, –59} x^3 + {10, 11} x^2 + {14, 15, 17, –143, 11, 12, 14, –146, 8, –3, –2, 0, –160, –6}.

So deg (p(x) + q(x)) = 5 as coefficient of x^8 is {0}.
So deg (p(x) + q(x)) < deg (p(x) and deg q(x).

Let us take S = {Collection of all subsets from the ring Z_{12}}.

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in s\}$$ be the subset polynomial semiring.

Take p(x) = {0, 4}x^5 + {1, 2, 3} x^3 + {6, 7} x + {8, 9, 10} and q (x) = {0, 3, 6, 9} x^8 + {3, 5, 7} x^2 + {1, 2, 0} $\in$ S[x].

We find p(x) × q(x) = {0, 4} × {0, 3, 6, 9} × $x^5 \times x^8$ + {1, 2, 3} × {0, 3, 6, 9} $x^3 \times x^8$ +{6, 7} × {0, 3, 6, 9} x × x^8 + {8, 9, 10} × {0, 3, 6, 9} x^8 + {0, 4} × {3, 5, 7} $x^5 \times x^2$ + {1, 2, 3} × {3, 5, 7} $x^3 \times x^2$ + {6, 7} × {3, 5,7} x × x^2 + {8, 9, 10} × {3, 5, 7} x^2 + {0, 4} {1, 2, 0} x^5 + {1, 2, 3} × {1, 2, 0} × x^3 + {6, 7} × {1, 2, 0} x + {8, 9, 10} × {1, 2, 0}

= {0} x^{13} + {0, 3, 6, 9} x^{11} + {0, 6, 9, 3} x^9 + {0, 3, 6, 9} x^8 + {0, 8, 4} x^7 + {3, 5, 7, 6, 10, 2, 9} x^5 + {6, 9, 11, 1} x^3 + {0, 4, 8, 3, 9, 6, 2, 10} x^2 + {0, 4, 8} x^5 + {1, 2, 3, 4, 6} x^3 + {6, 0, 7, 2} x + {8, 4, 9, 6, 10}.

Clearly deg (p(x) × q(x)) = 11 < deg p(x) + deg q(x) and
deg (p(x) + q(x)) = deg q(x) = 8.

Take p(x) = {7} x^8 + {2,1, 3, 4} x^6 + {0, 5, 6} x^2 + {1, 2, 0, 9, 11} and

q(x) = {5}x^8 + {1, 2} x^7 + {4, 8} x^6 + {7} x^2 + {3} $\in$ S [x].

We see p(x) + q(x) = ({7} + {5}) x^8 + {1, 2} x^7 + {1,2,3,4} x^6 + {4, 8} x^6 + ({0, 5, 6} + {7}) x^2 + {1,2,0,9,11} +{3} = {0} x^8 + {1,2}x^7 + {1, 4, 2, 3, 8} x^6 + {0, 5, 6, 7} x^2 + {1, 2, 3, 9, 11, 0}.

deg q(x) = 8 but
degree of (p(x) + q(x)) = 7 < deg p(x) and (deg q(x)).

Now having seen how the degree of subset polynomial work out and also having seen how product and sum work out we now proceed onto work with subset polynomials degree in S[x] of a group and a semigroup under the '$\cup$' and '$\cap$' operations.

When coefficient of any power of x is empty we do not take that element to exist.

Let
S = {Collection of all subsets of the group $G = Z_7 \setminus \{1\}$}, S[x] be the subset polynomial semiring with coefficients from S in the variable x.

Let $p(x) = \{3, 2\} x^6 + \{1,2\}x^3 + \{1,5,6\}$ and $q(x) =\{4,6,1\}x^8 + \{3,4\} x^4 + \{1,2\}$ be in S[x].

We find $p(x) \cap q(x) = (\{3,2\} \cap \{4,6,1\}) (x^6 \times x^8) + (\{1,2\} \cap \{4,6,1\}) (x^3 \times x^8) + (\{1,5,6\} \cap \{4,6,1\}) (x^8) + (\{3,2\} \cap \{3, 4\}) (x^6 \times x^4) + (\{1,2\} \cap \{3,4\}) (x^3 \times x^4) + (\{1,5,6\} \cap \{3,4\}) (x^3 \times x^4) + (\{1,5,6\} \cap \{3,4\}) (x^4) + (\{3,2\} \cap \{1,2\}) (x^6) + (\{1,2\} \cap \{1,2\}) x^3 + (\{1,5,6\} \cap \{1,2\}) = \phi x^{14} + \{1\} x^9 + \{1,6\}x^8 + \{3\}x^{10} + \{\phi\}x^7 + \{\phi\}x^4 + \{2\}x^6 + \{1,2\}x^3+\{1\} \in S[x]$.

Clearly deg $(p(x) \times q(x))$ is 9 for the coefficient of x^{14} is ϕ.

Now one can find $p(x) \cup q(x) = \{4, 6, 1\}x^8 + \{3, 2\}x^6 + \{3, 4\}x^4 + \{1,2\}x^3 + \{1, 2, 5, 6\} \in S[x]$ and deg of $(p(x) + q(x))$ is 8.

We have seen that in case of subset polynomial semirings over semigroup or group we may have the non commutative nature of the semigroup.

We will illustrate this situation by some example.

Let S = {Collection of all subsets from symmetric semigroup S(3)}.

$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in S\}$ be the subset polynomial semigroup of the symmetric semigroup S(3).

Let $p(x) = \{p_1\}x^3 + \{p_2, p_1\}x + \{p_4\}$ and $q(x) = \{p_2\}x^2 + \{p_1\}x + \{p_3\} \in S[x]$.

Now $p(x) \cap q(x) = \{p_1\} \times \{p_3\}x^3 + \{p_2, p_1\} \times \{p_2\}x \times x^2 + \{p_2, p_1\} \times \{p_1\}x \times x + \{p_4\} \{p_1\}x + \{p_1\} \times \{p_3\}x^3 + \{p_2, p_1\} \times \{p_3\}x + \{p_4\} \{p_3\}$

$= \{p_5\}x^5 + \{e\}x^4 + \{p_4\}x^3 + \{e, p_5\}x^3 + \{p_4, e\}x^2 + \{p_2\}x + \{p_4\}x^3 + \{p_4, p_5\}x + \{p_1\}$.

$= \{p_5\}x^5 + \{e\}x^4 + \{e, p_4, p_5\}x^3 + \{p_4, e\}x^2 + \{p_2, p_4, p_5\}x + \{p_1\}$.

We find now $q(x) \times p(x) = (\{p_2\}x^2 + \{p_1\}x + p_3) (\{p_1\}x^3 + \{p_2, p_1\}x + \{p_4\})$

$= \{p_2\ p_1\}x^2 \times x^5 + \{\ p_1^2\ \}x^4 + \{p_3\ p_1\}x^3 + \{\ p_2^2, p_2\ p_1\}x^3 + \{p_1\ p_2,\ p_1^2\ \}x^2 + \{p_1\ p_4\}x + \{p_3, p_1\}x^3 + \{p_3, p_2, p_3, p_1\}x + \{p_3\ p_4\}$

$= \{p_4\}x^5 + \{e\}x^4 + \{p_5\}x^3 + \{e, p_4\}x^3 + \{p_5, e\}x^2 + \{p_3\}x + \{p_5\}x^3 + \{p_5, p_4\}x + \{p_2\}$

$= \{p_4\}x^5 + \{e\}x_4 + \{p_5, e, p_4\}x^3 + \{p_5, e\}x^2 + \{p_3, p_4, p_5\}x + \{p_2\}$.

Clearly $p(x) \times q(x) \neq q(x) \times p(x)$.

So we see this subset polynomial semigroup is non commutative but of infinite order.

Now we consider the subset polynomial semigroup over the group A_4.

Let

S = {Collection of all subsets from the alternating group A_4}; $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ be the subset polynomial semigroup over A_4. Clearly S[x] is a non commutative semigroup of infinite order.

$$\text{Let } p(x) = \left\{\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}\right\}x + \left\{\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\right\} \text{ and}$$

$$q(x) = \left\{\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\right\}x + \left\{\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}\right\} \text{ be in S[x].}$$

$$p(x) \times q(x) = \left\{\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\right\} \times x^2$$

$$+ \left\{\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\right\} x +$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}\right\} x + \left\{\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}\right\} x^2 + \left\{\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}\right\} x +$$

$$\left\{\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\right\} x + \left\{\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\right\}x^2 + \left\{\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\right\}x +$$

$$\left\{\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\right\} \in S[x].$$

$q(x) \times p(x) =$

$$\left(\left\{\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\right\}x + \left\{\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\right\}\right)\times$$

$$\left(\left\{\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}\right\}x + \left\{\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\right\}\right)$$

$$= \left\{\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix},\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}\right\}x^2 +$$

$$\left\{\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}\times\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}\right\}x +$$

$$\left\{\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\times\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\right\}x + \left\{\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\right\}x^2 + \left\{\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}\right\}x +$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}\right\}x + \left\{\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\right\}.$$

$p(x) \times q(x) \neq q(x) \times p(x).$

We see if we are to get subset polynomial semigroups which are non commutative that is to get non commutative subset polynomial semigroups we should use non commutative semigroup or non commutative group over ring or field or semifields or semirings.

To get non commutative subset semiring using non commutative rings.

Let S[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in$ {Collection of all subsets from the group ring Z_2S_3} be the subset polynomial semiring which is clearly non commutative.

For take p(x) = $\{p_1, p_2\}x^2 + \{p_3\}$ and q(x) = $\{p_4\}x + \{p_1\}$ in S[x];

$p(x) \times q(x) = (\{p_1, p_2\}x^2 + \{p_3\}) \times (\{p_4\}x + \{p_1\}) = \{p_1 p_4, p_2 p_4\}x^3 + \{p_3 p_4\}x + \{p_1^2, p_2 p_1\}x^2 + \{p_3 p_1\}$

$= \{p_3, p_1\}x^3 + \{e, p_4\}x^2 + \{p_2\}x + \{p_5\} \in S[x].$

$q(x) \times p(x) = \{(\{p_4\}x + \{p_1\}) \times (\{p_1, p_2\}x^2 . \{p_3\})$
$= \{p_4\} \times \{p_1, p_2\}x^3 + \{p_1^2, p_1 p_2\} \times x^2 + \{p_4 p_3\}x + \{p_1 p_3\}$
$= \{p_2, p_3\}x^3 + \{e, p_5\}x^2 + \{p_5\}x + \{p_4\}.$

Clearly $p(x) \times q(x) \neq q(x)\, p(x)$

Thus S[x] is a non commutative subset polynomial semiring.

Let S[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in$ S = {Collection of all subsets of the semigroup ring ZS(7) where S(7) is the symmetric semigroup} be the subset polynomial semiring which is non commutative.

Let $p(x) = \{1 + p_1, p_4\}x + \{p_2 + p_3, 1\}$ and $q(x) = \{p_2 + 1 + p_3\}x + \{p_4 + p_1, p_3\}$ be in S[x].

$p(x) \times q(x) = (\{1 + p_1, p_4\} + \{p_2 + p_3, 1\})x^2 + \{1, p_2 + p_3\} \times \{p_2 + p_3 + 1\})x + \{1+p_1, p_4\} \{p_4 + p_1, p_3\}x + \{p_2 + p_3, 1\} \times \{p_1 + p_4, p_3\}$

$= \{p_2 + p_3 + 1, p_2 + p_3 + p_4 + p_5\}x + \{p_4 + p_3 + p_1, p_2 + p_4 + p_5 + p_2\}x^2 + \{p_3 + p_4, p_5 + p_2, e, e + p_1 + p_4 + p_3\}x + \{e + p_5, p_3, p_1 + p_4, p_1 + p_2\}$

$= \{p_4 + p_5, p_1 + p_3 + p_4\}x^2 + \{p_2 + p_3 + 1, p_2 + p_3 + p_4 + p_5, p_3 + p_4, p_2 + p_5, e, e + p_1 + p_4 + p_3\}x + \{e + p_5, p_3, p_1 + p_4, p_1 + p_2\}$.

Find $q(x) \times p(x)$ and test the commutativity of the product.

Now we have seen several examples of non commutative subset polynomial semirings. We can find subset polynomial subsemirings and subset polynomial ideals of these subset polynomial semiring.

Infact if we are using non commutative subset polynomial semirings to find right subset polynomial ideals and left subset polynomial ideal for subset polynomial semirings.

We will show this by some examples.

Let $S[x] = \left\{\sum a_i x^i \,\middle|\, a_i \in S\right.$ = {Collection of all subsets of the group ring $Z_5 S_3$} be the subset polynomial semiring.

Take the right ideal generated by the subset polynomial $p(x) = \{p_1 + 1\}x^2 + \{p_2\}x + \{p_3\}$, say I.

Clearly the left ideal generated by the subset polynomial p(x) be J. We see $I \neq J$.

Consider $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subset of the semigroup ring Z_7 S(5)} be the subset polynomial semiring S[x] has both left and right ideals.

For take the right ideal generated by the subset polynomial

$$p(x) = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 4 & 1 \end{pmatrix} \right\} x^2 +$$

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \right\} x +$$

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 5 & 1 & 2 \end{pmatrix} \right\}$$

generates both right subset polynomial ideal say J as well as left subset polynomial ideal say I; we see $I \neq J$.

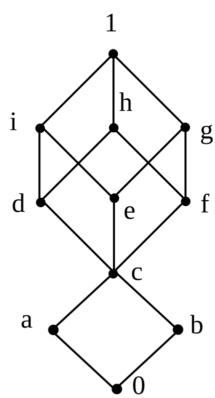
Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets of the group ring $ZD_{2,9}$} be the subset polynomial semiring of the ring $ZD_{2,9}$.

This has both right ideals as well as left ideals.

Let $D_{2,9} = \{a, b \mid a^2 = 1 = b^9, bab = a\}$ be the non commutative group. S[x] is a subset polynomial semiring of the group ring.

Let $p(x) = \{8 + 9b + 7b^2, 1, 1 + ab^2 + ba^2\}x^2 + \{-9b^3 + ab^2 + 9a, 1 + b^3a\} \in S[x]$. Let I be the right ideal generated by p(x) and J be the left ideal generated by p(x). Clearly $J \neq I$.

Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets of the group semiring LS_4 where L is the distributive lattice given by L =



Take p(x) = $\{as_1 + ds_2 + es_3 + fs_4\}x^3 + \{as_5, ds_7, es_3 + s_4, s_{12} + bs_{15}\} \in S[x]$, $s_i \in S_4$; $1 \leq i \leq 24$.

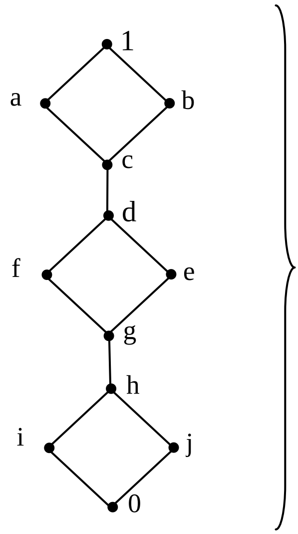
We find the right and left ideal generated by p(x).

We can also speak of maximal subset polynomial ideals and minimal subset polynomial ideals. This task is left as an exercise to the reader.

However one can also develop the notion of prime and principal subset polynomial ideals this task is a matter of routine so left as an exercise to the reader. Several problems in this direction are suggested at the end of this chapter.

We can define in a similar way the notion of Smarandache subset polynomial semiring, Smarandache subset polynomial subsemiring and Smarandache subset polynomial ideals of all types.

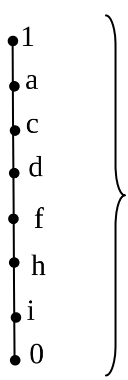
Example 3.20: Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\right.$Collection of all subsets of the semiring L; where L =



be the subset polynomial semiring of the semiring L. S[x] is a Smarandache subset polynomial semiring.

For take $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{\right.$Collection of all subsets of the chain sublattice T of L is as follows:

T :



to be the subset polynomial subsemiring of the subset polynomial semiring S[x].

Now M = {{1}, {a}, {c}, {d}, {f}, {g}, {h}, {i}, {0}} $\subseteq$ P is a semifield hence P is a S-subset polynomial subsemiring of S as M is a subset semifield.

Example 3.21: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the ring } Z\}\right\}$$

be the subset polynomial semiring of the ring Z.

$$P = \left\{ \sum b_i x^i \,\middle|\, b_i \in \{\text{Collection of all subsets of semiring } Z^+ \cup \{0\}\}\right\}$$

be the subset polynomial subsemiring $\subseteq$ S[x]. Clearly P is a Smarandache subset polynomial subsemiring of S[x]; for T = {{n} | n $\in$ {$Z^+ \cup$ {0}} $\subseteq$ S is a subset semifield.

Clearly S[x] is also S-subset polynomial semiring as T $\subseteq$ S[x] is a subset semifield. Hence S[x] is a S-subset polynomial semiring.

Example 3.22: Let S[x] = {$\sum a_i x^i$ | $a_i \in$ S = {Collection of all subsets from the ring $C(Z_{12})$ (g) with $g^2 = 0$}} be the subset polynomial semiring. S[x] is also a S-subset polynomial semiring.

Now having seen examples of Smarandache subset polynomial semirings we now proceed onto define topologies on subset polynomial semirings.

Let S[x] be the subset polynomial semiring.

Let T = {Collection of all subset ideals of S[x]}. Give operation $\cup$ and $\cap$ on T. For A, B $\in$ T, A $\cup$ B in general need not be an ideal so we generate the ideal so $\langle A \cup B\rangle$ is an ideal of T. However A $\cap$ B is always an ideal and is in T.

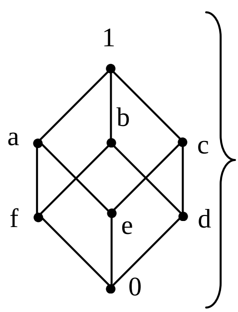
Thus {T, ∪, ∩} is a topological space defined as the subset polynomial ideal topological space semiring of S[x].

We see in general the semiring topological subset polynomial ideal space is of infinite cardinality.

Example 3.23: Let S[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets from the field Z_5}} be subset polynomial semiring of the field Z_5}.

We see if T = {Collection of all subset polynomial ideals of the semiring S[x]}; then {T, ∪, ∩} is a subset polynomial ideal topological space of the semiring S[x].

Example 3.24: Let S[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \right.$ {Collection of all subsets of the Boolean algebra



be the subset polynomial semiring.

T = {Collection of subset polynomial ideals of S[x]}.

{T, ∪, ∩} is a subset polynomial semiring ideal topological space of S[x].

Example 3.25: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subset of the field } C\}\right\}$$

be the subset polynomial semiring. T = {Collection of all subset polynomial ideals of the subset polynomial semiring S[x]}; T is a subset polynomial ideal topological space of S[x].

Now we can define for a subset polynomial semiring the notion of subset polynomial ideal new topological space of the semiring only in case when the coefficients of the polynomials are subsets from a ring or a field or a semiring or a semifield.

We consider the same T = {Collection of all subset polynomial ideals of the semiring S[x]; where S = {collection of all subsets from the field or ring or semiring or a semifield}}.

We define for any subset A, B $\in$ S, A $\cap_N$ B = C where $\cap_N$ is the product in the field or ring or semiring or a semifield.

Similarly A $\cup_N$ B where $\cup_N$ is the sum (addition) defined in the field or ring or semiring or a semifield.

We will first illustrate this situation by an example or two.

Example 3.26: Let

S = {Collection of all subsets from the ring Z_6}. S[x] be the subset polynomial semiring.

Let T = {Collection of all ideals of S[x]}, {T, $\cup_N$, $\cap_N$} is a semigroup.

If $p(x) = \{3, 2, 1\}x^3 + \{1, 4\}x + \{5, 2\}$ and $q(x) = \{4\}x^3 + \{2, 3, 0\}$ are in S[x].

$p(x) \cap_N q(x) = \{0, 2, 4\}x^6 + \{4\}x^4 + \{2\}x^3 + \{0, 4, 2, 3\}x^3 + \{2, 3, 0\}x + \{4, 0, 3\}$

$= \{0,2,4\}x^6 + \{4\}x^4 + \{0, 4, 2, 3\}x^3 + \{0, 2, 3\}x + \{0, 3, 4\} \in S[x]$.

We now find $p(x) \cup_N q(x)$; $p(x) \cup_N q(x) = \{5, 0, 1\}x^3 + \{1, 4\}x + \{1,5,2,4\} \in S[x]$.

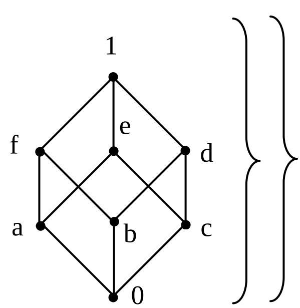
This is the way operations $\cup_N$ and $\cap_N$ are performed $\{T, \cup_N, \cap_N\}$ is defined as the subset polynomial ideal new topological semiring space.

Example 3.27: Let
S = {Collection of all subsets of the ring $Z_4(g)$}. S[x] be the subset polynomial semiring.

T = {Collection of all subset polynomial ideals of S[x]} $\{T, \cup_N, \cap_N\}$ is a subset polynomial new topological ideal space of the semiring S[x].

By taking $\phi \in T$, that is $T' = \{T \cup \phi\}$ we get $(T', \cup, \cap)$ the subset polynomial topological ideal space of the semiring S[x].

Example 3.28: Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets of the semiring



be the subset polynomial semiring.

T = {Collection of all ideals of the subset polynomial semiring}; (T′, ∪, ∩) and (T, $\cup_N$, $\cap_N$) is a subset polynomial ideal topological space semiring and subset polynomial new ideal topological space semiring respectively.

Thus we can have two topological spaces on subset polynomial semiring topological spaces and new topological subset polynomial semiring.

The authors have given several problems at the end of this chapter to solve.

Now we proceed onto define the notion of set subset polynomial ideal of a semiring or set ideal of the subset polynomial semiring.

DEFINITION 3.3: *Let S [x] = {{$\sum a_i x^i$ | $a_i \in$ S = {Collection of all subsets of the semiring ($Q^+ \cup \{0\}$ S_3}} be the subset polynomial semiring. Let P ⊆ S[x] be a subset of S[x]. T ⊆ S[x] be a subset polynomial subsemiring. If for p ∈ P and t ∈ T; pt and tp ∈ P, then we call P to be a set ideal subset polynomial semiring over the subset polynomial subsemiring T of S[x].*

We will first illustrate this by some examples.

Example 3.29: Let S [x] = {$\sum a_i x^i$ | $a_i \in$ S = {Collection of all subsets from the ring Z_4}} be the subset polynomial semiring. Take the subset polynomial subsemiring P = {{0}, {0,2}, {2}} ⊆ S[x].

T = {Collection of all set ideals of S[x] over P}
= {{{0}, {0,2}x, {2}x}, {{0}, {0,2,3}x^3 + {0,1,2}x + {0,3}, {0,2}x^3 + {0,2}x + {0,2}}, {{0}, {0,3} x^8 + {0,2,3}, {0,2}x^8 + {0,2}} and so on}.

We see finite subsets in S[x] can be set subset polynomial ideals of the semiring. This is impossible in case of usual ideals. Thus the main and interesting feature of set ideals is we have finite subsets of S[x] to be set ideals of S[x].

However if the subset polynomial subsemigroup P has even one x term certainly P will be of infinite cardinality, consequently all set ideal subset polynomial semirings will be of infinite order.

We will illustrate this situation by an example of two.

Example 3.30: Let S[x] = {Collection of all polynomials $\sum_{i=0}^{\infty} a_i x^i$ where $a_i \in$ S = {Collection of all subsets from the ring $Z_6(g)$ with $g^2 = 0$}} be the subset polynomial semiring.

P = {{0}, {0,3}, {3g,0}, {3+3g, 0}, {3, 3g, 3+3g, 0}} be the subset semiring of S[x]. We can have set ideals of finite subsets in S[x]. This is the advantage of using this P.

However if P has even a single x term P will be of infinite order if P is to be a subset polynomial subsemiring of S[x].

Example 3.31: Let S[x] = $\{\sum a_i x^i \mid a_i \in$ S = {Collection of all subsets of the semiring $Z^+ \cup \{0\}$}} be the subset polynomial semiring. We see every subset subsemiring of S[x] is of infinite order. So every set ideal of the subset polynomial semiring is of infinite cardinality. We cannot think of finite set ideals of S[x].

Even if we replace $Z^+ \cup \{0\}$ by $(Z^+ \cup \{0\})$ (g) or $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ all set ideals of the subset polynomial semiring will continue to be of infinite order.

Example 3.32: Let
S = {Collection of all subsets of the field Z_{43}}.

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S \right\}$$ be the subset polynomial semiring.

Take two subsets P = {0, 1} and A = $\{\{0\}, \{0, 40\}x^7 + \{0, 2, 5,$

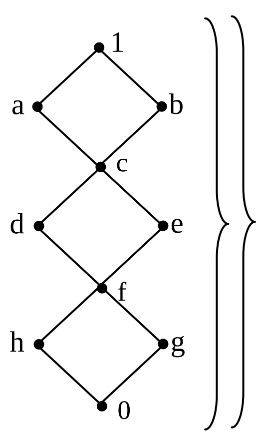
4, 3, 11}x^6 + {1, 5, 6, 7, 40, 41, 39}x^4 + {2, 3, 4, 5, 28, 27}} $\subseteq$ S[x]. A is a quasi set ideal of the subset polynomial semiring over the set P = {0, 1}. Infact S[x] has several such quasi set ideals.

Example 3.33: Let S[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {collection of all subsets of the ring Z_{14} (g) with g^2 = 0}} be the subset polynomial semiring. Take P = {0, 1, g} $\subseteq$ S[x] as a subset. We have several quasi set ideals of the subset polynomial semiring over the set P.

Let T = {Collection of all quasi set ideals of the subset polynomial semiring S[x] over the set P}; T is a quasi set ideal topological space of subset polynomial semiring.

By varying the subsets in S[x] we can get several quasi set ideal topological spaces of the subset polynomial semiring S[x], which is the main advantage of defining the notion of quasi set ideals of a subset polynomial semiring.

Example 3.34: Let S[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets of the semiring



be the subset polynomial semiring. Take P = {0, h, g} $\subseteq$ S[x], we can have several quasi set ideals of the subset polynomial semiring S[x] over P.

Infact if T = {Collection of all quasi set ideals of the subset polynomial semiring over P}; then T is a quasi set ideal topological space of the subset polynomial semiring.

Example 3.35: Let S[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets from the semiring ring $Z_6S(4)$}} be the subset polynomial semiring.

Let P = {{0}, {e}, {e, p_4, p_5}} $\subseteq$ S[x] be a subset in S[x]. T = {Collection of all quasi set ideals of the subset polynomial semiring over the set P} be the quasi set ideal topological space of the subset polynomial semiring over P.

Let P_1 = {{0}, {1}, {0,1}, {3}, {0, 3}, {0, 1, 3}, {3, 1}} $\subseteq$ S[x] be a subset of S[x]. T_1 = {Collection of all quasi set ideals of the subset polynomial semiring over the set P_1} is the quasi set ideal topological space of the subset polynomial semiring of S[x].

We see S[x] has several such quasi set ideal topological space of the subset polynomial semiring.

Example 3.36: Let S[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets from the group ring Z_8 S_7}} be the subset polynomial semiring.

We see for given any set P in S[x] we can define right quasi set ideal subset polynomial semiring and left quasi set ideal subset polynomial semiring. Now based on this we can have both left and right quasi set ideals of the subset polynomial semiring over sets in S[x].

Thus we can have several quasi set ideal topological spaces of subset polynomial semirings and also these have two operations ($\cup$, $\cap$) and ($\cap_N$, $\cup_N$).

We will proceed to give more examples and different types of Smarandache set ideal and quasi set ideal topological spaces of a subset polynomial semiring.

Example 3.37: Let $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the group ring } ZS_{10}\}\right\}$ be the subset polynomial semiring.

Take P = {{0,–1}, {–1,0,1}, {1}, {–1}, {0,1}, {0}} $\subseteq$ S[x], a subset in S[x].

We can have several set ideal topological spaces of the subset semiring and these spaces can opt for usual $\cup$ and $\cap$ or $\cup_N$ and $\cap_N$; the inherited operations from the ring.

Interested reader can solve different types of problems related with the subset polynomial semiring topological spaces over sets and subsemirings leading to S-topological spaces.

Now we proceed onto define the new notion of polynomial subsets and algebraic structures on them.

DEFINITION 3.4: *Let R[x] be a polynomial ring or a polynomial semiring where R is a ring or a field or a semiring or a semifield and x an indeterminate.*

P[x] = {Collection of all subsets of the polynomial ring or a polynomial semiring with empty set ϕ}. {P[x], $\cup$, $\cap$} is a semiring defined as the polynomial subset semiring. If {P[x] \ ϕ} is given the inherited operations $\cup_N$ and $\cap_N$ of the ring (semiring) R[x] then {{P[x] \ ϕ, $\cup_N$, $\cap_N$} is the new polynomial subset semiring.

We will illustrate this situation by some examples.

Example 3.38: Let $Z_5[x]$ be a polynomial ring $P[x]$ = {Collection of all subsets of $Z_5[x]$}. $P[x]$ is of infinite order and $\{P[x], \cup, \cap\}$ is a polynomial subset semiring.

For take $A = \{3x^2 + 2x + 1, 5x + 1, 2x^3 + 4x + 2, x^3 + 4\}$ and $B = \{x^3 + 4, 3x + 1, 4x^2 + 3, x^8 + 8, x^3 + 2\} \in P[x]$.

$A \cap B = \{x^3 + 4, 3x + 1\}$ and $A \cup B = \{3x^2 + 2x + 1, 3x + 1, 2x^3 + 4x + 2, x^3 + 4, 4x^2 + 3, x^8 + 1, x^3 + 2\} \in P[x]$.

$A \cup_N B = \{3x^2 + 2x + 1 + x^3 + 4, 3x^2 + 2x + 1 + 3x + 1, 3x^2 + 2x + 1 + 4x^2 + 3, 3x^2 + 2x + 1 + x^8 + 1, 3x^2 + 2x + 1 + x^3 + 2, 3x + 1 + x^3 + 4, 3x + 1 + 3x + 1, 3x + 1 + 4x^2 + 3, 3x + 1 + x^8 + 1, 3x + 1 + x^3 + 2, 2x^3 + 4x + 2 + x^3 + 4, 2x^3 + 4x + 2 + 3x + 1, 2x^3 + 4x + 2 + 4x^2 + 3, 2x^3 + 4x + 2 + x^8 + 1, 2x^3 + 4x + 2 + x^3 + 2, x^3 + 4 + x^3 + 4, x^3 + 4 + 3x + 1, x^3 + 4 + 4x^2 + 3, x^3 + 4 + x^8 + 1, x^3 + 4 + x^3 + 2\}$

$= \{x^3 + 3x^2 + 2x, 3x^2 + 2, 2x^2 + 2x + 4, 3x^2 + 2x + x^8 + 2, 3x^2 + x^3 + 2x + 3, x^3 + 3x, x + x, 4x^2 + 3x + 4, x^8 + 3x + 2, x^3 + 3x + 3, 3x^3 + 4x + 1, 2x^3 + 2x + 3, 2x^3 + 4x^2 + 4x, 2x^3 + 2x + 3, 2x^3 + 4x^2 + 4x, 2x^3 + x^8 + 3 + 4x, 3x^3 + 4x + 4, 2x^3 + 3, x^3 + 3x, x3 + 4x^2 + 2, x^8 + x^3, 2x^3 + 1\} \in P[x]$.

Clearly $A \cup B \neq A \cup_N B$.

Consider $A \cap_N B = \{(3x^2 + 2x + 1) (x^3 + 4), (3x^2 + 2x + 1) (3x + 1), (3x^2 + 2x + 1) (4x^2 + 3) (3x^2 + 2x + 1) (x^8 + 1), (3x^2 + 2x + 1) (x^3 + 2), (3x + 1) (x^3 + 4), (3x + 1)2 (3x = 1) (4x^2 + 3), (3x + 1) (x^8+1) (3x+1) (x^3+2), (2x^3 + 4x + 2) (x^3 + 4) (2x^3 + 4x + 2) (3x + 1) (2x^3 + 4x + 2) (4x^2 + 3) (2x^3 + 4x + 2) (x^8 + 2), (2x^3 + 4x + 2) (x^3 + 2) (x^3 + 4)^2, (x^3 + 4) (3x+1), (x^3+4) (4x^2 + 3), (x^3+4) (x^8+1), (x^3+4) (x^3+2)\} \in P(X)$.

However $A \cap B \neq A \cap_N B$. Thus we see $(P[x], \cup, \cap)$ is clearly a different polynomial new subset semiring from the polynomial subset semiring $\{P[x], \cup_N, \cap_N\}$.

Several important and interesting questions arise in the case of polynomial subset semiring P[x].

Example 3.39: Let Z[x] be a polynomial ring P[x] = {Collection of all subsets of the ring Z[x]} {P[x], $\cup$, $\cap$} and {P[x], $\cap_N$, $\cup_N$} are polynomial subset semiring of the ring Z[x].

Let A = $\{x^3 + 4, x^2 - 8\}$ and B = $\{x^3 + 4, x + 1\} \in$ P[x].

Now $A \cap B = \{x^3 + 4\}$ and $A \cup B = \{x^3 + 4, x + 1, x^2 - 8\}$ are in P[x].

Consider $A \cap_N B = \{(x^3 + 4)^2\ (x + 1)\ (x^3 + 4),\ (x^2 - 8),\ (x+1)\ (x^2 - 8)\ (x^3 + 4)\}$ and

$A \cup_N B = \{2x^3 + 8, x^3 + x^2 - 4, x^3 + 5 + x, x^2 + x - 7\}$.

Both $A \cup_N B$ and $A \cap_N B$ are in P[x].

However $A \cap_N B \neq A \cap B$ and $A \cup_N B \neq A \cup B$.

We see $A \cup A = A = \{x^3 + 4, x^2 - 8\}$ and $A \cap A = A = \{x^3 + 4, x^2 - 8\}$

$A \cap_N A = \{(x^3 + 4)^2, (x^2-8)2, (x^3+4)\ (x^2-8)\} \neq A$.
$A \cup_N A = \{2x^3 + 8, 2x^2 - 16, x^3 + x^2 - 4\} \neq A$.

Thus $\cup_N$ and $\cap_N$ are not idempotent operations.

Example 3.40: Let Z_{12} [x] be a ring of polynomials. P[x] = {Collection of all subsets of the ring Z_{12} [x]} is the polynomial subset semiring. Clearly the two semirings {P[x], $\cup$, $\cap$} and {P[x], $\cup_N$, $\cap_N$} are distinct.

Let A = $\{3x^2 + 6x + 9\}$ and B = $\{4x + 8\}$ be in P[x].

We see $A \cap B = \phi$, $A \cup B = \{3x^2 + 6x + 9, 4x + 8\}$ are in P[x]. Now $A \cap_N B = \{0\}$ and $A \cup_N B = \{3x^2 + 10x + 5\}$.

We see A $\cap_N$ B = {0} but A ≠ {0} and B ≠ {0} so we see in case of polynomial subset semirings we can have zero divisors.

We further see if X = {$3x^2 + 6x + 3$, $9x^2 + 3$, $6x^3 + 3$, $6x^7 + 6$} and Y = {$4x^7 + 8$, $4x^3$, $4x^9 + 8$, $4x^8 + 4$, $8x^9 + 8$} are in P[x]; X $\cap_N$ Y = {0}. So in P[x], X and Y are zero divisors.

Can we have idempotents in P[x] under $\cup_N$ and $\cap_N$.

Clearly only constant subsets can contribute to idempotents provided the basic coefficient ring R of R[x] has idempotents otherwise we cannot have idempotents under $\cap_N$.

However under $\cup$ and $\cap$ every set A $\in$ P[x] is an idempotent as A $\cap$ A = A and A $\cup$ A = A.

Example 3.41: Let R[x] be the polynomial ring where R is the field of reals. P[x] = {Collection of all subsets of the ring R[x]} be the polynomial subset semiring of the ring R[x].

{P[x], $\cap$, $\cup$} and {P[x], $\cap_N$, $\cup_N$} are subset polynomial semirings. Clearly [P[x], $\cap_N$, $\cup_N$] has no zero divisors or units.

Infact {P[x], $\cup_N$, $\cap_N$} is a semifield. Further P[x] is of infinite cardinality.

In this case we see one is interested in studying other related properties of these polynomial subset semirings.

One of the natural interest would be to find degree of the polynomial subset of a polynomial subset semiring.

We see if A $\in$ P[x] the polynomial subset degree of A in P[x] is the degree of the highest polynomial which has no zero coefficient.

For example if A = $\left\{\begin{array}{c} x^8 + 13x^3 + 8x + 7 \\ 3x^2 - 1 \\ 7x^2 + 4x^3 + 3 \end{array}\right\} \in$ P[x].

The degree of the polynomial subset is eight that is deg p(A) is eight. B = $\{9x^{19} + 3, 2x - 1. 8x^2 + 5x + 9\}$, deg p (B) = 19. Thus we can have as in case of polynomials in case of polynomial subset the degree of the polynomial subset.

Example 3.42: P[x] = {Collection of all subsets of the polynomial ring} be the polynomial subset semiring.

Let A = $\{8x^3 + 5x + 3, 9x^{12} + 4x + 8, 10x^8 + 4\}$ and B = $\{9x + 3, 8x^{19} + 3x + 8, 2x^3 + 10x + 15\} \in$ P[x]; deg p(A) = 12 and deg p(B) = 19.

Now we have the following properties about degree of polynomial subsets in semirings.

deg (p(A + B)) = 19 and
deg p(AB) = 27 < deg (P(A)) + deg (P(B)) = 12+19=31.

This is the way the sum and product of the degree of polynomial subsets semiring are defined.

Let A = $\{6x^9 + 3x + 1, 8x^2 + 3, 10x^7 + 3x^4 + 8\} \in$ P[x]. We see deg (p(A)) = 9. A $\cap_N$ A = $\{(6x^9 + 3x + 1)^2, (8x^2 + 3)^2, (10x^7 + 3x^4 + 8)^2, (6x^9 + 3x + 1)(8x^2 + 3), (6x^9 + 3x + 1) \times (10x^7 + 3x^4 + 8), (8x^2 + 3), (10x^7 + 3x^4 + 8)\}$. deg p(A $\cap_N$A) = 16. However deg p(A $\cup_N$A) = 19. Let A = $\{x^{25} + 1, 3x^{38} + 5x^{25} + 1, 2x^6 + 3x^3 + 15\} \in$ P[x] and deg p(A $\cup_N$ A) = 38.

Let B = $\{9x^{28} + 1, 3x^2 + 7x + 13, 13x^6 + 3\} \in$ P[x].

deg p(B $\cap_N$ B) = 66. Thus we see the deg p(A) for any A behaves in different way under ($\cup_N$, $\cap_N$) and under ($\cup$, $\cap$).

Example 3.43: Let Z[x] be the polynomial ring.

Let P[x] = {Collection of all subsets of the ring Z[x]} be the polynomial subset semiring of the ring Z[x].

If deg p(A) = n and deg p(B) = m then deg p $(A \cap_N B)$ = m+n.

deg p$(A \cup_N B)$ = m (if m > n and will be n if m < n).

This so happens because Z is an integral domain.

Example 3.44: Let $Z_p[x]$ (p a prime) be the polynomial ring. P[x] = {Collection of all subsets of the polynomial ring $Z_p[x]$} is the polynomial subset semiring.

Let A, B $\in$ P[x], deg p(A) = m and if deg p(B) = n then deg p$(A \cup_N B)$ = m (or n) according as m > n or m < n, deg p $(A \cap_N B)$ = m + n.

Inview of this we have the following theorem.

THEOREM 3.3: *Let R[x] be the polynomial ring (R is a field or an integral domain).*

P[x] = {Collection of all subsets of the polynomial ring R[x]} is the polynomial subset semiring. If deg p(A) = n and deg p(B) = m; A, B $\in$ P[x] then deg (p $(A \cap_N B)$) = m + n and deg (p $(A \cup_N B)$) = m (if m > n and will be n if m < n.

The proof is obvious from the very construction as R is a field or the integral domain.

Example 3.45: Let R[x] be a polynomial ring and R has zero divisors.

P[x] = {Collection of all subsets of the polynomial ring R[x]} be the polynomial subset semiring. Clearly if A, B $\in$

P[x] with deg p(A) = m and deg p(B) = n then deg p(A $\cap_N$ B) < deg p(A) + deg p(B).

deg p(A $\cup_N$ B) < deg p (A) or deg p(B) and so on.

THEOREM 3.4: *Let R[x] be the polynomial ring (R is a ring with zero divisors).*

P[x] = {Collection of all subsets of the polynomial ring R[x]} . If deg p(A) = m, deg p(B) = n then deg p(A $\cap_N$ B) < m + n and deg p (A $\cup_N$ B) < m (or n); A, B $\in$ P[x].

The proof is left as an exercise to the reader.

Example 3.46: Let Z_{12}[x] be the polynomial ring P[x] = {Collection of all subsets of the polynomial ring Z_{12} [x]} be the polynomial subset semiring.

We see if A = $\{4x^9 + 8, 8x^4 + 4x^2 + 8, 4x^7 + 4x^2 + 8\}$ and B = $\{3x^{12} + 3, 6x^{10} + 3x + 6\}$ $\in$ P[x] then (A $\cap_N$ B) = {0} so deg p(A $\cap_N$ B) = {0}.

(A $\cup_N$ B) = $6x^{10} + 3x + 6 + 4x^9 + 8$, $3x^{12} + 3 + 4x^9 + 8$, …, $8x^4 + 4x^2 + 8 + 3x^{12} + 3$, …, $6x^{10} + 4x^7 + 4x^2 + 3x + 2$}.
deg p(A $\cup_N$ B) = 12.

Example 3.47: Let Z_{10} [x] be the polynomial ring.

P[x] = {collection of all subsets of the polynomial ring Z_{10} [x]} be the polynomial subset semiring.

Let A = $\{5x^3 + 4x + 1, 5x^3 + 4x + 1, 5x^3 + 1\}$ $\in$ P[x].

Now we find A $\cup_N$ A = {8x + 2, 2, 8x+2 and so on}.

We see deg p (A $\cup_N$ (A) < deg p (A) = 3.

Infact deg p(A $\cup_N$ A) = 1 and deg p (A $\cap_N$A) = 6.

Example 3.48: Let C[x] be the complex polynomial ring P[x] = {Collection of all subsets of the complex polynomial ring} be the polynomial subset semiring.

$A = \{(3 + 4i)x^7 + 2x + (7 + 9i), 4ix^6 + 20x^2 + (40–3i)x + 9 + 5i\}$ and $B = \{(10+30i)x^3 + (7–3i), (5i+3)x^5 + 4x + 3+2i\} \in P[x]$.

Clearly $A \cap B = \phi$, $A \cap_N B = \{[(3 + 4i)x^7 + 2x + 7 + 9i]\, [(10 + 30i)x^3 + (7–3i)], [(3+4i)x^7 + 2x + (7+9i)] \times [(5i+3)x^5 + 4x + 3 + 2i], \ldots\} \in P[x]$.

$A \cup B \neq A \cup_N B$.

Now we are finally interested in solving or finding roots of polynomial subset.

In the first place we want to make it clear that we can solve only when on P[x] we take the operations $\cup_N$ and $\cap_N$ under the usual union $\cup$ and $\cap$ we do not have any meaning of solving the equations.

Thus while we are going to discuss about solving for roots of polynomial subset we use only the new operation $\cup_N$ and $\cap_N$ inherited from the basic set from which the subsets are taken. We solve polynomial subsets only in this way.

If A = B, A, B $\in$ P[x] where
P[x] = {Collection of all subsets of the polynomial ring R[x]} be the polynomial subset semiring.

Let $A = \{3x^2 + 2x + 1, 5x – 3\}$ and $B = \{x^2 + 8x – 7, 2x + 4\} \in P[x]$. For if A = B we in the first place can equate every element in A to every other element in B in the following way.

$$3x^2 + 2x + 1 = x^2 + 8x – 7 \quad \text{(i)}$$
$$3x^2 + 2x + 1 = 2x + 4 \quad \text{(ii)}$$
$$5x – 3 = x^2 + 8x – 7 \quad \text{(iii)}$$
$$5x – 3 = 2x + 4 \quad \text{(iv)}$$

Now we can solve these equations.

$2x^2 - 6 + 8 = 0$

$$x = \frac{+6 \pm \sqrt{36 - 4 \times 8 \times 2}}{2 \times 2} = \text{imaginary}$$

so no solution for the (i) equation

$3x^2 + 2x + 1 = 2x + 4$

$3x^2 = 3 \qquad x^2 = 1$

so $x = \pm\ 1$ is solvable.

$5x - 3 = x^2 + 8x - 7$

$x^2 + 3x - 4 = 0$

$$x = \frac{-3 \pm \sqrt{9 + 4 \times 4 \times 1}}{2}$$

$$= \frac{-3 \pm \sqrt{25}}{2} = \frac{-3 \pm 5}{2} = -4,\ 1.$$

Finally from equation (iv)

$5x - 3 = 2x + 4$

$3x = 1 \qquad x = 1/3.$

Thus we see of the four equations, three had solution and one has no solutions so we cannot find all solutions. So we cannot say the polynomial subset A = B is completely solvable we can only say they are partially solvable.

In the first place we have to keep on record that the way of equating these polynomial subsets is justified as these involve the indeterminate x. If both the sets are constants only constant subsets; we can never say they are equal unless both A and B are singleton sets and A and B are identical.

A = {3} then B = {3}.

If A = {3, 5} and B = {3, 5}

$A \neq B$ for our equating means 3 = 3, 3 = 5 this is impossible so equality in polynomial subsets is possible if and only if both A and B are identical and are singletons.

If A and B are two polynomial subsets and A = B that is if $A = \{a_1, \ldots, a_t\}$ and $B = \{b_1, \ldots, b_s\}$ then $a_1 = b_i$; $1 \leq i \leq s$.

$a_2 = b_i$; $1 \leq i \leq s$

and so on. $a_t = b_i$; $1 \leq i \leq s$ (however equating constants if any present in A and in B can n ever be equated).

Thus we have ts number of polynomials equated to zero. If every polynomial has a solution in the ts polynomials we say A = B is completely solvable over R; R[x] is the subsets of P[x] if only few has solution; we say A = B is partially solvable, if A = B has no solution we ay A = B is not solvable. This is way the usual solving of polynomials varies from the solving of polynomial subsets.

We will describe them by some more examples.

Example 3.49: Let Z[x] be the polynomial ring.

P[x] = {Collection of all subset of Z[x]} be the polynomial subset semiring.

Let $A = \{x + 7, 2x + 4\}$ and $B = \{8, 4x - 4, 8x + 7\} \in P[x]$.

A = B implies $x + 7 = 8$,

$x + 7 = 4x - 4$,

$x + 7 = 8x + 7$,

$2x + 4 = 8$,

$2x + 4 = 4x - 4$ and

$2x + 4 = 8x + 7$.

$\{x = 1, 3x = 11, x = 0, 2x = 4, 2x = 8, 6x + 3 = 0\}$

$= \{1, 11/3, 0, x = 2, -1/2\}$.

We see 11/3 and $-1/2 \notin Z$.

Thus A = B has the solution subset which is not complete only partial given by {1, 0, 2, 4}.

Let A = $\{x^2, 10x - 5\}$ and B = $\{25, 5\} \in P[x]$. Solution subset for the polynomial subset equation A = B is $x^2 = 25$

so $x = \pm 5$
$10x - 5 = 25$ $\quad$ $x = 3$
$x^2 = 5$ $\quad$ $x = \pm\sqrt{5}$.

Thus we have only partial solution to this equation A = B or it is partially solvable. The solution set is T = {5, –5, 3, 1}.

Consider A = {0} where A is a polynomial subset we call this type of equation as Reduced polynomial subset equation for if A = B then we get {A – B} = {0} with $t \times s$ equations if A has t elements and B has s elements.

So all polynomial subset equations can be got as the reduced polynomial subset equation. If A = B where B has only constants that is no polynomials we call this equation has constant reduced polynomial subset equation.

We will first give examples of them.

Example 3.50: Let Q[x] be the polynomial ring.

P[x] = {Collection of all subsets of Q[x]} be the polynomial subset semiring.

$$\text{Let } A = \left\{\begin{array}{cc} x^2+4x+4 & 5x+7 \\ 7x-3 & 4x^2-4 \\ x^2-9 & \end{array}\right\} \text{ and}$$

B = {0} be the polynomial subsets in P[x].

$$A = B \text{ gives } \left\{\begin{array}{cc} x^2+4x+4=0 & 5x+7=0 \\ 7x-3=0 & 4x^2-4=0 \\ x^2-9=0 & \end{array}\right\}.$$

The polynomial subset root subset is
R = {3/7, 2, 2, –7/5, 1, 1, 3, –3}.

Clearly R ∈ P[x] and A = B is completely solvable.

$$\text{Consider } B_1 = \begin{Bmatrix} 2, 5, -10, 8, 11 \\ 0, 6 \end{Bmatrix} \text{ and}$$

$$A_1 = \begin{Bmatrix} 5x-7, 8x-3, 18x-1 \\ 14+3x, 4+x, 2x \end{Bmatrix} \text{ be in P[x].}$$

Let $A_1 = B_1$ we get for the polynomial subset the solution set which is as follows.

5x – 7 = 2	5x – 7 = –10
5x – 7 = 5	5x – 7 = 8

5x – 7 = 11, 5x – 7 = 0, 5x – 7 = 6, 8x – 3 = 2, 8x – 3 = 5, 8x – 3 = –10, 8x – 3 = 8, 8x – 3 = 11, 8x – 3 = 0, 8x – 3 = 6, 18x – 1 = 2, 18x – 1 = 5, 18x – 1 = –10, 18 x –1 = 8, 18x – 1 = 11, 18x – 1 = 0, 18x – 1 = 6,

14 + 3x = 2, 14 + 3x = 5, 14 = 3x = –10

14 + 3x = 8, 14 + 3x = 11, 14 + 3x = 0, 14 + 3x = 6, 4 + x = 2, 4+x = 5, 4 + x = –10, 4 + x = 8, 4 + x = 11, 4 + x = 0, 4 + x = 6, 2x = 2, 2x = 5, 2x = –10, 2x = 8, 2x = 11, 2x = 0, 2x = 6.

The solution set is

C_1 = {9/5, 12/5, –3/5, 3, 18/5, 7/5, 13/5, 5/8, x = 1, x = –7/8, 11/8, 14/8, 3/8, 9/8, 3/18, 6/18, –9/18, 9/18, 12/18, 1/18, 7/18, –12/3, –9/3, –24/3, –6/3, –3/3, –14/3, –8/3, –2, –14, 1, 4, 7, –4, 2, 1, 5/2, –10/2, 4, 11/2, 0, 3}.

We see 42 equations occur by equating the polynomial subsets and C_1 is the solution set. This equation gives a complete solution of the polynomial subsets.

Consider the polynomial subset

$$A = \left\{\begin{matrix} x^2+3x+2, x^2-9 \\ x^2+4x-5, x^2-8 \end{matrix}\right\} \text{ and } B = \left\{\begin{matrix} 3,4,0 \\ 8 \end{matrix}\right\} \in P[x].$$

Let A = B be the polynomial subset equation.

$$\text{Now} \left\{\begin{matrix} x^2+3x+2=3 & x^2+4x-5=3 \\ x^2+3x+2=0 & x^2+4x-5=4 \\ x^2+3x+2=4 & x^2+4x-5=8 \\ x^2+3x+2=8 & x^2-8=0 \\ x^2-9=3 & x^2-8=8 \\ x^2-9=4 & x^2-8=3 \\ x^2-9=0 & x^2-8=4 \\ x^2-9=8 & \\ x^2+4x-5=0 & \end{matrix}\right\}.$$

Let C_2 denote the polynomial subset root;

C_2 = {(two roots are not in Q), –2, –1 (two roots are not in Q), (two roots are not in Q), (two roots of x^2 = 12 is not in Q), (two roots of x^2 = 13 are not in Q), ± 3, (x^2 = 17 roots are not in Q), (x^2 + 4x – 5 = 0 roots are imaginary not in Q), (x^2 + 4x – 8 = 0 roots are not in Q), (x^2 + 4x – 9 = 0 roots are not in Q). (x^2 + 4x – 13 = 0 roots are not in Q), (x^2 – 8 = 0 roots not in Q), ± 4, (roots of x^2 = 11 are not in Q), (roots of x^2 = 12 are not in Q)}.

Thus the polynomial subset equation has only partial solution for many does not contain the root in Q.

Example 3.51: Let R[x] be the polynomial ring.

P[x] = {Collection of all polynomial subsets of the polynomial ring R[x]} be the polynomial subset semiring.

Let A = $\begin{Bmatrix} x^2-4, x^2-7 \\ x^4+4x^2+4 \end{Bmatrix}$ and B = {0} $\in$ P[x].

Now the polynomial subset equation A = B gives $x^2 – 4 = 0$, $x^2 – 7 = 0$, $x^4 + 4x^2 + 4 = 0$. Thus the solution subset of the polynomial subset equation is {±2, ± $\sqrt{7}$. i$\sqrt{2}$, ± i$\sqrt{2}$ }.

We see A = B in P[x] has only a partial polynomial subset solution and not a complete polynomial subset solution.

Consider A = $\begin{Bmatrix} x^3-8, & x^3-27 \\ x^2-4 & x^2-10 \end{Bmatrix}$ and B = {0} in the polynomial subset semiring P[x].

Clearly $x^3 – 8 = 0$, $x^3 – 27 = 0$, $x^2 – 4 = 0$ and $x^2 – 10 = 0$.

The root polynomial subset of the polynomial subset equation is C = {±2 root of $x^3 – 8$ are imaginary, $x^3 – 27$ has two imaginary roots, 3, ± $\sqrt{10}$ }. Thus this polynomial subset equation A = B has only partial subset solution has only partial subset solution given by C.

However if Q[x] is replaced by C[x] then in the polynomial subset semiring P[x] every equation of polynomial subset is solvable. We call

CP[x] = {all subsets of the complex polynomial ring C[x]} polynomial subset semiring as the polynomial subset algebraically closed semifield.

Infact all other semifield of polynomial subsets are subsemifields of polynomials subsets of characteristic zero. Thus we have so far only seen examples of P[x] in which we have taken the ring Z[x] or Q[x] or R[x] or C[x].

Now we study the same question in case of R^+ [x] $\cup$ {0}, Q^+ [x] $\cup$ {0}, and Z^+ [x] $\cup$ {0}.

We will now study using examples the subset roots of the polynomial subset equation over $Z_n[x]$ (n prime or otherwise).

Example 3.52: Let Z_6 [x] be the polynomial ring.

P[x] = {Collection of all subsets of $Z_6[x]$} be the polynomial subset semiring.

Let A = $\begin{Bmatrix} 3x^2+2 & 4x+3 \\ 2x^2+1 & x+3 \end{Bmatrix}$ and B = {0} be in P[x].

$A = B$ gives $3x – 2 = 0$
$4x + 3 = 0$
$2x^2 + 1 = 0$ and $x + 3 = 0$.

We see $3x = 4$ $3x = 4$ is the solution only we can find values of x by this equation $4x + 3 = 0$ $4x = 3$.

This also has no value for x but $4x = 3$, $2x^2 = 5$, $2x^2 = 5$ is the only value $x = 3$. Thus in case of P[x] the polynomial subset semiring many a times we will not be in a position to find the value of x if the coefficient of the power of x is an idempotent or a zero divisor which is evident from the above examples as 4 and 2 are zero divisors in Z_6 and $3^2 = 3$ so 3 is an idempotent in Z_6.

Example 3.53: Let Z_{12} [x] be the polynomial ring.
P[x] = {Collection of all subsets of the polynomial ring Z_{12} [x]} be the polynomial subset semiring.

Let A = $\{5x^2 + 2, x + 7, 3x + 8, 4x + 1, 8x + 1\}$ and B = {0} ∈ P[x];

$5x^2 + 2 = 0$	$x^2 + 10 = 0$
$x+7 = 0$	$x^2 = 2$
$3x + 8 = 0$	$x = 5$
$4x + 1 = 0$	$4x = 11$
$8x + 9 = 0$	$8x = 3$

We see some has solution, many have no solution if the coefficients are zero divisors or idempotents.

Example 3.54: Let Z_5 [x] be polynomial ring.
P[x] = {Collection of all subsets of the polynomial ring Z_5 [x]} be the polynomial subset semiring.

Let A = $\{3x^2 + 4, 2x = 1, 3x + 2, 4x^2 + x + 3\}$ and B = {0} $\in$ P[x]. Suppose A = B to find the subset solution of the polynomial subset equation.

$$\begin{array}{ll} 3x^2 + 4 = 0 & x^2 + 8x^2 + 3 = 0, \\ 2x + 1 = 0 & x + 3 = 0, \\ 3x + 2 = 0 & x + 4 = 0 \text{ and} \\ 4x^2 + x + 3 = 0, & 4x^2 + x + 3. \end{array}$$

$$\{2, 1, x^2 = 2 \text{ and so on}\}.$$

Example 3.55: Let $C(Z_3)$ [x] be the polynomial complex finite modulo integer ring
P[x] = {Collection of all subsets of the ring $C(Z_3)$ [x]} be the polynomial subset semiring.

$$\text{Let A} = \left\{\begin{array}{lll} x+1 & 2x^2+1 & 2i_F x+1 \\ x+i_F & x+2i_F & \end{array}\right\} \text{ and}$$

B = {0} $\in$ P[x]. Let A = B be the polynomial subset equation.

$$\begin{array}{llll} x + 1 = 0 & x+i_F = 0 & 2i_F x = 2 & \\ 2x^2 + 1 = 0 & x + 2i_F = 0 & i_F x = 1 & 2x = i_F,\ x = 2i_F \\ 2i_F\, x + 1 = 0 & 2x^2 = 2 & x^2 = 1,\ x = 1, 2 & \\ x = 2 & x = 2i_F & x = i_F & \end{array}$$

C = $\{2, 2i_F, i_F, 1\}$ are the such that C is the solution set of A = B.

Now consider $A_1 = \{2i_F\, x^3 + 1 \quad x^4 + i_F, i_F x^2 + 2i_F x + 1\}$ and B_1 = {0} $\in$ P[x].

We have $A_1 = B_1$ that is

$$\begin{array}{ll} 2i_F x^3 + 1 = 0 & x^4 + i_F = 0 \\ i_F x^2 + 2i_F x + 1 = 0 & \\ 2 i_F^2 x^3 + i_F = 0 & x^3 + i_F = 0 \\ x^3 = 2i_F = 2 & x = i_F \text{ is a root.} \end{array}$$

Thus we have got at this juncture a open conjecture.

Conjecture 3.1: Can $C(Z_p)$ the finite complex modulo integer field be the algebraically closed? (for suitable p) [12].

What happens when p is a composite number or when $C(Z_p)$ is not a field?

We can solve polynomial subset equations and the result will be a subset of constant polynomials.

Example 3.56: Let $C(Z_{10})[x]$ be a complex polynomial finite modulo integer ring. $P[x]$ = {Collection of all subsets of the polynomial ring $C(Z_{10})$ [x]} be the polynomial subset semiring.

Let $A = \{4x^3 + 3, 5x + 4, 3x^2 + 7, 2x + 5\}$ and $B = \{x + 4, 3x + 8, 4x^2 + 3, 6x, 0\}$ be in P[x].

To solve the polynomial subset equation A = B.

A = B gives

$$\begin{cases} 4x^3 + 3 = x + 4 & 5x + 4 = x + 4 \\ 4x^3 + 3 = 3x + 8 & 5x + 4 = 3x + 8 \\ 4x^3 + 3 = 4x^2 + 3 & 5x + 4 = 4x^2 + 8 \\ 4x^3 + 3 = 6x & 5x + 4 = 6x \\ 4x^3 + 3 = 0 & 5x + 4 = 0 \end{cases}$$

$$\left.\begin{matrix} 3x^2+7=x+4 & 2x+5=x+4 \\ 3x^2+7=3x+8 & 2x+5=3x+8 \\ 3x^2+7=0 & 2x+5=4x^2+3 \\ 3x^2+7=4x^2+3 & 2x+5=6x \\ 3x^2+7=6x & 2x+5=0 \end{matrix}\right\}$$

= C is the subset solution sought set.

$$C = \left\{\begin{matrix} 4x^3+9x+9=0 & 4x^3+4x+3=0 \\ x=0 & 4x^3+3=0 \\ 4x^3+7x+5=0 & 5x=6 \\ 4x^3+6x^2=0 & x=4 \end{matrix}\right.$$

$$\left.\begin{matrix} 4x^2+5x+4=0 & 2x=4 \\ 3x^2+9x+3=0 & 3x^2+7x+9=0 \\ 3x^2+7=0 & x^2+6=0 \\ 3x^2+4x+7=0 & x=9 \\ 6x=5, 2x=5 & x=7 \end{matrix}\right\}.$$

The solvability of these equations are left as an exercise to the reader.

Now if we have polynomial semirings say $Z^+ \cup \{0\}$ [x] and so on our working is very different.

We will indicate this in a line or two.

Example 3.57: Let $(Z^+ \cup \{0\})$ [x] be a polynomial semiring. P[x] = {Collection of all subsets of the polynomial ring; $(Z^+ \cup \{0\})$ [x]} be the polynomial subset semiring.

How to solve polynomial subset equations in them.

Let A = $\{5x + 3, 10x^2 + 4\}$ and B = $\{0\} \in$ P[x].

We see $5x + 3 = \{0\}$ and $10x^2 + 4 = \{0\}$. Clearly we do not have any x in $Z^+ \cup \{0\}$ such that the equations are satisfied. Thus as far as solving polynomial subset equations in $(Z^+ \cup \{0\})[x]$ or for that matter over the semirings of characteristic zero which are not rings is not possible.

Thus we see we have limitations in solving polynomial subset equations when the polynomial subsets are from $(Z^+ \cup \{0\})[x]$.

Only if $A = \{5x, 4x, 10x^2, 11, x\}$ and $B = \{4, 5, 7, 0, 8, 1, 5\} \in P[x]$ then A = B has a complete or partial solution subset for 5x = 4 not solvable in $Z^+[x] \cup \{0\}$ but solvable in $Q^+[x] \cup \{0\}$ and so on.

Thus one cannot make a sweeping statement it is impossible to solve polynomial subset equations in many a cases we may have a partial subset solutions. However we do not say always for A = B a polynomial subset equation has a subset solution. Thus we see we have limitations in solving polynomials even in usual semirings.

We proceed on to propose some problems for this chapter.

Problems:

1. Enumerate some interesting features enjoyed by the subset polynomial semiring S[x] with coefficients from the power set P(X) = S.

2. Find for the subset polynomial semiring S[x] where S = P(X) with $X = \{a_1, a_2, \ldots, a_{12}\}$ two different sets of subsets polynomials so that their '$\cap$' (product) is the subset empty polynomial in S[x].

3. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \right.$ {Collection of all subsets of the subsets of $X = \{a_1, a_2, \ldots, a_{14}\}$ including X and $\phi\} = P(X) = S\}$ be the subset polynomial semiring.

   (i) Find two subset polynomial ideals of S[x].
   (ii) Find two subset polynomial subsemirings which are not subset polynomial ideals of S.
   (iii) Can S[x] have infinite number of subset polynomial subsemirings?
   (iv) Is it possible for S[x] to have infinite number of subset polynomial ideals?

4. Obtain some special and interesting features enjoyed by S[x] where S is a power set of a finite set X.

5. Can S[x] in problem (4) have more than one set of algebraic structures so that S[x] is a subset polynomial semiring where S = P(X); X is a finite set.

6. Does there exist a S[x], the subset polynomial semiring which has no subset polynomial ideals?

7. Does there exist a S[x] in which every subset polynomial subsemiring is an ideal of S[x]?

8. Does there exist a S[x] which has no subset polynomial subsemirings?

9. Let S[x] be the subset polynomial semiring with coefficients from the powerset P(X) where $X = \{1, 2, 3, 4, 5\}$.

   (i) Does S[x] contain a subset polynomial of degree greater than or equal to two which cannot be linearly written as product of linear subset polynomials.
   (ii) In how many ways can $p(x) = \{1, 2, 3\} x^2 + \{2, 5\}$ written as product of subset linear polynomials.

10. Show in general a second degree subset polynomial in S[x] where S = P(X); X = {1, 2, 3, 4} can be decomposed in more than one way.

11. Let S[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in$ S = {Collection of all subsets from the power set P(X) where X = {1, 2, …, 18}} be the subset polynomial semiring.

    (i) Is $\{1, 2, 5, 7, 8\}x^3 + \{2, 8, 1, 5, 10, 11\}x^2 + \{3, 8, 14\} x + \{2, 14, 16\}$ = p(x) reducible as linear factors?

    (ii) If p(x) is decomposable as linear factors; in how many ways can p(x) be decomposed?

12. Let S[x] = {Collection of all polynomials $\sum_{i=0}^{\infty} a_i x^i$ with $a_i \in$ S = {Collection of subset of power set of X; P(X) = S}} be the subset polynomial semiring (X = $\{a_1, a_2, \ldots, a_{24}\}$).

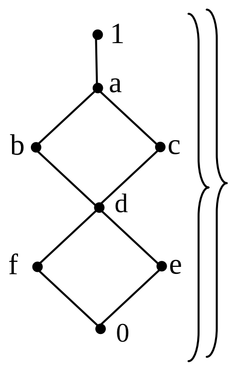
    (i) Can p(x) = $\{a_5, a_6, a_3, a_1\} x^4 + \{a_5, a_6, a_3, a_1\}$ the subset polynomial in S[x] be linearly decomposable as subset polynomials.

    (ii) How many ways can p(x) be decomposed into linear subset polynomials?

13. Find for S[x] in problem (12); three subset polynomial subsemirings which are not subset polynomial set ideals and three subset polynomial ideals.

14. Obtain some striking application of subset polynomial semirings with coefficients from a power set P(X) of a set X = $\{a_1, a_2, \ldots, a_n\}$.

15. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \right.$ {Collection of all subsets of the semiring L =



be the subset polynomial semiring.

(i) Find subset polynomial ideals of $\{S[x], \cap_L, \cup_L\}$.

(ii) Find subset polynomial subsemirings in $\{S[x], \cap_L, \cup_L\}$ which are not subset polynomial ideals of S[x].

(iii) Study (i) and (ii) in $\{S'[x], \cup, \cap\}$.

(iv) Compare the two subset polynomial semirings.

(v) Prove subset polynomial ideal in $\{S'[x], \cup, \cap\}$ need not be a subset polynomial ideal in $\{S[x], \cup_L, \cap_L\}$.

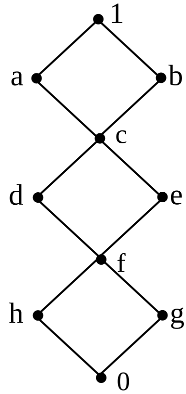
(vi) Can S[x] have zero divisors?

(vii) Is it possible for S[x] to have idempotents?

16. Obtain some interesting features enjoyed by S[x] where the subset polynomial semirings P takes its coefficients of subset polynomials from the semiring P.

17. Study $\{S'[x], \cup, \cap\}$ and $\{S[x], \cup_P, \cap_P\}$; compare and contrast them.

18. Find the difference between S[x] when
$S=\{P(X); X = (a_1, \ldots, a_n)\}$ and when
S_1 = {Collection of all subsets from the semiring L =



}.

S[x] the subset polynomial semiring.

(i) Which subset polynomial semiring can have zero divisors?

(ii) Find subset polynomial ideals and subsemirings of S[x] and $S_1[x]$.

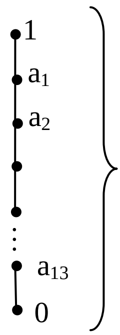
19. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right.$ = {Collection of all subsets of the semiring $Z^+ \cup \{0\}$} be the subset polynomial semiring.

(i) Find subset polynomial ideals of S[x].

(ii) Does S[x] contain subset polynomial subsemirings which are not ideals?

(iii) Can S[x] have zero divisors?

20. Let S = {Collection of all subsets of the semiring L = C_{15} =



S[x] be the subset polynomial semiring.

(i) Find subset polynomial ideals in S[x].

(ii) Can S[x] have zero divisors?

(iii) Can S[x] have idempotents?

(iv) Find subset polynomial subsemirings which are not subset polynomial ideals.

21. Let S[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right.$ = {Collection of all subsets of the ring $C(Z_{12})$}} be the subset polynomial semiring. Study questions (i) to (iv) of problem (20).

22. Let S[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right.$ = {Collection of all subsets of the ring $C(Z_7)$ (g_1, g_2) where $g_1^2 = g_2^2 = g_2 g_1 = g_1 g_2 = 0$}} be the subset polynomial semiring.

(i) Find S-zero divisors in S[x].

(ii) Can S[x] have S-idempotents?

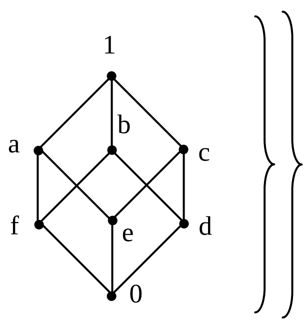
(iii) Find S-subset polynomial subsubrings in S[x].

(iv) Find S-subset polynomial ideals of S[x].

(v) Let T = {Collection of all subset polynomial ideals of S}. Prove (T, $\cup$, $\cap$} is a subset polynomial topological semiring space.

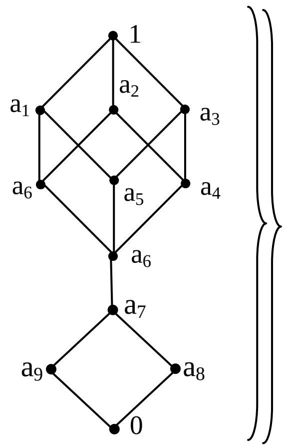
23. Let S[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the ring } C(Z_{24})\,(g) \mid g^2 = (0)\}\right\}$ be the subset polynomial semiring. Study questions (i) to (v) for this S[x] given in problem (22).

24. Let S[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the semiring}\right.$



be the subset polynomial semiring, study questions (i) to (v) given in problem 23.

25. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets from the semiring}\right.$



be the subset polynomial semiring.

(i) Study question (i) to (v) given in problem 22.

26. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the ring } z(g_1, g_2) \mid g_1^2 = g_2^2 = g_1g_2 = g_2g_1 = 0\}\right\}$ be the subset polynomial semiring.

Study questions (i) to (v) given in problem 22.

27. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{all subsets of the semiring } (Z^+ \cup \{0\}) S_3\}\right\}$ be the subset polynomial semiring.

(i) Prove S[x] has both left and right subset polynomial ideals.
(ii) Give two left subset polynomial ideals which are not right subset polynomial ideals.
(iii) Give an example of two S - right subset polynomial ideals which are not S-left subset polynomial ideals.
(iv) Can S[x] have subset polynomial subsemirings which are not S-subset polynomial susbemirings?

(v) Give 2 subset polynomial subsemirings which are not S-subset polynomial ideals.
(vi) Can S[x] have zero divisors or idempotents?

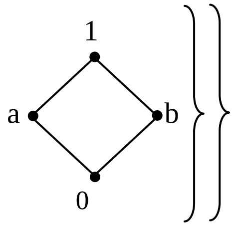
28. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets of the semigroup ring Z_{12} S(5)}} be the subset polynomial semiring.
Study questions (i) to (vi) of problem 27 for this S[x].

29. Find for the S[x] given in (28) the two topological spaces of subset polynomial semiring ideals.

30. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets of the semigroup semiring $(Z^+ \cup \{0\})P$ where P is a semigroup Z_{12}}} be the subset polynomial semiring.

(i) Prove S[x] has zero divisors.
(ii) Can S[x] have S-zero divisors?
(iii) Find S-ideals if any of the subset polynomial semiring.
(iv) Can S[x] have subset polynomial subsemiring which is not a subset polynomial ideal of the semiring?
(v) Let T = {Collection of all subset ideals of the polynomial semiring S[x]}}; Prove $(T', \cup, \cap)$ and $\{T, \cup_N, \cap_N\}$ are subset polynomial semiring topological spaces of the subset polynomial semiring S[x]. $(T' = T \cup \{\phi\})$.
(vi) Compare the two topological spaces $\{T, \cup_N, \cap_N\}$ and $\{T', \cup, \cap\}$ for any subset polynomial semiring.

31. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets of the semiring



be the subset polynomial semiring.

Study questions (i) to (vi) of problem 30 for this S[x].

32. Find some interesting properties enjoyed by $\{T, \cup_N, \cap_N\}$.

33. Distinguish between the two ideal topological spaces $\{T, \cup_N, \cap_N\}$ and $\{T' = \langle T \cup \phi\rangle, \cup, \cap\}$.

34. Does quasi set ideal topological spaces of $\{T, \cup_N, \cap_N\}$ and $\{T' = \langle T \cup \phi\rangle, \cup, \cap\}$ enjoy any special properties?

35. What is the situation if the problem in (34); the topological spaces are Smarandache.

36. What are the special features enjoyed by subset polynomial semirings?

37. Let
$S[x] = \{\sum a_i x_i \mid a_i \in S = \{\text{Collection of all subsets of } Z_{10}\}\}$ be a subset polynomial semiring.

    (i) Find zero divisors in S[x].
    (ii) Can S[x] have S-zero divisors?
    (iii) Can S[x] have idempotents?
    (iv) Can S[x] be a semifield?
    (v) Is S[x] a S-semiring?

38. Let $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets from the polynomial ring Z_7} be the subset polynomial semiring.

(i) Find ideals of the subset polynomial semiring.
(ii) Is S[x] a S-subset polynomial semiring?
(iii) Can S[x] have zero divisors?
(iv) Can S[x] have S-idempotents?
(v) Give a subset polynomial subsemiring which is not a S-ideal.
(vi) Can S[x] have ideals which are not S-ideals?

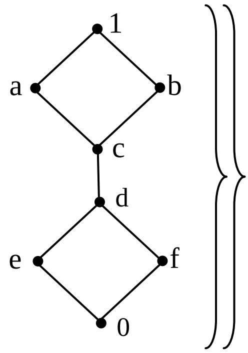
39. Let $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets of the polynomial semiring $Z^+ \cup \{0\}$} be the subset polynomial semiring.
Study questions (i) to (vi) mentioned in problem (38) in case of this S[x].

40. Let $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets from the polynomial ring Z}} be the subset polynomial semiring.

Study questions (i) to (vi) given in problem (38) for this S[x]. Compare S[x] in problem (39) and this S[x].

41. Draw any interesting feature enjoyed by S[x], the subset polynomial semiring when semiring is used instead of ring.

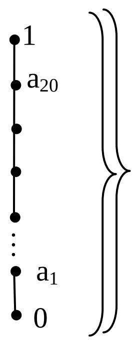
42. Let $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets from the lattice



be the subset polynomial semiring.

(i) Study questions (i) to (vi) given in problem 38.

43. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets}\right.$ from the lattice L =



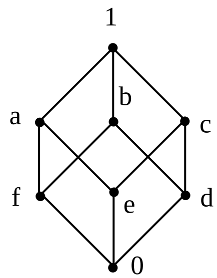
be the subset polynomial semiring.

Study questions (i) to (vi) given in problem 38.

44. Compare S[x] given in problems (38) and (39).

45. Let $S[x] = \{\sum a_i x_i \mid a_i \in S = \{$Collection of all subsets from the ring $Z_{20} \times Z_8\}\}$ be the subset polynomial semiring.
Study questions (i) to (vi) given in problem 38 for this S[x].

46. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of}\right.$ the semiring



$\}\}$

be the subset polynomial semiring.
Study questions (i) to (vi) given in problem (38) for this S[x].

47. Obtain some special features enjoyed by polynomial subset semirings.

48. Compare the polynomial subset semiring with subset polynomial semiring when same ring is used.

49. Let $S[x] = \{\sum a_i x_i \mid a_i \in S = \{$Collection of all subsets of the ring $Z_{12}\}$ be the subset polynomial semiring. $P[x] = \{$Collection of all subsets from the polynomial ring $Z_{12}[x]\}$ be the polynomial subset semiring. Compare S[x] and P[x].

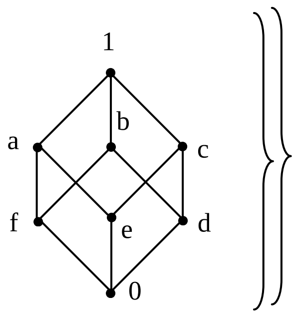
50. Let $P[x] = \{$Collection of all subsets of the ring $Z_6[x]\}$ be the polynomial subset semiring.

(i) Can P[x] have zero divisors?
(ii) Can P[x] have S-zero divisors?
(iii) Can P[x] have idempotents?
(iv) Find polynomial subset subsemirings.
(v) Is P[x] a S-polynomial subset semiring?
(vi) Can P[x] have polynomial subset ideals which are not S-ideals?

(vii) Can in P[x], A = B have a complete subset solution?
(viii) Find in P[x], A = B which has only partial solution.

51. Let P[x] = {Collection of all subsets of the polynomial semiring $(Z^{+} \cup \{0\})$ [x]} be the polynomial subset semiring. Study questions (i) to (viii) given in problem (50) in case of this P[x].

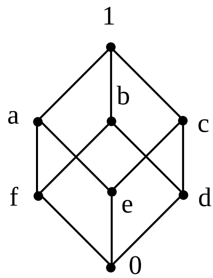
52. Let P[x] = {Collection of all subsets of the polynomial semiring L [x] where L =



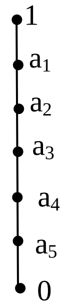
be the polynomial subset semiring.

Study questions (i) to (viii) given in problem (50) for this P[x].

53. If L =



is replaced by the chain lattice



in problem (52) study for P[x] questions (i) to (viii) given in problem 50.

54. Let P[x] = {Collection of all subsets of the polynomial ring Z_{18} [x]} be the polynomial subset semiring. Study questions (i) to (viii) in problem 50 for this P[x].

55. Let P[x] = {Collection of all subsets of the polynomial ring C(Z_{18}) [x]} be the polynomial subset semiring.

    (i) Study questions (i) to (viii) in problem 50 for this P[x].
    (ii) Compare P[x] in problems 54 and 55.
    (iii) Can one say all polynomial subset equations A = B is solvable in P[x] given in this problem (55) in comparison with P[x] given in problem 54? (A, B $\in$ P[x])
    (iv) Is A = B for all A, B $\in$ P[x] in problem 55 completely subset solvable?
    (v) Solve A = B where A = $\{3i_F x^2 + 3 + 2i_F, 9i_F x^4 + 2i_F, 8 i_F^3 x^3 + 16i_F x^2 + 16i_F x + 8\}$ and B = $\{i_F, 8, 0, 4 + 3i_F\}$, A, B $\in$ P[x].

56. Let P[x] = {Collection of all subset polynomials from the polynomial ring Z[x]} be the polynomial subset semiring.

    (i) Prove P[x] has polynomial subset equations A = B which are not completely solvable.
    (ii) Study questions (i) to (viii) given in problem 50 completely.

57. Let P[x] = {Collection of all subsets of the polynomial ring $C(Z_{11})$ [x]} be the polynomial subset semiring.

    (i) Study questions (i) to (viii) in problem 50 for this P[x].
    (ii) Is every polynomial subset equation A = B completely solvable? (A, B $\in$ P [x]).

58. Let P[x] = {Collection of all subset of the polynomial ring Z_{19}[x]} be the polynomial subset semiring.

    (i) Study question (i) to (viii) of problem 50 for this P[x].
    (ii) Prove every polynomial subset equation A = B need not in general be completely solvable. (A, B $\in$ P[x]).
    (iii) Let A = $\{3x^9 + 2x^3 + 5, 3x + 4x^2 + 6x + 1, 8x^3 + 5x + 2\}$ and B = $\{8x + 3, 0, 9x + 40 + 18x^2 + 3x + 1, 4x^5 + 2, 3\}$ be in P[x]. Solve A = B.

59. Let P[x] = {Collection of all subsets of the polynomial ring $C(Z_9)$ [x]; p a prime} be the polynomial subset semiring. Is every polynomial subset equation A = B, A, B $\in$ P[x] completely solvable?

60. Can we say P[x] = {Collection of all subset of the polynomial ring C[x]}, the polynomial subset semiring is an algebraically closed semifield? Is every A = B polynomial subset equation in P[x] completely solvable for A, B $\in$ P[x]?

Chapter Four

SUBSET POLYNOMIAL SEMIVECTOR SPACES AND SUBSET MATRIX SEMIVECTOR SPACES

In this chapter we for the first time using the subset polynomial semirings (polynomial subset semirings) and subset matrix semirings construct semivector spaces. We describe, define and develop these concepts. This technique gives us lots of scope in building differently semivector spaces.

We have already described building semivector spaces using subsets [19]. Here we mainly concentrate on building semivector spaces using subset matrices and subset polynomials.

DEFINITION 4.1: *Let M = {Collection of all $m \times n$ matrices whose entries are subsets from a semifield S} be the semigroup subset of $m \times n$ matrices under +. We see M is a subset matrix semivector space over the semifield S. ($m \neq n$; or $m = n$, $m = 1$ or $n = 1$ can also occur).*

We give examples of this.

Example 4.1: Let M = {2 × 3 matrices with entries from subsets of the semifield S = $Z^+ \cup \{0\}$}.

M is a subset matrix semivector space over S = $Z^+ \cup \{0\}$.

$$\text{Let } A = \begin{pmatrix} \{0,2\} & \{5,7,8\} & \{10,11,0\} \\ \{1,2\} & \{4,5\} & \{6,7,8\} \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} \{0\} & \{3,4\} & \{2\} \\ \{0,5\} & \{0\} & \{4,7,0\} \end{pmatrix} \in M.$$

We get

A + B =

$$\begin{pmatrix} \{0,2\} & \{8,9,10,11,12\} & \{2,12,13\} \\ \{1,2,6,7\} & \{4,5\} & \{6,7,8,10,11,12,13,14,15\} \end{pmatrix} \in M.$$

Take 10 ∈ $Z^+ \cup \{0\}$.

$$10A = \begin{pmatrix} \{0,20\} & \{50,70,80\} & \{100,110,0\} \\ \{10,20\} & \{40,50\} & \{60,70,80\} \end{pmatrix} \text{ is in M.}$$

Thus M is a subset semivector space over the semifield $Z^+ \cup \{0\}$.

Example 4.2: Let $T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in S = \{\text{Collection of all} \right.$

subsets of the semifield $Q^+ \cup \{0\}$} be a subset matrix semivector space over $Q^+ \cup \{0\}$.

$$\text{Let } A = \begin{bmatrix} \{0,3/2,7/9\} \\ \{8/5,4,20\} \\ \{9/11,10,14/3\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{0,3/2,8\} \\ \{0,4/5,1\} \\ \{9/11,0,12\} \end{bmatrix} \in T$$

we see

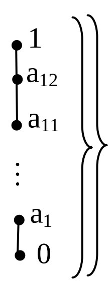
$$A + B = \begin{bmatrix} \{0,3/2,8,7/9,41/18, \\ 19/2,79/9\} \\ \{8/5,4,20,12/5,24/5,5, \\ 104/5,21,13/5\} \\ \{9/11,10,14/3,18/11,119/11, \\ 181/33,22,141/11,50/3\} \end{bmatrix} \text{ is in T.}$$

Take $5/7 \in Q^+ \cup \{0\}$; we find

$$5/7\, A = \begin{bmatrix} \{0,15/14,5/9\} \\ \{8/7,20/7,100/7\} \\ \{45/77,50/7,10/3\} \end{bmatrix} \in T.$$

This is the way operations are performed on T. Thus T is a subset matrix semivector space.

Example 4.3: Let N = $\{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in$ {Collection of all subsets from the chain lattice L =



be the subset row matrix semivector space over the semifield L.

Let A = ({0, a_1, a_3}, {1, a_2, a_5}, {a_3, a_6}, {a_7}, {a_{10}}, {a_1, a_2, a_{11}, a_7}) and

B = ({1, a_2}, {a_3, a_4, a_6}, {a_8}, {a_7, a_6, a_{10}, a_{11}, a_1}, {a_{10}, a_1, a_2, a_3} {a_{11}}) be in N.

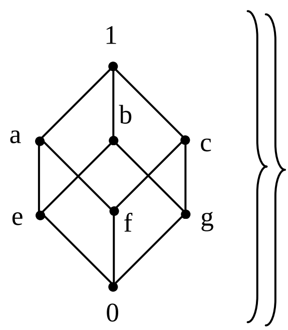
We find A + B = A $\cup$ B

= ({1, a_2, a_3}, {1, a_3, a_4, a_6, a_5}, {a_8}, {a_7, a_{10}, a_{11}}, {a_{10}}, {a_{11}}) $\in$ N.

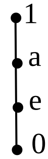
Take $a_3 \in$ L we find $a_3 \times A = a_3 \cap A$ = ({0, a_1, a_3}, {a_3}, {a_3}, {a_3}, {a_3}, {a_1, a_2, a_3}) $\in$ N.

Thus N is a subset matrix semivector space over L.

Example 4.4: Let M = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets from the lattice



be a subset matrix semivector space over the semifield

S = 

Consider A = $\begin{bmatrix} \{0,a,b\} & \{0,e\} \\ \{1,a,e\} & \{e,f,g\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \{0,e,f\} & \{1,g,e\} \\ \{c,1\} & \{0,b\} \end{bmatrix} \in M.$$

We find A + B = A ∪ B

$$= \begin{bmatrix} \{0,e,f,a,b,1\} & \{g,e,1,b\} \\ \{1\} & \{e,f,g,b,1\} \end{bmatrix} \in M.$$

Take e ∈ S. We find eA = e ∩ A = $\begin{bmatrix} \{0,e\} & \{0,e\} \\ \{0,e\} & \{0,e\} \end{bmatrix} \in M.$

Thus M is a subset matrix semivector space over S.

Example 4.5: Let T = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in \right.$ {Collection of all subsets of the semifield $R^+ \cup \{0\}$} $1 \leq i \leq 12$} be a subset matrix semiring.

Let A = $\begin{bmatrix} \{0\} & \{1,2\} & \{3,4\} \\ \{5\} & \{6\} & \{0,8\} \\ \{9,10\} & \{1,11,0\} & \{1,0\} \\ \{2,5,6\} & \{3,1,2\} & \{4,6,7\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \{9\} & \{0,13,6\} & \{6\} \\ \{0,14\} & \{8\} & \{7\} \\ \{1,2,3\} & \{4,6\} & \{8,9\} \\ \{10,1,4\} & \{11\} & \{6,8\} \end{bmatrix} \in T.$$

A + B =

$$\begin{bmatrix} \{9\} & \{1,2,14,15,7,8\} & \{9,10\} \\ \{5,19\} & \{14\} & \{7,15\} \\ \{10,11,12,13\} & \{5,7,4,6,15,17\} & \{10,9,8\} \\ \{3,6,7,12,15,16,10,9\} & \{14,12,13\} & \{10,12,13,14,15\} \end{bmatrix} \text{ is in T.}$$

Let $5 \in R^+ \cup \{0\}$,

$$5 \times A = \begin{bmatrix} \{0\} & \{5,10\} & \{15,20\} \\ \{25\} & \{30\} & \{0,40\} \\ \{45,50\} & \{0,5,55\} & \{0,5\} \\ \{10,25,30\} & \{5,10,15\} & \{20,30,35\} \end{bmatrix} \in T.$$

Example 4.6: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in \{\text{Collection of all subsets of the semiring } Z^+ \cup \{0\},\ 1 \leq i \leq 10\} \right.$$

be the subset matrix semiring over the semifield $Z^+ \cup \{0\}$.

$$\text{Let } A = \begin{bmatrix} \{0,1\} & \{3,4,5,6\} \\ \{10,15\} & \{9\} \\ \{0\} & \{1\} \\ \{4,8,7\} & \{9,12,19\} \\ \{4,0\} & \{1,0,15\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{6,9,8\} & \{1,2\} \\ \{0\} & \{0,4,8\} \\ \{9,1,8\} & \{4,10,6\} \\ \{14,0\} & \{0\} \\ \{1\} & \{5,4\} \end{bmatrix} \in M;$$

we find A + B = $\begin{bmatrix} \{6,9,8,9,10,7\} & \{4,5,6,7,8\} \\ \{10,15\} & \{9,13,17\} \\ \{9,1,8\} & \{5,11,7\} \\ \{4,8,7,18,22,21\} & \{9,12,19\} \\ \{5,1\} & \{5,6,20,4,19\} \end{bmatrix}$.

Take $3 \in R^+ \cup \{0\}$,

$$3 \times A = 3A = \begin{bmatrix} \{0,3\} & \{9,12,15,18\} \\ \{30,45\} & \{27\} \\ \{0\} & \{3\} \\ \{12,24,21\} & \{27,36,57\} \\ \{12,0\} & \{3,0,45\} \end{bmatrix} \in M.$$

We can as in case of semivector spaces define the notion of subset matrix semivector subspaces.

We will only illustrate this situation by some simple examples.

Example 4.7: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \text{ where } a_i \in S = \{\text{Collection of all subsets of the semifield } Z^+ \cup \{0\}, 1 \le i \le 9\} \right\}$$

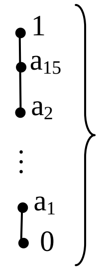
be the subset matrix semivector space over the semifield $F = Z^+ \cup \{0\}$.

$$\text{Take } P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \ a_i \in S_1 = \{\text{Collection of all subsets of the set } 2Z^+ \cup \{0\}, 1 \le i \le 9\} \right. \subseteq T;$$

P is clearly a subset matrix semivector subspace over the semifield $F = Z^+ \cup \{0\}$.

Infact T has infinitely many subset matrix semivector subspaces over the semifield $Z^+ \cup \{0\}$.

Example 4.8: Let $M = \{(a_1, a_2, \ldots, a_9) \mid a_i \in$ {Collection of all subsets of the semifield F =

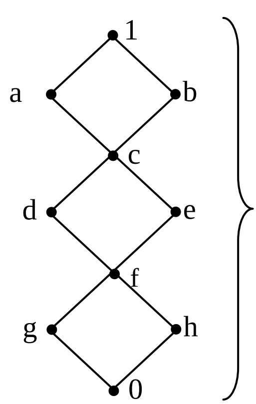


$1 \leq i \leq 9 = S\}$ be the subset matrix semivector space over F.

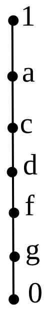
Take $P = \{(a_1, a_2, 0, 0, \ldots, 0) \mid a_1, a_2 \in S\} \in M$. Clearly P is a subset matrix semivector subspace of M over F.

Infact number of elements in M is finite, so M has only finite number of subset matrix semivector subspaces over F.

Example 4.9: Let N = {Collection of all subsets from the semiring



P = {all 2 × 5 matrices with subset entries from N} be the subset matrix semivector space over the semifield L =



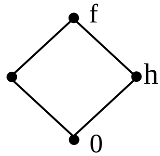
$$A = \begin{pmatrix} \{1,a\} & \{b\} & \{c,d\} & \{f\} & \{0\} \\ \{0\} & \{1\} & \{g,h\} & \{0\} & \{a,b,c\} \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} \{1\} & \{a\} & \{b\} & \{c\} & \{d\} \\ \{0\} & \{f\} & \{d,c\} & \{1,a\} & \{0,g\} \end{pmatrix} \in P.$$

We see A + B =

$$A \cup B = \begin{pmatrix} \{1\} & \{1\} & \{b\} & \{c\} & \{d\} \\ \{0\} & \{1\} & \{d,c\} & \{1,a\} & \{a,b,c\} \end{pmatrix} \in P.$$

$$\text{Take } L = \left\{ \begin{pmatrix} \{0\} & \{0\} & \{0\} & \{a_1,a_2\} & \{a_1\} \\ \{0\} & \{0\} & \{0\} & \{a_1\} & \{a_1\} \end{pmatrix} \right|$$

$a_1, a_2 \in g$  $\Bigg\} \subseteq P;$

L is a subset matrix subsemivector subspace of P over the semifield L.

Now before we proceed onto define other types of semivector spaces using subset matrices we just give examples of subset matrix semilinear algebra over semifields.

Example 4.10: Let M = $\{(a_1, a_2, a_3) \mid a_i \in$ {Collection of all subsets of the semifield $Q^+ \cup \{0\}$}; $1 \leq i \leq 3\}$ be the subset row matrix semivector space over the semifield $Z^+ \cup \{0\}$.

M is a subset row matrix semilinear algebra over the semifield $Z^+ \cup \{0\}$.

Take $A = \left(\begin{Bmatrix} 0,18,9 \\ 4,8,6 \end{Bmatrix}, \{0, 1, 2, 4\}, \{5, 7, 9, 6, 0\}\right)$ and

$$B = \left(\begin{Bmatrix} 0,1,2 \\ 3,4 \end{Bmatrix}, \{0, 2\}, \{5, 0, 1\}\right) \in M.$$

A + B =({0, 18, 9, 4, 8, 6, 1, 19, 10, 5, 7, 20, 2, 11, 3, 21, 12, 22, 13}, {0, 1, 2, 4, 3, 6}, {0, 5, 6, 7, 9, 1, 8, 10, 12, 11, 14}) $\in$ M.

We can also find A $\times$ B = ({0, 18, 9, 4, 8, 6} $\times$ {0, 1, 2, 3, 4},{0, 1, 2, 4}$\times${0, 2},{0, 5, 6, 7, 9} $\times$ {0, 1, 5})

= ({0, 18, 9, 4, 8, 6, 36, 12, 24, 27, 54, 16, 32, 72}, {0, 2, 4, 8},{0, 5, 6, 7, 9, 25, 30, 35, 45}) $\in$ M.

Thus M is a subset matrix semilinear algebra over the semifield $Z^+ \cup \{0\}$.

Example 4.11: Let

$$M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in \{\text{Collection of all subsets of the semiring } Z^+ \cup \{0\}, 1 \leq i \leq 4\}\right\}$$

be a subset matrix semigroup under '+'. M is the subset semivector space over the semifield $Z^+ \cup \{0\}$.

M is not a subset matrix semilinear algebra.

Note: If on M we can define the natural product $\times_n$ of subset matrices then M will be a subset matrix semilinear algebra.

Let $A = \begin{bmatrix} \{3,4,7\} \\ \{0\} \\ \{8,9,2\} \\ \{1,0,4\} \end{bmatrix}$ and $B = \begin{bmatrix} \{2,1,3,4\} \\ \{4,8,9,25\} \\ \{1,2,4\} \\ \{5,7,8\} \end{bmatrix} \in M$;

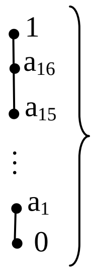
$$\text{we find } A + B = \begin{bmatrix} \{5,6,9,4,8,7,10,11\} \\ \{4,8,9,25\} \\ \{9,10,3,11,4,12,13,6\} \\ \{5,7,8,6,9,11,12\} \end{bmatrix} \in M.$$

We can find

$$A \times_n B = \begin{bmatrix} \{3,4,7,6,8,14,9,12,21,16,28\} \\ \{0\} \\ \{8,9,2,16,18,4,32,36\} \\ \{5,20,0,7,28,8,32\} \end{bmatrix}.$$

We see under natural product $\times_n$; M is a subset matrix semilinear algebra, otherwise M is only a subset matrix semivector space as we cannot define usual product of column matrices, more so in subset column matrices.

Example 4.12: Let N = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the semifield F =



$1 \leq i \leq 6\}$ be the subset matrix semivector space over the semifield F.

We cannot define usual product on subset matrices as there are 2×3 matrices so for us to make N into a subset matrix semilinear algebra we can define the natural product $\times_n$ on N. $\{N, \times_n\}$ will be a subset matrix semilinear algebra over the semifield F.

Inview of this we have the following result the proof of which is left as an exercise to the reader.

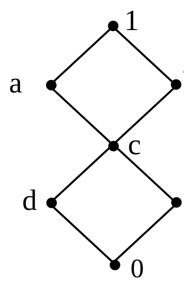
THEOREM 4.1: *Let*
M = {All m × n matrices with entries from subsets of the semifield F} be the subset matrix semilinear algebra over the semifield F; then M in a subset matrix semivector space. If M is a subset matrix semivector space then M is general is not a matrix subset matrix semilinear algebra.

Now we proceed onto define subset matrix semivector space of type I.

DEFINITION 4.2: *Let*
M = {Collection of all m × n matrices whose entries are from the semiring S} we see if sA ∈ M for all s ∈ S and A ∈ M, then we define M to be a subset matrix semivector space over the semiring of type I.

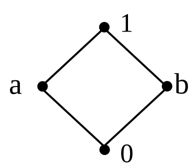
We will illustrate this situation by some examples.

Example 4.13: Let M = $\left\{ \begin{bmatrix} a_1 & a_5 \\ a_2 & a_6 \\ a_3 & a_7 \\ a_4 & a_8 \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the semiring S,



} $1 \leq i \leq 8$} be the subset matrix semivector space over the semiring S of type I.

Example 4.14: Let N = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the semiring S =



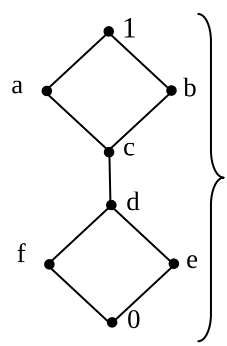
} $1 \leq i \leq 12$} be the subset matrix semivector space of type I over the semiring S.

Example 4.15: Let T = $\left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the semiring S = $Z^+ \cup \{0\} \times Z^+ \cup \{0\}$}, $1 \leq i \leq 16$}

be the subset matrix semivector space of type I over the semiring S.

Example 4.16: Let M = $\left\{\left[\begin{array}{cc} a_1 & a_{13} \\ a_2 & a_{14} \\ \vdots & \vdots \\ a_{12} & a_{24} \end{array}\right] \right|$ $a_i \in$ {Collection of all subsets from the semiring S = $Q^+ \cup \{0\} \times R^+ \cup \{0\} \times R^+ \cup \{0\}$}; $1 \leq i \leq 24$} be the subset matrix semivector space of type I over the semiring S.

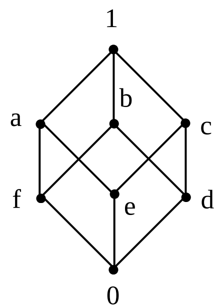
Example 4.17: Let T = $\left\{\left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{array}\right] \right|$ $a_i \in$ {Collection of all subsets from the semiring S =



$1 \leq i \leq 9$} be the subset matrix semivector space of type I over the semiring S.

Now having seen examples of subset matrix semivector space of type I over the semiring S, we can now define substructures in them which is a matter of routine so it is left as an exercise to the reader. However we will give some examples.

Example 4.18: Let P = {Collection of all subsets from the semiring S =



}.

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix} \middle| \; a_i \in P, 1 \leq i \leq 40 \right\}$$ is the subset matrix semivector space of type I over the semiring S.

$$\text{Take } N = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix} \middle| \; a_i \in P; 1 \leq i \leq 20 \right\},$$

the subset matrix semivector subspace of T of type I over S. We can have several such subset matrix semivector subspaces of T. However we keep on record, as $o(T) < \infty$ we can have only finite number of subset matrix semivector subspaces of T over S.

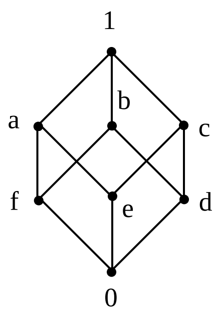
Example 4.19: Let $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in \right.$ {Collection of all subsets of the semiring $P = Z^+ \cup \{0\} \times R^+ \cup \{0\}, 1 \leq i \leq 10\}$ be the subset matrix semivector space over the semiring P of type I.

Let $T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the semiring $P = Z^+ \cup \{0\} \times \{0\}\}, 1 \leq i \leq 10\} \subseteq S$; T is a subset matrix semivector subspace of P over the semiring P of type I.

In this case S has infinite number of subset matrix semivector subspaces of type I over the semiring P.

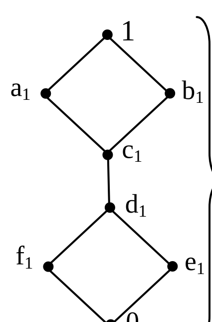
Example 4.20: Let $V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the semiring

$L = P_1 \times P_2$ where $P_1 =$



and

$P_2 =$



$, 1 \leq i \leq 4\}$

be a subset matrix semilinear algebra over the semiring L of type I.

Clearly if the usual product is defined on V, we see V is a non commutative subset matrix semilinear algebra of type I.

However if natural product $\times_n$ is taken the subset matrix semilinear algebra of type I is commutative.

We see o(V) < ∞ so we have only finite number of subset matrix subsemilinear algebra of type I.

Let A = $\begin{bmatrix} \{(a,d_1),(0,0)\} & \{(a,f_1)\} \\ \{(1,a_1),(0,1)\} & \{(f,0)\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \{(0,1),(1,0)\} & \{(b,0)\} \\ \{(1,1)\} & \{(1,a_1)\} \end{bmatrix} \in V.$$

We find $A \times_n B = \begin{bmatrix} \{(0,d_1),(0,0),(a,0)\} & \{(f,0)\} \\ \{(1,a_1),(0,1)\} & \{(f,0)\} \end{bmatrix}$.

Consider

$$A \times B = \begin{bmatrix} \{(a,f_1),(a,d_1)\} & \{(a,f_1)\} \\ \{(f,1),(1,0),(f,0),(f,a_1)\} & \{(f,0),(b,0)\} \end{bmatrix}$$

$A \times B \neq A \times_n B$.

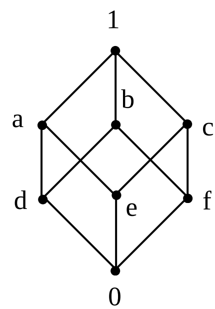
Consider

$$B \times A = \begin{bmatrix} \{(0,0),(0,d_1),(a,0),(b,d_1),(1,0)\} & \{(f,f_1),(a,0)\} \\ \{(0,a_1),(a,a_1),(1,a_1)\} & \{(a,f_1)\} \end{bmatrix}.$$

We see A × B ≠ B × A, thus V is not a subset matrix non commutative semilinear algebra.

Example 4.21: Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right|$ $a_i \in$ {Collection of all subsets from the semiring S = $Z^+ \cup \{0\} \times S_1$; where

S_1 =



; $1 \leq i \leq 15\}$

be a subset matrix semilinear algebra under natural product $\times_n$ of matrices of type I over S.

Clearly o(M) = ∞ but M has subset matrix semilinear algebras of finite order.

For take P = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right|$ $a_i \in$ {Collection of all subsets from the semiring $\{0\} \times S_1\}$, $1 \leq i \leq 15\} \subseteq$ M; P is a subset matrix subsemilinear algebra of finite order.

Thus we can have for a subset matrix semilinear algebra / semivector space of infinite order a subset matrix subsemilinear algebra / subsemivector space of finite order (order is used in a sense as the number of elements).

Now having seen examples of subset matrix semilinear algebra / semivector spaces and their substructure we now proceed onto describe other types of subset matrix semilinear algebras / semivector spaces.

Suppose we have M = {Collection of all subset matrices with subsets from a ring R} then M is defined as the special subset matrix semivector space of type II over the ring R; that is for B, A $\in$ M we have A + B $\in$ M and for all A $\in$ M and r $\in$ R, rA and Ar are in M.

We will illustrate this by some examples.

Example 4.22: Let M = {Collection of all subset 1 $\times$ 7 matrices

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_7 \end{bmatrix} \text{ where}$$

$a_i \in$ {Collection of all subsets of the ring Z, $1 \leq i \leq 7$}} be the subset 1 $\times$ 7 matrix special semivector space of type II over the ring Z.

Clearly o(M) = ∞.

Example 4.23: Let N = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets from the ring Z_{12}}; $1 \leq i \leq 12$} be the subset matrix special semivector space of type II over the ring Z_{12}.

Clearly $o(N) < \infty$.

Thus we can have both finite and infinite special subset semivector spaces of type II over the ring Z_{12}.

Example 4.24: Let $P = \{(a_1, a_2, a_3, a_4) \mid a_i \in$ {Collection of all subsets of the group ring Z_2S_3}; $1 \leq i \leq 4\}$ be the special subset semivector space of type II over the ring Z_2S_3.

Let $A = (\{0, p_1\}, \{p_2, p_3\}, \{p_4\}, \{p_3\})$ and $B = (\{p_2\}, \{p_1\}, \{p_3\}, \{p_2\}) \in P$.

We see

$A + B = (\{0, p_1+p_2\}, \{p_1+p_2, p_1+p_3\}, \{p_4 + p_3\}, \{p_3 + p_2\}) \in P$.

Now $A \times B = (\{0, p_5\}, \{p_4, p_5\}, \{p_1\}, \{p_4\})$ and

$B \times A = (\{0, p_4\}, \{p_5, p_4\}, \{p_2\}, \{p_5\})$.

We see $A \times B \neq B \times A$ so P is a non commutative subset matrix semilinear algebra.

Take for $A \in P$ an element in Z_2S_3 say p_2 in Z_2S_3.

$p_2A = (\{0,p_5\}, \{1, p_4\}, \{p_1\}, \{p_4\})$ and

$Ap_2 = (\{0, p_4\}, \{1, p_5\}, \{p_3\}, \{p_5\})$.

We see $p_2 A \neq Ap_2$ for $p_2 \in Z_2S_3$ and $A \in P$.

Thus the special subset semivector space of type II is doubly non commutative.

Example 4.25: Let $T = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \right.$ $a_i \in$ {Collection of all subsets of the group ring ZD_{29}}; $1 \leq i \leq 4$} be the special subset matrix semivector space type II.

Clearly $o(T) = \infty$ and T is also a doubly non commutative special subset matrix semivector space of type II.

Example 4.26: Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \end{pmatrix} \middle| \; a_i \in \{\text{Collection of all} \right.$$

subsets of the group semiring $(Z^+ \cup \{0\})S_7$}, $1 \leq i \leq 14$}

be the subset matrix semivector space of type I over the semiring $(Z^+ \cup \{0\})S_7$. Clearly M is doubly non commutative. Further if $(Z^+ \cup \{0\})S_7$ is replaced by $Z^+ \cup \{0\}$, M is only a non commutative subset matrix semilinear algebra and $xA = Ax$ for all $x \in Z^+ \cup \{0\}$ and $A \in M$; however $A \times_n B \neq B \times_n A$ even under natural product.

Example 4.27: Let

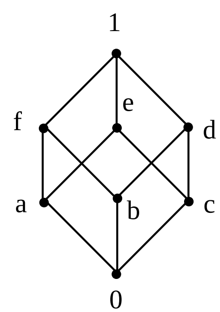
$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \middle| \; a_i \in \{\text{Collection of all subsets} \right.$$

from the semigroup semiring $(Z^+ \cup \{0\})S(10)$}, $1 \leq i \leq a_{20}$}

be the subset matrix semivector space of type I over the semiring $(Z^+ \cup \{0\})(S(10))$. Clearly P is a doubly non commutative subset matrix semilinear algebra under natural product $\times_n$ of type I over the semiring $(Z^+ \cup \{0\})S(10)$.

If we replace $(Z^+ \cup \{0\})S(10)$ by $Z^+ \cup \{0\}$ we see P is only a non commutative subset matrix semilinear algebra over the semifield $Z^+ \cup \{0\}$. However P is not doubly non commutative only non commutative.

Example 4.28: Let W = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets from the group lattice LS_6 where L =

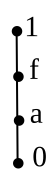


$1 \leq i \leq 21\}$ be the subset matrix semivector space over the semiring L of type I.

W is only non commutative as a subset matrix semilinear algebra of type I under the natural product $\times_n$ over L.

However if L is replaced by LS_3 certainly W is a doubly non commutative subset matrix semilinear algebra of type I over the semiring LS_3.

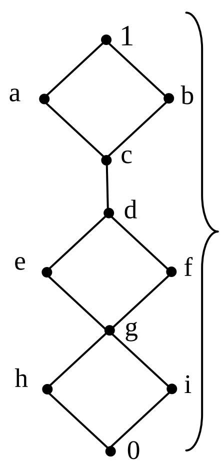
If L is replaced by



(say L_1) a chain lattice (semifield); we see W is a subset matrix semivector space over L_1. However W is a subset matrix non commutative semilinear algebra over the semifield L_1 under natural product $\times_n$, but is not doubly commutative as $xA = Ax$ for all $x \in L_1$ and $A \in W$ but $A \times_n B \neq B \times_n A$ for $A, B \in W$. Hence the claim.

Example 4.29: Let M = $\left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{pmatrix} \right|$ $a_i \in$ {Collection

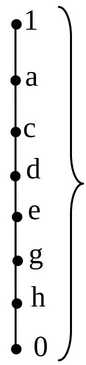
of all subsets of the semigroup semiring LS(9) where L is a lattice



$1 \leq i \leq 3$} be the subset matrix semivector space of type I over the semiring LS(9). Clearly M is a doubly non commutative subset matrix semilinear algebra of type I over the semiring LS(9) even under natural product $\times_n$ of matrices.

If LS(9) is replaced by the lattice L of course M will not be doubly commutative for $xA = Ax$ for all $x \in L$ and $A \in M$ but $A \times_n B \neq B \times_n A$ for $A, B \in M$.

If L is replaced L_1 where L_1 =



then M is just a subset matrix non commutative semilinear algebra over the semifield L_1 under natural product $\times_n$.

Thus from this example we see a doubly non commutative subset matrix semilinear algebra of type I which is doubly non commutative can contain subset matrix semilinear algebra of type I which are not doubly commutative but changing the semiring over which it is defined to be commutative but of type I. Finally if we change the base semiring to be a semifield we get subset matrix semivector space over the semifield which is a subset of the semiring.

Example 4.30: Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \right.$ $a_i \in$ {Collection of all subsets from the semigroup semiring S = $(Q^+ \cup \{0\})$ (S(12)}, $1 \leq i \leq 6\}$ be the subset matrix semivector space of type one over the semiring S.

M is doubly non commutative.

Take N = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{bmatrix} \middle| \right.$ $a_i \in$ {Collection of all subsets from the semigroup semiring S}, $1 \leq i \leq 3\} \subseteq M$; N is a subset matrix semivector subspace of M over S of type I.

Example 4.31: Let T = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{12} & a_{24} \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the group semiring $S = (Z^+ \cup \{0\})\, A_4\}$, $1 \leq i \leq 24\}$ be the subset matrix semivector space of type I over the semiring S. T has infinitely many subset matrix semivector subspaces. Infact T is a Smarandache subset matrix semivector space over the subset $Z^+ \cup \{0\} \subseteq (Z^+ \cup \{0\})A_4$, where $Z^+ \cup \{0\}$ is a semifield.

Example 4.32: Let W = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets from the semiring $R^+ \cup \{0\}\}$, $1 \leq i \leq 6\}$ be the subset matrix semivector space of type I over $5Z^+ \cup \{0\}$. W has infinitely many subset matrix semivector subspaces.

Now we proceed onto give examples of subset matrix semivector spaces of type III over a field F.

Example 4.33: Let M = $\left\{ \begin{bmatrix} a_1 & a_3 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of a field $Z_7\}$; $1 \leq i \leq 6\}$ be the special strong subset matrix semivector space over the field Z_7 of type III. Clearly $o(M) < \infty$.

Example 4.34: Let M = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{12} & a_{24} \end{bmatrix}\right|$ $a_i \in$ {Collection of all subsets from the field R}; $1 \leq i \leq 24$} be the special strong subset matrix semivector space of type III over the field R. Infact o(M) = ∞.

Suppose R is replaced by $R^+ \cup \{0\}$. M will only be a subset matrix semivector space over the semifield $R^+ \cup \{0\}$. If R is replaced by Z, M will be a subset matrix semivector space of type II over the ring Z.

Now we see M has special subset matrix semivector spaces of type II also apart from the subset matrix semivector spaces.

Example 4.35: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| \; a_i \in \{\text{Collection of all}$$

$$\text{subsets form the ring } QS_5\},\ 1 \leq i \leq 9\}$$

be the special strong subset matrix semivector space of type III over the field Q. M is also a special strong subset matrix semilinear algebra of type III over the field Q.

Infact o(M) = ∞ and M is a non commutative special strong subset matrix semilinear algebra of type III over the field Q. We can consider M as a double non commutative special subset matrix semilinear algebra of type II over the ring QS_5.

Infact M is also a Smarandache doubly non commutative special matrix semilinear algebra of type II over the ring QS_5.

Example 4.36: Let W = $\left\{\begin{pmatrix} a_1 & a_2 & ... & a_{11} \\ a_{12} & a_{13} & ... & a_{22} \\ a_{22} & a_{23} & ... & a_{33} \end{pmatrix} \middle| \right.$ $a_i \in$ {Collection

of all subsets from the semigroup ring RS(3)}, $1 \leq i \leq 3$} be the strong special subset matrix semivector space of type III over the field R.

Infact W is a special subset matrix semilinear algebra of type II over the ring RS(3) which is doubly non commutative and is of infinite order.

Example 4.37: Let M = {Collection of all 6 × 6 matrices with entries from the subsets of the groupring $Z_{19}D_{2,17}$} be the special strong subset matrix semivector space over the field Z_{19} of type III.

If Z_{19} is replaced by the ring $Z_{19}D_{2,17}$ we have M to be a special subset matrix semivector space over the ring $Z_{19}D_{2,17}$ of type II.

Clearly $o(W) < \infty$.

Example 4.38: Let W = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \right.$ $a_i \in$ {Collection of

all subsets from the field Z_5}, $1 \leq i \leq 15$} be the special strong subset matrix semilinear algebra of type III over the field Z_5.

Clearly $o(W) < \infty$.

Example 4.39: Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the group ring $Z_{11}S_3$}, $1 \leq i \leq 9$} be the strong special subset matrix semivector space of type III over the field Z_{11}. Clearly as a semilinear algebra of type III, M is non commutative. If Z_{11} is replaced by $Z_{11}S_3$ we see M is a special subset matrix semivector space of type II.

Infact M is a doubly non commutative subset matrix linear algebra of type II over the ring $Z_{11}S_3$.

Example 4.40: Let

W = {Collection of all subset of the groupring $Z_{13}S_7$}.

$$P = \left\{\begin{bmatrix} a_1 & a_{11} & a_{21} \\ a_2 & a_{12} & a_{22} \\ \vdots & \vdots & \vdots \\ a_{10} & a_{20} & a_{30} \end{bmatrix} \right| a_i \in W, 1 \leq i \leq 30\} \text{ be a strong}$$

special subset matrix semivector space over the semifield Z_{13} of type III; $o(P) < \infty$. If Z_{13} is replaced by $Z_{13}S_7$ we get special subset matrix semivector space of type II over the ring $Z_{13}S_7$.

We see the type will affect the basis of the structure. To this end we define the following properties about these subset matrix semivector spaces of all types.

Let M be a subset matrix semivector space. We say A and B in M are linearly subset dependent if A = cB; c $\in$ ring or field or semiring or semifield over which M is defined.

If for no c we can write A = cB then we say A and B linearly subset independent in M.

Let A = $\begin{pmatrix} \{0,5,8,3\} & \{1,2\} \\ \{0\} & \{0\} \end{pmatrix}$ and B = $\begin{pmatrix} \{0\} & \{0\} \\ \{1,2,3,4\} & \{0\} \end{pmatrix} \in$ M

= {all 2 × 2 matrices with the subsets from the semifield $Z^+ \cup \{0\}\}$ and M is a subset matrix semivector space over the semifield $Z^+ \cup \{0\}$. Clearly A ≠ cB for any $c \in Z^+ \cup \{0\}$ so A and B are subset linearly independent in M over $Z^+ \cup \{0\}$.

Let A = $\begin{pmatrix} \{0,3,6\} & \{0\} \\ \{0\} & \{3\} \end{pmatrix}$ and B = $\begin{pmatrix} \{0,1,2\} & \{0\} \\ \{0\} & \{1\} \end{pmatrix} \in$ M.

We see A = 3B so A and B are subset linearly subset dependent matrices.

Given any pair of matrices in M we may have them to be linearly subset independent. A collection of linearly subset independent matrices which is capable of generating the subset matrix semivector space is defined to be the subset matrix basis of M over the appropriate algebraic structure.

We will illustrate this situation by an example or two.

Example 4.41: Let M = $\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \middle| \right.$ $a_1, a_2 \in$ {Collection of subsets of the field $Z_3\}\}$ be the special strong subset matrix semivector space over the field Z_3.

The subset matrices A = $\begin{bmatrix} \{1\} \\ \{0\} \end{bmatrix}$ and B = $\begin{bmatrix} \{2\} \\ \{0\} \end{bmatrix}$ in M are linearly dependent as 2A = B; that is B = 2A. However if A = $\begin{bmatrix} \{0\} \\ \{1,2\} \end{bmatrix}$ and B = $\begin{bmatrix} \{0,2\} \\ \{0\} \end{bmatrix}$ are subset linearly independent matrices, can A, B in M generate M.

If S is the generated by A, B.

$$\text{If } A = \begin{bmatrix} \{0\} \\ \{1,2\} \end{bmatrix} \in S \text{ then } \begin{bmatrix} \{0\} \\ \{0\} \end{bmatrix} \in S.$$

$$A + A = \begin{bmatrix} \{0\} \\ \{2,0,1\} \end{bmatrix} \in S.$$

$$\text{Now } B \in S \text{ so } B + B = \begin{bmatrix} \{0,1,2\} \\ \{0\} \end{bmatrix} \in S.$$

$$2B = \begin{bmatrix} \{0,1,2\} \\ \{0\} \end{bmatrix} \in S.$$

However $S \neq M$; that is A, B cannot generate M as subset matrix linearly independent set for $\begin{bmatrix} \{1\} \\ \{1\} \end{bmatrix}, \begin{bmatrix} \{2\} \\ \{2\} \end{bmatrix} \notin S$.

Now consider the sets $\left\{ \begin{bmatrix} \{0\} \\ \{1\} \end{bmatrix}, \begin{bmatrix} \{1\} \\ \{0\} \end{bmatrix}, \begin{bmatrix} \{0,1\} \\ \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} \\ \{0,1\} \end{bmatrix} \right\} \in M.$

We see this can be a basis of M and are linearly subset independent over the field Z_3.

Example 4.42: Let $M = \{(a_1, a_2, a_3) \mid a_i \in$ {Collection of all subsets from the ring Z_4}, $1 \leq i \leq 3\}$ be the special subset matrix semivector space over the ring Z_4.

$$\text{Consider } P = \left\{ \begin{bmatrix} \{1\} \\ \{0\} \\ \{0\} \end{bmatrix}^t, \begin{bmatrix} \{0\} \\ \{1\} \\ \{0\} \end{bmatrix}^t, \begin{bmatrix} \{0\} \\ \{0\} \\ \{1\} \end{bmatrix}^t \right\},$$

P is a linearly subset independent collection. However P cannot form a basis of M.

T = {({1, 0}, {0}, {0}), ({0}, {0,1}, {0}), ({0}, {1}, {0}), ({0}, {0}, {1})} $\subseteq$ M is a subset linearly independent matrix. Infact T is a subset matrix basis of M over Z_4.

We can have another set of subset linearly independent subsets which forms a basis of M over Z_4.

Example 4.43: Let W = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle|\right.$ $a_i \in$ {Collection of all subsets of the ring Z_6}, $1 \leq i \leq 4$} be a special subset matrix semivector space over the ring Z_6.

$$\text{Take P} = \left\{\begin{bmatrix} \{0\} & \{0\} \\ \{1\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{1\} \\ \{0\} & \{0\} \end{bmatrix},\right.$$

$$\begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{1\} \end{bmatrix}, \begin{bmatrix} \{0,1\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0,1\} \\ \{0\} & \{0\} \end{bmatrix},$$

$$\left.\begin{bmatrix} \{0\} & \{0\} \\ \{0,1\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0,1\} \end{bmatrix}\right\} \subseteq W$$

is a collection of subset matrix independent set which is a basis of W over Z_6.

Clearly subset dimension of W over Z_6 is eight.

Example 4.44: Let W = $\left\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle|\right.$ $a_i \in$ {Collection of all subsets from the ring Z_{12}}, $1 \leq i \leq 6$} be the special subset matrix semivector space over the ring Z_{12}.

$$\text{Take P} = \left\{\begin{pmatrix} \{1\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{1\} & \{0\} \\ \{0\} & \{0\} & \{0\} \end{pmatrix},\right.$$

$$\begin{pmatrix} \{0\} & \{0\} & \{1\} \\ \{0\} & \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{1\} & \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{1\} & \{0\} \end{pmatrix},$$

$$\begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{1\} \end{pmatrix}, \begin{pmatrix} \{0,1\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0,1\} & \{0\} \\ \{0\} & \{0\} & \{0\} \end{pmatrix},$$

$$\begin{pmatrix} \{0\} & \{0\} & \{0,1\} \\ \{0\} & \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{0,1\} & \{0\} & \{0\} \end{pmatrix}, \begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{0,1\} & \{0\} \end{pmatrix},$$

$$\left.\begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0,1\} \end{pmatrix}\right\} \subseteq W$$

is a subset matrix basis of W over Z_6. The subset dimension of W over Z_{12} is 12.

It is left as an exercise for the reader to find subset basis of subset matrix semivector spaces.

Example 4.45: Let W = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right|$ $a_i \in$ {Collection of all subsets of the field Z_7}, $1 \leq i \leq 8$} be the strong special subset matrix semivector space over the field Z_7.

Let P = $\left\{ \begin{bmatrix} \{1\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{1\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{1\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{1\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \right.$

$$\begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{1\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{1\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{1\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{1\} \end{bmatrix},$$

$$\begin{bmatrix} \{0,1\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0,1\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0,1\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0,1\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix},$$

$$\left.\begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0,1\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0,1\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0,1\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0,1\} \end{bmatrix}\right\} \subseteq W$$

is a special strong subset matrix semivector basis of W over the field Z_7 and the special strong subset dimension is 16.

Example 4.46: Let $M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| a_i \in \right.$ {Collection of all subsets from the semifield $Z^+ \cup \{0\}$}; $1 \leq i \leq 6\}$ be the subset matrix semivector space over the semifield $Z^+ \cup \{0\}$.

$$P = \left\{ \begin{bmatrix} \{1\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{1\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{1\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \right.$$

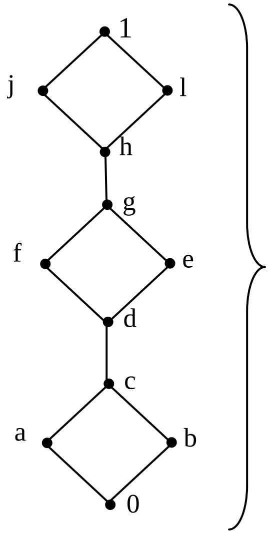
$$\begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{1\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{1\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{1\} \end{bmatrix}, \begin{bmatrix} \{0,1\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix},$$

$$\begin{bmatrix} \{0\} & \{0,1\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0,1\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0,1\} \\ \{0\} & \{0\} \end{bmatrix}, \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0,1\} & \{0\} \end{bmatrix},$$

$$\left.\begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0,1\} \end{bmatrix}\right\} \subseteq M$$

is a subset matrix basis of the semivector of M over the semifield $Z^{+} \cup \{0\}$. Clearly the subset matrix semivector space is dimension is 12.

Example 4.47: Let M = {Collection of all 5 × 5 matrices with subset entries from the semiring



be the subset matrix semivector space over the semiring L of type I.

Find a basis subset matrix of M over L

Now we proceed onto define yet a new type of subset matrix semivector space over semirings or semifields of subsets.

DEFINITION 4.3: *Let S = {Collection of all subsets of a field or a ring or a semiring or a semifield}.*

M = {m × n matrices with elements from S}. M is defined as the super subset matrix semivector space over the subset semiring / semifield S.

We will first illustrate this situation by some examples.

Example 4.48: Let M = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \right.$ $a_i \in$ S = {Collection of all subsets from the ring Z_9}; $1 \leq i \leq 4$} be the super subset matrix semivector space over the semiring / semifield S.

Let P = $\begin{bmatrix} \{0,3\} & \{0,4,2\} \\ \{0,6,8\} & \{1\} \end{bmatrix}$ ∈ M and x = {0, 2, 7} ∈ S.

$$xP = \begin{bmatrix} \{0,2,7\}\times\{0,3\} & \{0,2,7\}\times\{0,4,2\} \\ \{0,2,7\}\times\{0,6,8\} & \{0,2,7\}\times\{1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,6,3\} & \{0,4,5,8,1\} \\ \{0,3,7,6,2\} & \{0,2,7\} \end{bmatrix} \in M.$$

This is the way we make super subset matrix semivector space.

Example 4.49: Let
S = {Collection of all subsets from the semifield $Z^+ \cup \{0\}$} be the subset semifield. Let W = {3 × 2 matrices with entries from S}, be the semivector space over the subset semifield S.

Let A= $\begin{bmatrix} \{0,1,2,4,8,9\} & \{0,9,2\} \\ \{4,8,9,11\} & \{0\} \\ \{14,2,0\} & \{7,19,1\} \end{bmatrix}$ ∈W and x={0,9,12,3,4,1}∈S.

$$xA = \begin{bmatrix} \begin{matrix}\{0,1,2,4,8,9,18,3,6 \\ 12,24,27,16,32,36, \\ 72,81,48,96,108\}\end{matrix} & \begin{matrix}\{0,9,2,6,27,8,36, \\ 18,81,24,108\}\end{matrix} \\ \begin{matrix}\{4,8,9,11,12,24, \\ 27,33,16,32,36,44,0, \\ 81,72,99, \\ 48,96,108,132\}\end{matrix} & \{0\} \\ \begin{matrix}\{0,2,14,6,42,8,56, \\ 18,126,24,168\}\end{matrix} & \begin{matrix}\{1,7,19,0,3,21,57, \\ 4,28,76,9,63,171,12, \\ 84,228\}\end{matrix} \end{bmatrix} \text{ is in W.}$$

Example 4.50: Let M = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \right.$ $a_i \in$ {Collection of all subsets from the semifield $Q^+ \cup \{0\} = S$; $1 \leq i \leq 4$} be the super subset matrix semivector space over the subset semifield S.

M is infact a non commutative super subset matrix semilinear algebra over the subset semifield S.

Example 4.51: Let B = $\{(a_1, a_2, a_3, a_4, a_5, a_6, \ldots, a_{10}) \mid a_i \in$ S = {Collection of all subsets of the ring Z_{14}}; $1 \leq i \leq 10\}$ be the super subset matrix semivector space over the semiring S.

Clearly $o(B) < \infty$ and also the dimension of the super subset matrix semivector space is finite.

Example 4.52: Let M = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix} \middle| \right.$ $a_i \in$ S = {Collection of all subsets of the group ring Z_3S_3}, $1 \leq i \leq 8\}$ be the super subset matrix semivector space over the subset semiring S. Clearly M is doubly non commutative as a super subset matrix semilinear algebra over the subset semiring S.

Infact for $A \in S$ and $P \in M$; $AS \neq SA$ in general.

Example 4.53: Let
S = {Collection of all subsets from the semigroup ring $Z_4S(3)$}, $M = (a_1, a_2, a_3, a_4, a_5) \mid a_i \in S; 1 \leq i \leq 5\}$ be the super subset matrix semivector space over the subset semiring S. Clearly M as a super subset matrix semilinear algebra which is doubly non commutative. For if $X \in S$ and $A \in M$, $AX \neq XA$ in general and $AB \neq BA$ in general for $A, B \in S$.

Example 4.54: Let
S = {Collection of all subsets of the semiring $(Z^+ \cup \{0\})S(7)$}.

$$M = \left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix}\right| a_i \in S; 1 \leq i \leq 8\right\}$$

be the super subset matrix semivector space over the subset semiring S.

Clearly for $X \in S$ and $A \in M$ $XA \neq AX$. This is a special type of non commutative semivector space.

One is very well aware of a fact in general if V is a vector space (or a semivector space) and F a field (or a semifield) then for $v \in V$ and $a \in F$ $av = va$ but this is not in general true in case of subset matrix semivector spaces more so in super subset matrix semivector spaces.

This is the main difference between usual vector spaces (semivector spaces) and the subset matrix semivector spaces and super subset matrix semivector spaces.

We will illustrate this situation also by an example or two.

Example 4.55: Let M = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| a_i \in S = \{\text{Collection of subsets}\right.$ of the group ring $Z_4 S_3\}$, $1 \leq i \leq 4\}$ be the super subset matrix semivector space over the subset semiring S.

Take X = $\{p_1, p_2, 0, 3p_3\} \in S$ and A = $\begin{bmatrix} \{p_1, p_2\} \\ \{p_3\} \\ \{2p_4\} \\ \{p_2 + p_1\} \end{bmatrix} \in M$.

We find XA = $\begin{bmatrix} \{p_1, p_2, 0, 3p_3\}\{p_1, p_2\} \\ \{p_1, p_2, 0, 3p_3\}\{p_3\} \\ \{p_1, p_2, 0, 3p_3\}\{2p_4\} \\ \{p_1, p_2, 0, 3p_3\}\{p_2 + p_1\} \end{bmatrix}$

$$= \begin{bmatrix} \{1, 0, p_5, 3p_5, p_4, 3p_4\} \\ \{3, 0, p_4, p_5\} \\ \{0, 2p_3, 2p_1, 2p_2\} \\ \{0, 1+p_5, 1+p_4, 3p_4 + 3p_5\} \end{bmatrix} \in M.$$

Consider AX = $\begin{bmatrix} \{p_1, p_2\}\{p_1, p_2, 0, 3p_3\} \\ \{p_3\}\{p_1, p_2, 0, 3p_3\} \\ \{2p_4\}\{p_1, p_2, 0, 3p_3\} \\ \{p_2 + p_1\}\{p_1, p_2, 0, 3p_3\} \end{bmatrix}$ = XA, so only we use the term in general AX ≠ XA.

Now consider X = $\{p_1, 0\}$ ∈ S and A = $\begin{bmatrix} \{p_2\} \\ \{p_3\} \\ \{p_4, 1\} \\ \{p_5, 0\} \end{bmatrix}$ ∈ M,

we find XA = $\begin{bmatrix} \{0, p_1\}\{p_2\} \\ \{0, p_1\}\{p_3\} \\ \{0, p_1\}\{p_4, 1\} \\ \{0, p_1\}\{0, p_5\} \end{bmatrix} = \begin{bmatrix} \{0, p_5\} \\ \{0, p_4\} \\ \{0, p_1, p_3\} \\ \{0, p_2\} \end{bmatrix}$ ∈ M.

Consider AX = $\begin{bmatrix} \{p_2\}\{0, p_1\} \\ \{p_3\}\{0, p_1\} \\ \{p_4, 1\}\{0, p_1\} \\ \{0, p_5\}\{0, p_1\} \end{bmatrix} = \begin{bmatrix} \{0, p_4\} \\ \{0, p_5\} \\ \{0, p_1, p_2\} \\ \{0, p_3\} \end{bmatrix}$ ∈ M.

Clearly XA ≠ AX.

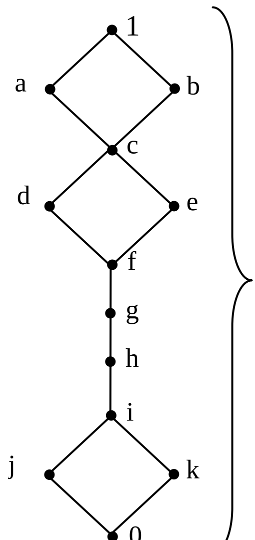
Example 4.56: Let M = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \right|$ a_i ∈ S = {Collection of all subsets from the semiring $Z^+ \cup \{0\}$ (S_7)}, 1 ≤ i ≤ 12} be a super subset matrix semivector space defined over the subset semiring S of type I.

We see o(M) is infinite, however the reader is left with the task of finding dimension of M over S.

Example 4.57: Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \end{bmatrix} \right|$ $a_i \in S =$ {Collection of all subsets of the group ring $Z_{11} S_6$}, $1 \leq i \leq 30\}$ be a super subset matrix semivector space over the subset semiring S.

We see $o(W) < \infty$ and W is finite dimensional over S.

Example 4.58: Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right|$ $a_i \in S =$ {Collection of all subsets of the group lattice LS_3 where L is the following lattice, L =



$1 \leq i \leq 15\}$ be a super subset matrix semivector space over the subset semiring S.

Clearly o(M) < ∞ and M is a commutative super subset matrix semilinear algebra over S.

Example 4.59: Let

S = {Collection of all subsets of the group ring $Z_{45} S_8$}.

$$M = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{array}\right] \middle| \; a_i \in S, 1 \leq i \leq 30 \right\}$$

be the super subset matrix semiring over the subset semiring S.

Clearly o(S) < ∞ but S is doubly non commutative as super subset matrix semilinear algebra over S.

Further S is also non commutative as a super matrix semivector space as $xA \neq Ax$ in general for all $x \in S$ and $A \in M$.

Example 4.60: Let

S = {Collection of all subsets of the field Z_{43}}.

$$M = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{24} \end{array}\right] \middle| \; a_i \in S; 1 \leq i \leq 24 \right\}$$

be the super subset matrix semivector space over the subset semiring S. o(S) < ∞.

M is infact commutative as a super subset matrix semivector space as well as super subset matrix semilinear algebra over S.

Example 4.61: Let

S = {Collection of all subsets of the group ring $Z_{43} D_{2,7}$}.

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in S, 1 \leq i \leq 16 \right\}$$

be a super subset semivector space over the subset semiring $Z_{43}D_{2,7}$.

o(W) < ∞. But W is non commutative as a super subset matrix semivector space and doubly non commutative as a super subset matrix semilinear algebra over a subset semiring.

We see the concept of linear independence and subset basis of a super subset matrix semivector space over a subset semiring.

Example 4.62: Let
S = {Collection of all subsets of the field Z_{11}}.
W = {(a_1, a_2, a_3, a_4) | $a_i \in$ S; $1 \leq i \leq 4$} be the super subset matrix semivector space over the subset semifield S.

Take A = ({1}, {0}, {0}, {0}) and
B = ({0}, {0,5}, {0}, {0}) ∈ W.

We see A and B are subset linearly independent.

Consider A = ({4, 6, 2}, {2}, {0}, {4}) and B = ({1, 3, 2}, {1}, {0}, {2}) in W. We see 2B = A thus A and B are subset linearly dependent.

Now consider

B = {({1}, {0}, {0}, {0}), ({0}, {1}, {0}, {0}), ({0}, {0}, {1}, {0}), ({0}, {0}, {0}, {1}), ({0,1}, {0}, {0}, {0}), ({0}, {0,1}, {0}, {0}), ({0}, {0}, {0,1}, {0}), ({0}, {0}, {0}, {0,1})}.

B is a not a subset basis of W for ({1}, {0}, {0}, {0}) and ({0,1}, {0}, {0}, {0}) are subset linearly dependent.

B would have been the subset basis of the W as not considered as a super subset matrix semivector space.

However for the super subset matrix semivector space W the super subset basis is B = {({1}, {0}, {0}, {0}), ({0}, {1}, {0}, {0}), ({0}, {0}, {1}, {0}), ({0}, {0}, {0}, {1})}.

Thus dimension of W over S is four.

Example 4.63: Let
S = {Collection of all subsets of the semiring $R^+ \cup \{0\}$}.

$$M = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \end{array} \right] \middle| \ a_i \in S, 1 \leq i \leq 36 \right\}$$

be the super subset matrix semivector space over the subset semiring S. o(S) = ∞ but super subset dimension of M over S is just 36.

Example 4.64: Let
S = {Collection of all subsets of the semifield $R^+ \cup \{0\}$}.
M = {Collection of all 7 × 7 matrices with entries from the subset semiring $R^+ \cup \{0\}$} is the super subset matrix semivector space of dimension 49 over S.

Inview of this we have a nice theorem.

THEOREM 4.2: *Let S = {Collection of all subsets from a field or a ring or a semifield or a semiring or a group ring or semigroup ring or a group semiring or a semigroup semiring}. M = {Collection of all m × n matrices with entries from S}. M is a super subset matrix semivector space of super subset dimension m × n over S.*

Proof is direct and hence left as an exercise to the reader. Whatever be the order of M we see super subset dimension of M is the same.

It is simple and is a matter of routine to get super subset matrix semivector subspaces of a super subset matrix semivector space over a subset semiring.

Now we proceed onto define describe and develop the notion of subset polynomial semivector spaces and the notion of super subset polynomial semivector spaces.

DEFINITION 4.4: *Let S = {Collection of all subsets of a semiring or a semifield or ring or a field}.*

$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$. *M is defined as the subset polynomial semivector space of a stipulated type depending on the structure over it is defined.*

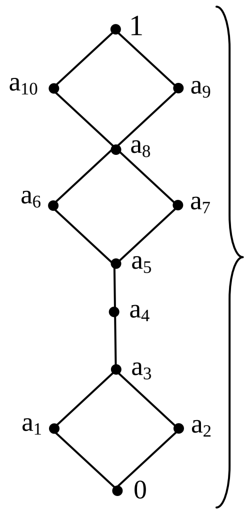
We will first illustrate this situation by some examples.

Example 4.65: Let

S = {Collection of all subsets of the semifield $Z^+ \cup \{0\}$}.

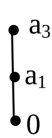
$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$; M is a subset polynomial semivector space over the semifield, $Z^+ \cup \{0\}$.

Example 4.66: Let S = {Collection all subsets of the lattice L =



$M = \left\{ \sum_{i=0}^{\infty} d_i x^i \,\middle|\, d_i \in S \right\}$ be the subset polynomial semivector space over the semifield

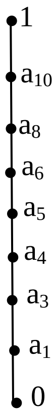
F =



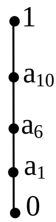
If M is a subset polynomial semivector space over the semiring L then we call M to be a subset polynomial semivector space of type I over L.

Infact by varying the semifields or chain lattices in L we can get several different subset polynomial semivector spaces over F_i, $1 \leq i \leq 4$.

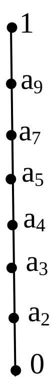
For take F_1 =



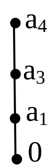
or F_2 =



or F_3 =



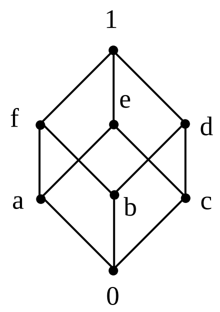
or F_4 =



and so on. We see F_i's are just chain lattices so they are nothing but semifields.

Also we can get type I subset polynomial semivector spaces over other sublattices which are not chain lattices.

Example 4.67: Let S = {Collection of all subsets of the semiring

B = 

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\};$$

M is a subset polynomial semivector space / subset polynomial semilinear algebra of type I over the semiring B.

Example 4.68: Let S = {Collection of all subsets of the semiring R = $(Z^+ \cup \{0\})S_7$ (group semiring)}.

$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ is a subset polynomial semivector space of type I over the semiring R.

The speciality of this space is that if a $\in$ R and p(x) $\in$ M then ap(x) $\neq$ p(x)a in general.

That is why these subset polynomial semivector spaces are non commutative of type I. Further as a subset polynomial semilinear algebras over R of type I they are doubly non commutative over R.

We will give some more examples of them.

Example 4.69: Let S = {Collection of all subsets of the semigroup semiring T = $(R^+ \cup \{0\})$ (S(4))}.

$M = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S\right\}$ is a subset polynomial semivector space over $R^+ \cup \{0\}$ and a subset polynomial semivector space of type I over T.

We see as a subset polynomial semivector space of type I over T, M is non commutative and doubly non commutative as a subset polynomial semilinear algebra of type I over T.

For take $a = \left\{\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix}\right\}$ in S and

$$p(x) = \left\{\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&2&3\end{pmatrix}\right\}x^2 +$$

$$\left\{\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&4&4&1\end{pmatrix}\right\} \in M.$$

$$ap(x) = \left\{\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}.\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix},\right.$$

$$\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}.\begin{pmatrix}1&2&3&4\\1&1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix}.\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix},$$

$$\left.\begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix}.\begin{pmatrix}1&2&3&4\\1&1&2&3\end{pmatrix}\right\}x^2 +$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}.\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}.\begin{pmatrix}1&2&3&4\\4&4&4&1\end{pmatrix},\right.$$

$$\begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix}.\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix},\begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix}.\begin{pmatrix}1&2&3&4\\4&4&4&1\end{pmatrix}\right\}.$$

$$= \left\{\begin{pmatrix}1&2&3&4\\2&2&1&1\end{pmatrix},\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}1&2&3&4\\3&3&2&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&2&1&1\end{pmatrix}\right\} x^2+$$

$$\left\{\begin{pmatrix}1&2&3&4\\2&2&3&3\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}1&2&3&4\\4&4&2&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix}\right\} \in M.$$

Now consider p(x) a = $\left\{\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}.\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix},\right.$

$$\begin{pmatrix}1&2&3&4\\1&1&2&3\end{pmatrix}.\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}.\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix},$$

$$\left.\begin{pmatrix}1&2&3&4\\1&1&2&3\end{pmatrix}.\begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix}\right\}x^2 +$$

$$\left\{\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}.\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}.\begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}1&2&3&4\\4&4&4&1\end{pmatrix}.\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&4&4&1\end{pmatrix}.\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}\right\}$$

$$= \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 1 \end{pmatrix} \right\} x^2 +$$

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 1 \end{pmatrix} \right\} \in M.$$

Clearly $ap(x) \neq p(x)a$.

That is why we say M is a non commutative subset polynomial semivector space of type I over the semiring T.

Example 4.70: Let
S = {Collection of all subsets of the semigroup ring $Z_{12}S(3)$}.
M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ be the special subset polynomial semivector space of type II over the ring $Z_{12}S(3)$.

Clearly M is a non commutative special subset polynomial semivector space of type II. M is a doubly non commutative special subset polynomial semilinear algebra of type II over $Z_{12}S(3)$.

Example 4.71: Let
S = {Collection of all subsets of the groupring Z_7S_8}.
M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ be the special subset polynomial semivector space of type II over Z_7S_8 or M can also be realized

as a special strong subset polynomial semivector space of type III over Z_7.

In the first case it is non commutative as special subset polynomial semivector space of type II and doubly non commutative as a special subset polynomial semilinear algebra of type II over Z_7S_8.

However it is a special strong polynomial semivector space over of type III over Z_7 but is a non commutative special strong polynomial semilinear algebra of type III as
$p(x)\, q(x) \neq q(x)\, p(x)$ in general for $p(x), q(x) \in M$.

Example 4.72: Let S = {Collection of all subsets of the field R}. $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ is a special strong subset polynomial semivector space of type III over the field R.

Infact M is also a special strong subset polynomial semilinear algebra of type III over the field R.

M can be realized as a subset polynomial semivector space over the field $R^+ \cup \{0\}$. M can be realized as a special polynomial semivector space over the ring $Z \subseteq R$ of type I.

We see just by varying subsets of the field R we can get different types of subset polynomial semivector spaces.

Example 4.73: Let
S = {Collection of all subsets of the group ring $R = Z_5D_{27}$}. $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ is the special subset polynomial semivector space over the ring R of type II.

If M is considered over the field Z_5 we see M is a special strong subset polynomial semivector space of type III over the field Z_5.

Example 4.74: Let
S = {Collection of all subsets of the semifield $Z^+ \cup \{0\}$}.

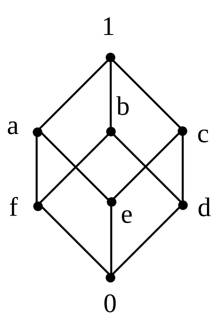
M = $\left\{ \sum_{i=0}^{9} a_i x^i \,\middle|\, a_i \in S;\ 0 \le i \le 9 \right\}$ is the subset polynomial semivector space over the semifield $Z^+ \cup \{0\}$.

Clearly M is not a subset polynomial semilinear algebra over the semifield $Z^+ \cup \{0\}$ as for p(x), q(x) $\in$ M in general p(x) $\times$ q(x) $\notin$ M.

Example 4.75: Let
S = {Collection of all subsets of the semiring T = $(Z^+ \cup \{0\})$ S_3}. W = $\left\{ \sum_{i=0}^{7} a_i x^i \,\middle|\, a_i \in S;\ 0 \le I \le 7 \right\}$ is a subset polynomial semivector space over the type I.

Clearly W is not a subset polynomial semilinear algebra over T for if p(x), q(x) $\in$ W then p(x) $\times$ q(x) $\notin$ W in general.

Example 4.76: Let S = {Collection of all subsets of the semiring B =



}

M = $\left\{ \sum_{i=0}^{4} a_i x^i \,\middle|\, a_i \in S;\ 0 \le i \le 4 \right\}$ be the subset polynomial semivector space over the semiring B of type I.

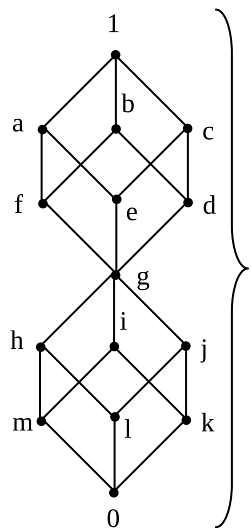
Clearly M is not a subset polynomial semilinear algebra over B of type I.

Example 4.77: Let
S = {Collection of all subsets of the ring Z_{40}}.

$M = \left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in S,\ 0 \leq i \leq 10 \right\}$ be the special subset polynomial semivector space of type II over the ring Z_{40}. M is not a special subset polynomial semilinear algebra of type II over Z_{40}.

Example 4.78: Let S = {Collection of all subsets of the group lattice LS_4 where

L =



$M = \left\{ \sum_{i=0}^{15} a_i x^i \,\middle|\, a_i \in S,\ 0 \leq i \leq 15 \right\}$ be a subset polynomial semivector space of type I over the semiring LS_4.

Clearly M is non commutative as a subset polynomial semivector space of type I over L. Further M is not a subset polynomial semilinear algebra of type I over L.

Now we have seen all subset polynomial semivector spaces in general are not subset polynomial semilinear algebras what ever be the type; however all subset polynomial semilinear algebras are always subset polynomial semivector spaces.

In view of this we have the following theorem the proof of which is left as an exercise to the reader.

THEOREM 4.3: *Let S = {Collection of all subsets of a semifield or a semiring or a field or a ring}. M =* $\left\{\sum_{i=0}^{n} a_i x^i \,\middle|\, a_i \in S;\ 0 \leq i \leq n,\ n < \infty\right\}$ *be a subset polynomial semivector space over a ring (or semifield or semiring or a field). Then M is never a subset polynomial semilinear algebra over the ring (or semifield or semiring or a field) of any type.*

Inview of this we have another theorem.

THEOREM 4.4: *Let M =* $\left\{\sum_{i=0}^{n} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of a ring or a semiring or a semifield or a field}\}\right\}$ *be a subset polynomial semivector space over ring or field or semiring or a semifield. M in general need not be a subset polynomial semilinear algebra of any type.*

The result is obvious if n < ∞ certainly M is never a subset polynomial semilinear algbra of any type.

We can as in case of usual subset semivector spaces define the notion of subset semivector subspaces, subset linearly independent elements and subset basis.

We only illustrate this situation by some examples.

Example 4.79: Let

$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the ring } Z_{10}\}\right\}$ be the special subset polynomial semivector space of type II over the ring Z_{10}.

Let $p(x) = \{0,5,8\}x + \{2,1\}$ and $q(x) = \{9,3\}x + \{8,0\} \in M$

We see p(x) and q(x) are subset linearly independent polynomials in M.

Consider $p(x) = \{0,2,4\}x + \{6, 8\}$ and $q(x) = \{0,1,2\}x + \{3, 4\}$ in M. We see p(x) and q(x) are subset linearly dependent polynomials in M for $p(x) = 2q(x)$.

Now as in case of usual vector spaces we see in case of subset polynomial semivector spaces also the basis B will form a linearly independent set, that is the basis B will be a subset linearly independent set.

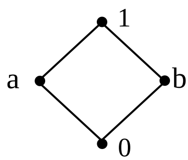
Take $B = \{\{1\}, \{1\}x, \ldots, \{1\}x^n, \ldots, \{0, 1\}, \{0,1\}x, \{0,1\}x^2, \ldots, \{0,1\}x^n, \ldots\}$ forms a subset basis of the special subset polynomial semivector space of type II over the ring Z_{10}.

Example 4.80: Let $M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the semifield } Z^+ \cup \{0\}\}\right\}$ be the subset polynomial semivector space over the semifield $Z^+ \cup \{0\}$.

We can vizulize M to be a subset polynomial semilinear algebra over the semifield $Z^+ \cup \{0\}$, then $p(x) \times q(x)$ is defined and is in M.

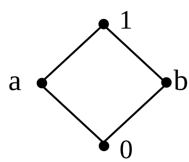
Now a subset polynomial basis of M over $Z^+ \cup \{0\}$ is given by $B = \{\{1\}, \{0, 1\}, \{0,1\}x, \{1\}x, \ldots, \{1\}x^n, \{0, 1\}x^n, \ldots\}$ over the semifield $Z^+ \cup \{0\}$.

Example 4.81: Let S = {Collection of all subsets of the semiring L =

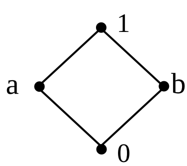


}

$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ is a subset polynomial semilinear algebra of type I over the semiring L =



Consider B = {{0, 1}, {1}, {1}x, {0,1}x {1}x^2, {0, 1}x^2, …} $\subseteq$ M is a subset basis of M over the semiring

L =  of type I.

Example 4.82: Let

$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the field } Z_7\} \right\}$ be the special strong subset polynomial semilinear algebra over the semifield Z_7 of type III.

B = {{1}, {0,1}, {0,1}x, {1}x …} is a subset polynomial basis of M over the field Z_7 of type III.

Example 4.83: Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of semiring } (Z^+ \cup \{0\})S_3\} \right\}$ be the subset polynomial semivector space of type I over the semiring $(Z^+ \cup \{0\})S_3$.

Let W = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P\right.$ = {Collection of all subsets of the semiring $(2Z^+ \cup \{0\})\, S_3\} \subseteq M$ be the subset polynomial semivector subspace of type I over $(Z^+ \cup \{0\})\, S_3$.

Example 4.84: Let M = {Collection of all polynomial $\sum_{i=0}^{\infty} a_i x^i \,\Big|$ $a_i \in S$ = {Collection of all subsets of the ring $Z_{11}\, A_4$}} be special subset polynomial semivector space of type I over the ring $Z_{11}A_4$.

$$\text{Take } N = \left\{\sum_{i=0}^{20} a_i x^i \,\middle|\, a_i \in S\right\} \subseteq M$$

is a special subset polynomial semivector space of type I over the ring $Z_{11}A_4$.

Now we proceed onto define super subset polynomial semivector spaces of all the three types.

DEFINITION 4.5: *Let S = {Collection of all subsets of the semiring or semifield or field or ring}.*

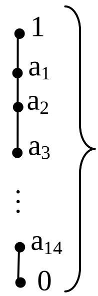
M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ *be the subset polynomial semivector space of the appropriate type over S. We define M to be the super subset polynomial semivector space over S.*

We will illustrate this situation by some examples.

Example 4.85: Let S = {Collection of all subsets of the semifield $Q^+ \cup \{0\}$}.

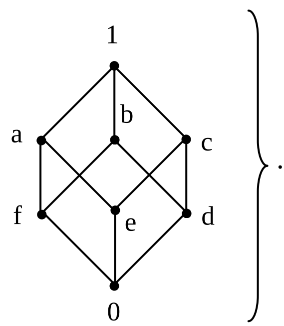
M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ be the super subset polynomial semivector space over the subset semiring S.

Example 4.86: Let S = {Collection of all subsets of the semifield L =



and M $= \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ be the super subset polynomial semivector space over the semiring S.

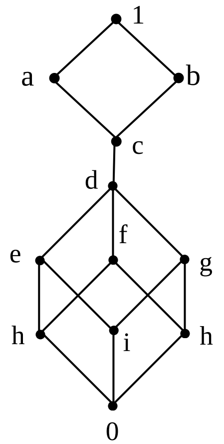
Example 4.87: Let S = {Collection of all subsets of the semiring



M $= \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ be the super subset polynomial semivector space over S of type I.

Example 4.88: Let S = {Collection of all subsets of the semiring $(Z^+ \cup \{0\})\, A_5$}. M $= \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ be the super subset polynomial semivector space of type I over S. Clearly even as a super subset polynomial semivector space, M is non commutative.

Example 4.89: Let S = {Collection of all subsets of the semiring LS_7 where L is a lattice



}.

M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ is a super subset polynomial semivector space of type I over S.

However M is a non commutative as a super subset polynomial semivector space and doubly non commutative as a super subset polynomial semilinear algebra. For we see if s ∈ S and p(x) ∈ M, sp(x) ≠ p(x)s in general and for p(x), q(x) ∈ M, p(x) q(x) ≠ q(x) p(x).

Example 4.90: Let M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right.$ = {Collection of all subsets of the ring Z_{42}}} be the special super subset polynomial semilinear algebra over S of type II.

We can have several such special super subset polynomial semilinear algebras over S of type II.

Example 4.91: Let M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all}\right.$ subsets of the ring Z_{12} $S_3\}\}$ be the special super subset polynomial semilinear algebra over S of type II.

Clearly M is non commutative special super subset polynomial semilinear algebra over S. Infact M is doubly commutative.

Example 4.92: Let
S = {collection of all subsets from the field Z_{43}}.

Let M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ be the special strong super subset polynomial semivector space over the semiring S of type III. Clearly M is commutative as a special strong super subset polynomial semilinear algebra over S.

$B_1 = \{\{1\}, \{1\}x, \{1\}x^2, \ldots, \{1\}x^n, \ldots\}$ is a subset basis of M as a special strong super subset polynomial semivector space over S.

Example 4.93: Let S = {Collection of all subsets of the group semiring $(Q^+ \cup \{0\})$}.

M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S\right\}$ be the super subset polynomial semivector space over S of type I. Infact for $p(x) \in M$ and $s \in S$ we have in general $sp(x) \neq p(x)s$.

Further M is a super subset polynomial semilinear algebra of type I over S which doubly non commutative.

$B_1 = \{\{1\}, \{1\}x, \{1\}x^2, \{1\}x^3, \ldots, \{1\}x^3, \ldots\}$ be the subset basis of M as a semivector space over S.

Example 4.94: Let
S = {Collection of all subsets of the semigroup ring $Z_5S(8)$} and
M = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S\right\}$ be the special super subset polynomial semivector space of type I over S.

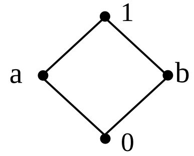
Clearly M is non commutative even as a special super subset polynomial semivector space of type I over S for sp(x) ≠ p(x)s in general for s ∈ S and p(x) ∈ M. Further if M is realized as a special super subset polynomial semilinear algebra then also then M is doubly non commutative as for p(x), q(x) ∈ M, p(x) q(x) ≠ q(x) p(x) in general. Finding basis is a matter of routine.

Example 4.95: Let

S = {Collection of all subsets of the group ring $Z_{70}S_3$}.

M = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S \text{ and } n < \infty\right\}$ be the special super subset polynomial semivector space of type I. Clearly M is non commutative, however M is not a special super subset polynomial semilinear algebra as for p(x), q(x) ∈ M, we see p(x) q(x) ∉ M in general.

Example 4.96: Let S = {Collection of all subsets of the lattice ring LS_3 where L is



and S_3 is a permulation group}. M = $\left\{\sum_{i=0}^{7} a_i x^i \;\middle|\; a_i \in S; 0 \leq i \leq 7\right\}$ be a super subset polynomial semivector space over S.

Clearly M is non commutative as sp(x) $\neq$ p(x)s for s $\in$ S and p(x) $\in$ M.

Further M is not a semilinear algebra.

Example 4.97: Let M = $\left\{\sum_{i=0}^{5} a_i x^i \right|$ $a_i \in$ S = {Collection of all subset of the group ring $Z_{12}S_4$}; $0 \leq i \leq 5$} be the super subset polynomial semivector space over the subset semiring S. o(M) < ∞.

We see M is non commutative but M is not a super subset polynomial semilinear algebra over S; for if p(x), q(x) $\in$ M then p(x) q(x) $\notin$ M.

Example 4.98: Let M = $\left\{\sum_{i=0}^{9} a_i x^i \right|$ $a_i \in$ S = {Collection of all subsets of the group semiring $(Z^+ \cup \{0\})$ A_5}, $0 \leq i \leq 9$} be the super subset polynomial semivector space over S which is non commutative, but o(M) = ∞.

Example 4.99: Let M = $\left\{\sum_{i=0}^{3} a_i x^i \right|$ $a_i \in$ S = {Collection of all subsets of the ring ZD_{27}}; $0 \leq i \leq 3$} be the super subset polynomial semivector space over S which is non commutative and o(M) = ∞.

Example 4.100: Let W = $\left\{\sum_{i=0}^{8} a_i x^i \right|$ $a_i \in$ S = {Collection of all subsets of the ring $Z_{18}D_{29}$} $0 \leq i \leq 8$} be the super special subset polynomial semivector space over the subset semiring S.

Clearly W is not a super special subset polynomial semilinear algebra over the subset semiring S.

Example 4.101: Let S = {Collection of all subsets of the ring $C(Z_{12})$ (g_1, g_2) where $g_1^2 = 0$ and $g_2^2 = g_2$, $g_1g_2 = 0 = g_2g_1$} be the subset semiring.

M = $\left\{ \sum_{i=0}^{18} a_i x^i \right|$ $a_i \in S$; $0 \leq i \leq 10$} be a super subset polynomial semivector space of M over S of type I.

Basis B of M over S is B = {{1}, {1}x, $\{1\}x^2$, …, $\{1\}x^{18}$}. Clearly M is a finite dimensional super subset semivector space over S.

Example 4.102: Let S = {Collection of subsets of Q} and M = $\left\{ \sum_{i=0}^{27} a_i x^i \right|$ $a_i \in S$, $0 \leq i \leq 27$} be the super subset polynomial semivector space over S. Clearly M is finite dimensional, but is not a super subset polynomial semilinear algebra over S.

Example 4.103: Let S = {Collection of all subsets of R} and M = $\left\{ \sum_{i=0}^{27} a_i x^i \right|$ $a_i \in S$, $0 \leq i \leq 27$} be the super subset polynomial semivector space over the subset semiring.

P = {Collection of all subsets of Q} $\subseteq$ S.

Clearly dimension of M over P is infinite.

Further M is not a super subset polynomial semilinear algebra over P. Finally we have super subset polynomial semivector spaces which are not super subset polynomial semilinear algebras.

Inview of this we have the following theorem.

THEOREM 4.5: *Let S = {Collection of all subsets over a semifield Q or R or ring or semiring or field}. M be a super special subset polynomial semivector space over S. Clearly M is not a super special subset polynomial semilinear algebra over S.*

Proof is direct and is left as an exercise to the reader.

Example 4.104: Let
S = {Collection of all subsets of the group ring $Z_{43}S_8$}.
M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ be the super subset polynomial semivector space over the subset semiring.

P = {Collection of all subsets of the field Z_{43}} $\subseteq$ S.

M is commutative as a super subset polynomial semivector space over P but M is a non commutative super subset polynomial semivector space over S. M is a doubly non commutative super subset polynomial semilinear algebra over S.

Example 4.105: Let

S = {Collection of all subsets of the field C}.

M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ is a super subset polynomial semivector space (as well as semilinear algebra) over T = {Collection of all subsets of Q or R} $\subseteq$ S.

M can also be a special super subset polynomial semilinear algebra over S.

Example 4.106: Let
S = {Collection of all subsets of the ring $C(Z_{24})$}.

$M = \left\{\sum_{i=0}^{9} a_i x^i \;\middle|\; a_i \in S;\ 0 \le i \le 9\right\}$ be the super subset polynomial semivector space over the subset semiring S.

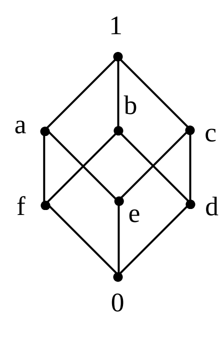
Example 4.107: Let S = {Collection of all subsets of the semiring $(Z^+ \cup \{0\})$ (g_1, g_2, g_3) where $g_1^2 = 0$, $g_2^2 = g_2$ and $g_3^2 = -g_3$, $g_ig_j = 0$, if $i \ne j$, $1 \le i, j \le 3$} be the subset semiring.

$M = \left\{\sum_{i=0}^{21} a_i x^i \;\middle|\; a_i \in S;\ 0 \le i \le 21\right\}$ is a super subset polynomial semivector space over S. Clearly M is of finite dimension over S. However M is commutative.

Example 4.108: Let
S = {Collection of all subsets of the semiring $(Q^+ \cup \{0\})$ S(5)}.
$M = \left\{\sum_{i=0}^{8} a_i x^i \;\middle|\; a_i \in S;\ 0 \le i \le 8\right\}$ be the super subset polynomial semivector space over S which is non commutative.

Infact M is not a super subset polynomial semilinear algebra over S.

Example 4.109: Let S = {Collection of all subsets of the semiring LS(12) where L =



be the subset semiring. $W = \left\{ \sum_{i=0}^{19} a_i x^i \,\middle|\, a_i \in LS(12); 0 \leq i \leq 19 \right\}$ be the super subset polynomial semivector space over the subset semiring S.

Clearly W is non commutative over S. W is not a super subset polynomial semivector space and is not a super subset polynomial semilinear algebra over S.

Example 4.110: Let
S = {Collection of all subsets of the semiring $(Q^+ \cup \{0\})S_9$} be the subset semiring. $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ is a doubly non commutative super subset polynomial semilinear algebra over the subset semiring S.

Example 4.111: Let
S = {Collection of all subsets of the ring $Z_8S(8)$}.
$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ be the super subset polynomial semilinear algebra over the subset semiring S.

M is doubly non commutative. Finding semilinear transformation and semilinear operators are a matter of routine. Let S = {collection of all subsets of the semifield $Z^+ \cup \{0\}$}.

$$M = \left\{ \sum_{i=0}^{8} a_i x^i \,\middle|\, a_i \in S, 0 \leq i \leq 8 \right\} \text{ and}$$

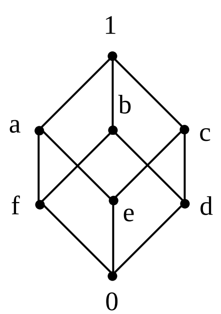
$N = \left\{ \sum_{i=0}^{16} a_i x^i \,\middle|\, a_i \in S; 0 \leq i \leq 16 \right\}$ be subset polynomial semivector spaces over the semifield $Z^+ \cup \{0\}$.

We can define $T : M \to N$ where T is a semilinear transformation of M to N.

Thus if we need to define semilinear transformation of subset polynomial semivector spaces M and N we must have M and N to be defined over the semiring or semifield or field or ring and in case of super subset polynomial semivector spaces over the same subset semiring S.

The definition and working is similar to usual semivector spaces.

Example 4.112: Let M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\right.$Collection of all subsets of the semiring L =



$\}$

be the subset polynomial semivector space over the semiring L of type I.

We can define T : M $\to$ M, semilinear operators on M.

We can also define semilinear functional T : M $\to$ L by

T(p(x)) = $\sum \sum_i g_i$ where p(x) = $\sum_{i=0}^{\infty} a_i x^i$, $g_i \in a_i \in S$ and $\sum g_i$ is the sum of all the elements in the set a_i.

We find the sum of the sums which is clearly in L.

Example 4.113: Let W = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the semiring } Z^+ \cup \{0\}\}\right\}$ be the subset polynomial semivector space over the semifield $Z^+ \cup \{0\}$.

Let p(x) = $\{8, 3, 4, 0, 11, 3, 5, 10, 9\}x^3 + \{1, 2, 3, 6\}x^2 + \{1, 2, 3, 4, 5, 6\} = a_1x^3 + a_2x^2 + a_3 \in W$.

Let $f : W \to Z^+ \cup \{0\}$.

$$f(p(x)) = \sum (8+3+4+0+11+3+5+10+9) + (1+2+3+6) + (1+2+3+4+5+6)$$

$$= \sum (53 + 12 + 21)$$
$$= 86 \in Z^+ \cup \{0\}.$$

Thus f is a subset semilinear functional of W.

Example 4.114: Let

M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the ring } Z_8\}\right\}$ be the super subset semivector space over the subset semiring S. We define $f : M \to S$ as follows:

If $p(x) = \{0, 1, 2\}\, x^3 + \{3, 4, 5\}\, x + \{3, 6\}$

then $f(p(x)) = (\{0, 1, 2\} + \{3, 4, 5\} + \{3, 6\})$
$= \{3, 4, 5, 6, 7\} + \{3, 6\}$
$= \{6, 7, 0, 1, 2, 3, 4, 5\}$
$= Z_8 \in S.$

This is the way super semilinear functionals are defined.

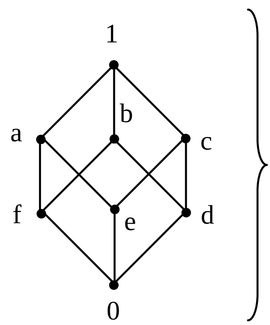
We suggest some problems for the reader.

Problems

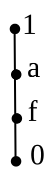
1. Obtain some important and interesting features enjoyed by subset matrix semivector spaces.

2. Distinguish the subset matrix semivector spaces from subset semivector spaces.

3. Let M = {(a_1, a_2, a_3, …, a_6) | a_i ∈ {Collection of all subsets from the semiring



$1 \leq i \leq 6$} be a subset matrix semivector space over the semifield L =

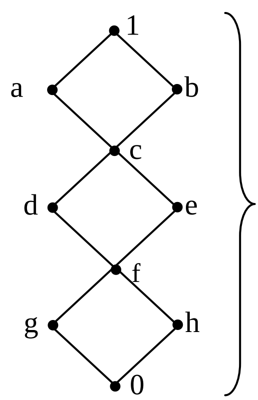


(i) Find the number of elements in M.
(ii) Find a semibasis for M.
(iii) Can M have subset matrix subsemivector subspaces?
(iv) Can M be a Smarandache subset matrix semivector space over the semifield L?
(v) How many semibasis can M have over L?

4. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle|\right.$ $a_i \in$ {Collection of all subsets of the semifield $Z^+ \cup \{0\}\}$ $1 \leq i \leq 4\}$ be the subset matrix semivector space over the semifield $Z^+ \cup \{0\}$.

   (i) Study question (ii) to (v) for this M.
   (ii) Prove M is non commutative under usual product $\times$ and commutative under the natural product $\times_n$.
   (iii) Can M have many sets of basis?

5. Let S = {Collection of all subsets of the semiring L =



be a subset semiring T = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle|\right.$ $a_i \in S; 1 \leq i \leq 12\}$

be the subset matrix semivector space over the semiring L of type I.

(i) Prove $o(T) < \infty$.
(ii) Find a subset semibasis of S.
(iii) Is T a S-subset matrix semivector space?
(iv) Find subset matrix semivector subspaces of T.
(v) How many basis can T have over L?

(vi) Can T be made into a subset matrix semilinear algebra over L?

(vii) If T is a subset matrix semilinear algebra over L will the subset dimension of T over L vary when T is just a subset matrix semivector space over L.

6. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right|$ $a_i \in S$ = {Collection of all subsets of the group semiring $P = (Z^+ \cup \{0\})(S_3)\}$, $1 \leq i \leq 9\}$ be the subset matrix semivector space of type I over the group semiring $(Z^+ \cup \{0\})S_3$.

(i) Show $o(M) = \infty$.
(ii) Find a subset basis of M over the semiring P.
(iii) Is it finite dimensional?
(iv) Can M have several basis over P?
(v) Obtain some interesting features enjoyed by M.
(vi) Can M be doubly non commutative?

7. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{bmatrix} \right|$ $a_i \in S$ = {Collection of all subsets from the group ring $Z_5S_3\}$, $1 \leq i \leq 20\}$ be a special subset matrix semivector space over the group ring Z_5S_3 of type II.

(i) Find o(M).

(ii) Can M have several basis over Z_5S_3?
(iii) Find a basis of M over Z_5S_3.
(iv) Find some subset matrix semivector subspaces of M over Z_5S_3.
(v) Can M be made into a subset matrix semilinear algebra over Z_5S_3?
(vi) Prove M is a non commutative subset matrix semivector space of type II over the ring Z_5S_3.

8. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \right.$ $a_i \in S$ = {Collection of all subsets from the semigroup ring R = $Z_7S(3)$}, $1 \leq i \leq 9$} be a subset matrix semivector space over the ring R of type II.

(i) Study questions (i) to (vi) given in problem 7.

9. Let M = {$(a_1, a_2, \ldots, a_{10})$ | $a_i \in$ {subsets of the semiring $(Z^+ \cup \{0\})$ (g)} $g^2 = 0$, $1 \leq i \leq 10$} be the subset matrix semivector space over the semifield $Z^+ \cup \{0\}$.

(i) Find a semibasis of M.
(ii) Can M have more than one basis?
(iii) Can M be a subset matrix semilinear algebra over $Z^+ \cup \{0\}$?
(iv) If the semifield $Z^+ \cup \{0\}$ is replaced by $(Z^+ \cup \{0\})$ (g) will M be a subset matrix semivector space of type I over $(Z^+ \cup \{0\})$ (g).

10. Let M be a subset matrix semivector space of type I defined over a semiring. Find some special features enjoyed by M.

11. Let M be a special subset matrix semivector space of type II over a ring R. What are the special features associated with M?

12. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_5 & a_6 \\ a_3 & a_4 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \right.$ $a_i \in S$ = {collection of all subsets of the ring Z_{15}}, $1 \leq i \leq 16$} be the special subset matrix semivector space over the ring Z_{15} of type II.

(i) Can M be commutative?

(ii) Prove M as a special subset matrix semilinear algebra is doubly non commutative?

(iii) Find a basis B of M as a special subset matrix semivector space of type I.

(iv) Find a basis B of M as a subset matrix semilinear algebra of type II.

(v) Compare the basis B_1 and B_2.

(vi) Is $o(B_1) > o(B)$ or $o(B) > o(B_1)$?

(vii) Does M have only one basis or several basis?

13. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{23} & a_{24} \end{bmatrix} \middle| \right.$ $a_i \in S$ = {Collection of all subsets of the groupring Z_7S_4}, $1 \leq i \leq 24$} be a special subset matrix semivector space of type II over the ring Z_7S_4.

    (i) Prove M is non commutative.

    (ii) Study questions (i) to (vii) of problem 12.

14. Let S = {Collection of all subsets of the ring $C(Z_{12})$}.

    $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \right.$ $a_i \in S$; $1 \leq i \leq 9$} is the special subset matrix semivector space of type II.

    Study questions (i) to (vii) of problem 13.

15. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix} \middle| \right.$ $a_i \in S$ = {Collection of all subsets of the field Q}} be the special strong subset matrix semivector space of type III.

    Study questions (i) to (vii) of problem 12 for this M.

16. Derive some interesting features enjoyed by strong special subset matrix semivector spaces of type III.

17. Distinguish between type II and type III subset semivector spaces?

18. Compare all the four subset semivector spaces.

19. Let M = $\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \middle| \right.$ $a_i \in S$ = {Collection of all subsets from $Z_{11}S_3$}; $1 \leq i \leq 12$} be the special subset matrix semivector space over the ring $Z_{11}S_3$.

    Study questions (i) to (viii) of problem 12.

20. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{20} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{40} \\ a_{41} & a_{42} & a_{43} & \cdots & a_{60} \\ a_{61} & a_{62} & a_{63} & \cdots & a_{80} \end{bmatrix} \middle| \right.$ $a_i \in S$ = {Collection of all subsets of the field Z_{11}}, $1 \leq i \leq 80$} be the special strong matrix semivector space over the field of type III over Z_{11}.

    (i) Find o(M).

    (ii) Find a basis of M over Z_{11}.

    (iii) Can M have more than one basis?

    (iv) Study questions (i) to (viii) of problem 12.

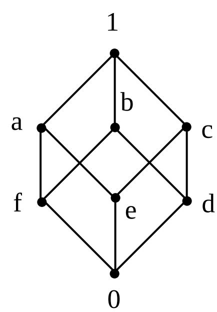
21. Let M = {Collection of all 5×3 matrices with entries from subsets of the semifield $Z^+ \cup \{0\}$} be the subset matrix semivector space over the semifield $Z^+ \cup \{0\}$.

    (i) Find a basis of M over $Z^+ \cup \{0\}$.

    (ii) Can M have more number of basis?

(iii) Is M finite or infinite dimension?

22. Let $M = \left\{ \begin{bmatrix} a_1 & a_{16} & a_{31} \\ a_2 & a_{17} & a_{32} \\ \vdots & \vdots & \vdots \\ a_{15} & a_{30} & a_{45} \end{bmatrix} \middle| \right.$ $a_i \in S$ = {Collection of all subsets over the semiring P =



}

$1 \leq i \leq 45$} be the subset matrix semivector space of type I over the semiring P.

(i) Find o(M).

(ii) Is $o(M) < \infty$?

(iii) Find a basis of M over P.

(iv) Can M have more than one basis?

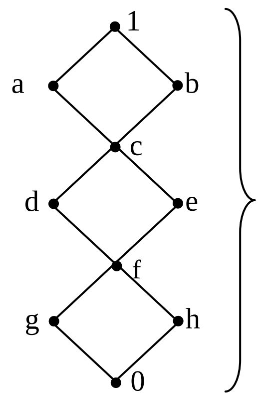
(v) Is M a commutative subset matrix semilinear algebra of type I over P?

(vi) Study M as a subset matrix semivector space of type I over

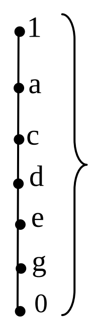
.

23. Let $M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{40} \end{bmatrix} \right|$ $a_i \in S$ = {Collection of all subsets of

the semiring T =



$1 \leq i \leq 40$} be a subset matrix semivector space over the semifield L =



(i) Find o(M).

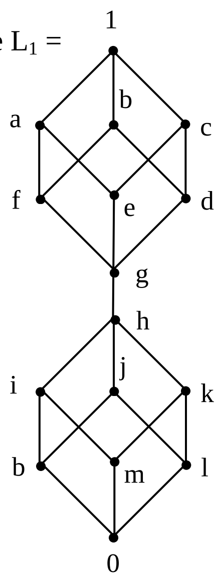
(ii) Find a basis of M over L.

(iii) If in M L is replaced by T is M_1 is a subset matrix semivector space of type I compare M and M_1.

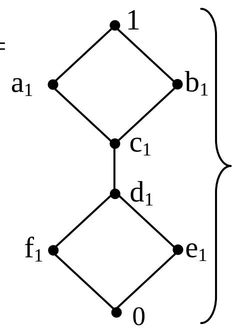
24. Let $B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in S = \{\text{Collection of all subsets} \right.$

of the semiring $L_1 \times L_2$

where $L_1 =$



and $L_2 =$



$1 \leq i \leq 9\}$ be a subset matrix semivector space of type I over L.

(i) Find o(B).

(ii) Find a basis of B over L.

(iii) Prove B is commutative.

(iv) How many basis can B have?

(v) Under usual operations of B can B be non commutative as a subset matrix semilinear algebra?

25. Let $T = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right|$ $a_i \in S$ = {Collection of all subsets of the semiring $C(Z_{10})$} $1 \leq i \leq 4$} be the subset matrix semilinear algebra over the semiring $C(Z_{10})$.

    (i) Find a basis of T over $C(Z_{10})$.

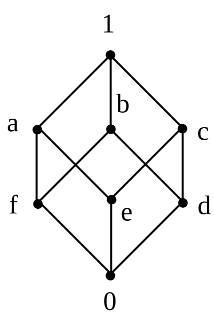
    (ii) How many basis can T have over $C(Z_{10})$?

    (iii) Find o(T).

    (iv) Show T is non commutative under usual product as a semilinear algebra.

    (v) Prove T under natural product $\times_n$ is a subset matrix semilinear algebra over $C(Z_{10})$.

26. Let $N = \left\{ \begin{bmatrix} a_1 & a_2 & ... & a_8 \\ a_9 & a_{10} & ... & a_{16} \end{bmatrix} \right|$ $a_i \in S$ = {Collection of all subsets of the semiring LS_5 where L =



}

$1 \leq i \leq 16\}$ be a super subset matrix semivector space over the semiring S.

(i) Find o(N).

(ii) Find a basis of N over S.

(iii) How many basis of N over S exist?

(iv) If N is not a super subset matrix semivector space over the semiring L. Compare both spaces.

27. Let M be a super subset semivector space of type I over a semiring.
Find some special properties enjoyed by the super subset semivector space of type I.

28. Let T = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \end{bmatrix} \middle| \right.$ $a_i \in S$ = {Collection of all subsets of the ring $Z_{14}(g)$, $g^2 = 0\}$, $1 \leq i \leq 16\}$ be the special super subset matrix semivector space of type II over the semiring S.

(i) Find a basis of T over S.

(ii) Find o(T).

(iii) How many basis can T have over S?

(iv) Prove T is commutative.

(v) Is W = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| \right.$ $a_i \in S$ = {Collection of all subsets of the ring $Z_{14}(g)$; $g^2 = 0\}$, $1 \leq i \leq 8\} \subseteq T$ a special super subset semivector subspace of T over S?

29. Let W = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}\right|$ $a_i \in S$ = {Collection of all subsets of the ring Z_5S_7}; $1 \leq i \leq 8$} be a special super subset semilinear algebra of type II over the semiring S.

(i) Prove W is doubly non commutative.

(ii) Find o(W).

(iii) Find a basis of W over S.

(iv) How many basis of W over S exist?

30. Let P = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \end{bmatrix}\right|$ $a_i \in S$ = {Collection of all subsets of the field Z_{19}}; $1 \leq i \leq 18$} be the special strong super subset matrix semivector of type III over the semiring S.

(i) Find o(P).

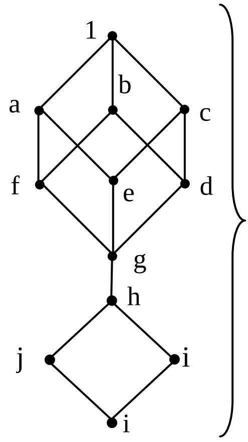
(ii) Find a basis of P over S.

(iii) How many basis P has over S?

(iv) Is P commutative as a special strong super subset matrix semilinear algebra?

31. Let V = $\left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix}\right| a_i \in S = \{\text{Collection of all}\right.$

subsets of the semiring



$1 \leq i \leq 18\}$ be the super subset matrix semivector space over S.

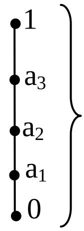
(i) Find o(V).

(ii) Find a basis of S over V.

(iii) Can S have more number of basis?

32. Let M = $\left\{\left.\sum_{i=0}^{\infty} a_i x^i \right| a_i \in S = \{\text{Collection of all subsets of}\right.$

semifield L =



be a subset matrix semivector space over L.

(i) Find a basis of M over L as a semivector space.

(ii) Find a basis of M over L as a semilinear algebra over L.

33. Let M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\right.$Collection of all subsets of the semifield $Z^+ \cup \{0\}\}$; $0 \leq i \leq 9\}$ be the subset matrix semivector space over the semifield $Z^+ \cup \{0\}$.

(i) Find a basis of M.

(ii) Prove M is not a subset matrix semilinear algebra over M.

(iii) Can M have more than one basis?

(iv) Find subspaces of M.

34. Study any special / interesting feature of a subset polynomial semivector space over a semifield.

35. Let V = $\left\{\sum_{i=0}^{8} a_i x^i \,\middle|\, a_i \in S = \{\right.$Collection of all subsets of the ring $Z_{11}S_8\}$, $0 \leq i \leq 8\}$ be the special subset semivector space over the ring $Z_{11}S_8$ of type II.

(i) Is $o(V) < \infty$?

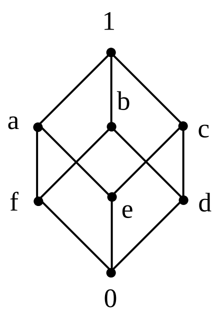
(ii) Find a basis of V over $Z_{11}S_8$.

(iii) How many basis of V over $S_{11}S_8$ exist?

(iv) Can V be a commutative special subset semivector space over $Z_{11}S_8$ of type II?

36. Let M = $\left\{ \sum_{i=0}^{8} a_i x^i \;\middle|\; a_i \in S = \{\right.$Collection of all subsets of the

semiring P =



$0 \le i \le 8\}$ be the subset polynomial semivector space over the semiring of type I over P.

(i) Find a basis of M over P.

(ii) Prove M is not a subset polynomial semilinear algebra over P.

(iii) Can M have more than a basis?

37. Let W = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \;\middle|\; a_i \in S = \{\right.$Collection of all subsets

of the ring Z_{12}}; $1 \leq i \leq 6$} be the special subset matrix

semivector space over Z_{12} of type I. $M = \left\{\left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{array}\right]\right|$

$a_i \in S$ = {Collection of all subset of the ring Z_{12}} be the special subset matrix semivector space over Z_{12} of type I.

(i) Define semi linear transformation of W to M.

(ii) Define semilinear operators $\eta_1 : W \to W$ and $\eta_2 : M \to M$.

(iii) Find semilinear functional of M and W; $f_1 : M \to Z_{12}$ and $f_2 : W \to Z_{12}$.

(iv) If R = {Collection of all semilinear transformation of W to M}, what is the algebraic structure enjoyed by R?

(v) If B = {Collection of all semilinear transformations from M to W}; what is the algebraic structure enjoyed by B?

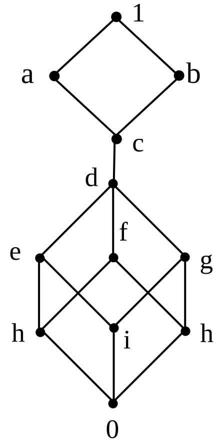
(vi) Let T_1 = {Collection of all semilinear operators of M to M}; find the algebraic structure enjoyed by T_1.

(vii) Let T_2 = {Collection of all semilinear operators from W to W}; find the algebraic structure enjoyed by T_2.

(viii) Let F_1 = {Collection of all semilinear functional from $M \to Z_{12}$} find the algebraic structure enjoyed by F_1.

(ix) Let F_2 = {Collection of all semilinear functionals from W to Z_{12}} find the algebraic structure enjoyed by F_2.

38. Let S = {Collection of all subsets of the lattice L =



}

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{bmatrix} \middle| \; a_i \in S;\ 1 \leq i \leq 20 \right\} \text{ and}$$

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \end{bmatrix} \middle| \; a_i \in S;\ 1 \leq i \leq 20 \right\} \text{ be two}$$

subset matrix semilinear algebras over the semiring L of type I.
Study questions (i) to (ix) of problem 37.

39. Let
S = {Collection of all subsets of the semifield $Q^+ \cup \{0\}$}, M = {Collection of all 5 × 5 matrices with entries from S} and W = {Collection of all 7 × 7 matrices with entries from S} be two subset matrix semivector space over the semifield $Q^+ \cup \{0\}$.

Study questions (i) to (ix) of problem 37 for this M and W.

40. Let S = {Collection of all subsets of the ring Z_{45} S(10)} M = {Collection of all 3 × 7 matrices with entries from S} and W = {Collection of all 8 × 3 matrices with entries from S} be two special subset matrix semivector space of type II over the ring Z_{45} S(10).

    Study questions (i) to (ix) of problem (37) for this M and W.
    Prove both M and W are non commutative.

41. Let S = {Collection of all subsets from the field C}. M = {Collection of all 7 × 2 matrices with entries from S} and W = {Collection of all 3 × 4 matrices with entries from S} be two strong special subset matrix semivector spaces of type III over the complex field C.

    Study questions (i) to (ix) of problem 37 in case of this M and N.

42. Let S = {Collection of all subsets from the semigroup ring C(S(12))}.

    M = {Collection of all 3 × 8 matrices with entries from S} and
    W = {Collection of all 6 × 4 matrices with entries from S} be special subset matrix semivector spaces over the ring C(S(12)) of type II.

    Study questions (i) to (ix) of problem 37 for this M and W.

    (i) Prove M and W are non commutative as special subset matrix semivector spaces of type II and are doubly non commutative as special subset matrix semilinear algebra of type II.

43. Let
S = {Collection of all subsets of the semifield $R^+ \cup \{0\}$}.

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S \right\}$$ be the subset polynomial

semivector space over the semifield $R^+ \cup \{0\}$.

(i) Find a basis of M over $R^+ \cup \{0\}$.

(ii) If $R^+ \cup \{0\}$ is replaced by $Q^+ \cup \{0\}$ find a basis of M over $Q^+ \cup \{0\}$.

(iii) If $R^+ \cup \{0\}$ is replaced by $Z^+ \cup \{0\}$ find a basis of M over $Z^+ \cup \{0\}$.

(iv) Is the basis unique or can M have many distinct basis?

(v) Can M be made into a subset polynomial semilinear algebra over $R^+ \cup \{0\}$ (or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$)?

(vi) Find subset polynomial subsemivector spaces of M.

(vii) Find T : {Collection of all $M \to M$}; what is the algebraic structure enjoyed by T?

44. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S = \{\text{Collection of all subsets of the}\right.$

semigroup semiring $P = (Z^+ \cup \{0\})\ (S(5))\}\}$ be the subset polynomial semivector space of type I over P.

Study questions (i) to (vii) from problem (43) for this M.

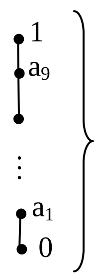
45. Let $M_1 = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S = \{\text{Collection of all subsets of} \right.$ the ring $R = Z_{42}\ D_{29}\}$ be the special subset polynomial semivector space over the ring R of type II. Study questions (i) to (vii) from problem (43) for this M_1.

46. Let $M_2 = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S = \text{Collection of all subsets of the} \right.$ field C}} be special strong subset polynomial semivector space over the field C of type III. Study questions (i) to (vii) from problem 43 for M_2.

47. Let $M_3 = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S = \{\text{Collection of all subsets of} \right.$ the semifield $R^+ \cup \{0\}\}\}$ be the subset polynomial semivector space over the semifield $R^+ \cup \{0\}$.

Study questions (i) to (vii) from problem 43 for M_3.

48. Let $M_4 = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S = \{\text{Collection of all subsets of} \right.$ the semifield L =

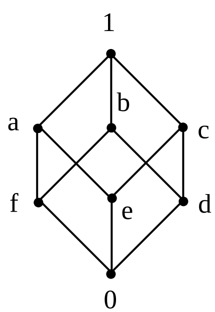


be the subset polynomial semivector space over the semifield L.

(i) Study questions (i) to (vii) of problem 43.

(ii) Find the set ideal subset polynomial topological vector subspace of M over any set.

49. Let $M_5 = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of}\right.$ the semiring LS_7 where L =



}

be the subset polynomial subset semivector space over the semiring LS_7 of type I.

Study questions (i) to (vii) of problem (43) study question (ii) problem 48 for this M_5.

50. Let $M_6 = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of}\right.$ the ring Z_{12} $S_7\}\}$ be the special subset polynomial semivector space of type II over the ring R.

Study questions (iv) to (vii) in problem 43 for M_6.

51. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = \{\text{Collection of all subsets of the}\right.$ field $Z_{43}\}\}$ be the strong special subset polynomial semivector space of type III over the field Z_{43}.

Study questions (i) to (vii) in problem 43 for this M.

52. Let
S = {Collection of all subsets from the semifield $Z^+ \cup \{0\}$}
be the subset semiring $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right\}$ be the super subset polynomial semivector space over the subset semiring over S.

(i) Find a basis of M over S.

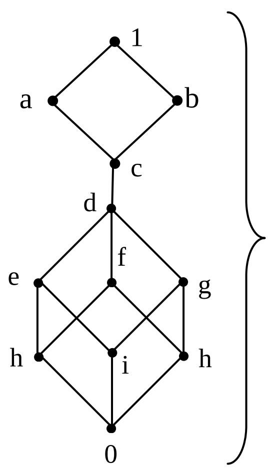
(ii) How many basis can M have over S?

(iii) Is M a super subset polynomial semilinear algebra over S?

(iv) Let T = {Collection of all super semilinear operators on M}. Study the algebraic structure enjoyed by T.

(v) Find super subset polynomial semivector subspaces of M over S.

53. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S \right.$ = {Collection of all subsets of the semiring



be the super subset polynomial semivector space over the subset semiring S of type I.

Study questions (i) to (v) of problem 52 for this M.

54. Let M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets of the ring R = Z_{45} S_7}} be the special super subset polynomial semivector space over the subset semiring S.

(i) Study questions (i) to (v) of problem 52 for this M.

(ii) If F = {collection of all super semilinear functional of M to S} find the algebraic structure enjoyed by F.

55. Let M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in S$ = {Collection of all subsets of the field C} be the strong special super subset polynomial semivector space over the subset semiring S of type III.

(i) Study question (i) to (v) of problem 52 for this M.

(ii) Let F = {Collection of all super semilinear functions of M to S}; study the algebraic structure enjoyed by F.

56. Let M = $\left\{ \sum_{i=0}^{7} a_i x^i \right| a_i \in S$ = {Collection of all subsets of the semifield $R^+ \cup \{0\}$}} be the super subset semivector space over the subset semiring S.

(i) Find a basis of M over S.

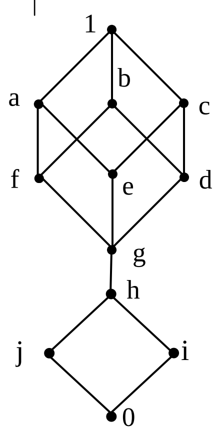
(ii) If T = {Collection of all super subset semilinear operators on M}, find the algebraic structure enjoyed by T.

(iii) Let F = {Collection of all maps from M $\rightarrow$ S} find the algebraic structure enjoyed by F.

(iv) Prove M is not a super subset polynomial semilinear algebra over S.

(v) Can M have more than one basis over S?

57. Let M = $\left\{ \sum_{i=0}^{6} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets of the semiring



}} be the super subset polynomial semivector space over the subset semiring S of type I.

Study questions (i) to (v) of problem 56 for this M.

58. Let M = $\left\{ \sum_{i=0}^{4} a_i x^i \,\middle|\, a_i \in S = \right.$ {Collection of all subsets of the ring Z_{12} S_8}} be the super special subset polynomial semivector space over the subset semiring of type II.

(i) Prove $o(M) < \infty$.

(ii) Study questions (i) to (iv) of problem 56 for this M.

59. Let $M = \left\{ \sum_{i=0}^{9} a_i x^i \;\middle|\; a_i \in S = \right.$ {Collection of all subsets of the field Z_{23}}, $0 \leq i \leq 9\}$ be the special strong super subset semivector space over the subset semiring S of type III.

    (i) Study questions (i) to (iv) of problem 56 for this M.

    (ii) Prove $o(M) < \infty$.

60. Find some interesting features enjoyed by super subset semivector spaces over a subset semiring.

61. Find for any subset polynomial semivector space M the subset topological polynomial semivector subspace T of M over a semifield F. T = {Collection of all subset polynomial semivector subspaces of M over a semifield} be the subset topological polynomial semivector subspace. Study T for all types of subset of subset polynomial semivector subspaces.

62. Let W = {Collection of m × n matrices with entries from the subsets of the semifield $R^+ \cup \{0\}$} be the subset semivector space over the semifield $R^+ \cup \{0\}$. Find the subset matrix topological semivector subspace of W over $R^+ \cup \{0\}$.

63. Study problem 62 in case of all the three types of subset matrix semivector spaces.

64. Study problem (62) if W is a super subset semivector subspace over S.

65. Define and describe set subset matrix (polynomial) topological semivector subspace of M over a set.

66. Study question 62 in case of super subset matrix (polynomial) semivector space over a subset semiring.

FURTHER READING

1. Baum J.D, *Elements of point set topology*, Prentice-Hall, N.J, 1964.

2. Birkhoff. G, *Lattice theory*, 2nd Edition, Amer-Math Soc. Providence RI 1948.

3. Simmons, G.F., *Introduction to topology and Modern Analysis*, McGraw-Hill Book 60 N.Y. 1963.

4. Smarandache. F. (editor), *Proceedings of the First International Conference on Neutrosophy*, Neutrosophic Probability and Statistics, Univ. of New Mexico-Gallup, 2001.

5. Sze-Tsen Hu, *Elements of general topology*, Vakils, Feffer and Simons Pvt. Ltd., Bombay, 1970.

6. Vasantha Kandasamy, W.B., *Smarandache Semigroups*, American Research Press, Rehoboth, 2002.

7. Vasantha Kandasamy, W.B., *Smarandache Rings*, American Research Press, Rehoboth, 2002.

8. Vasantha Kandasamy, W.B., *Smarandache Semirings, Semifields and Semivector spaces*, American Research Press, Rehoboth, 2002.

9. Vasantha Kandasamy, W.B., *Linear Algebra and Smarandache Linear Algebra*, Bookman Publishing, US, 2003.

10. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Neutrosophic Rings*, Hexis, Arizona, 2006.

11. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Set Linear Algebra and Set Fuzzy Linear Algebra*, InfoLearnQuest, Phoenix, 2008.

12. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Finite Neutrosophic Complex Numbers*, Zip Publishing, Ohio, 2011.

13. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Dual Numbers*, Zip Publishing, Ohio, 2012.

14. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Special dual like numbers and lattices*, Zip Publishing, Ohio, 2012.

15. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Special quasi dual numbers and groupoids*, Zip Publishing, Ohio, 2012.

16. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Natural Product $\times_n$ on matrices*, Zip Publishing, Ohio, 2012.

17. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Set Ideal Topological Spaces*, Zip Publishing, Ohio, 2012.

18. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Quasi Set Topological Vector Subspaces*, Educational Publisher Inc, Ohio, 2012.

19. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Algebraic Structures using Subsets*, Educational Publisher Inc, Ohio, 2013.

20. Voyevodin, V.V., *Linear Algebra*, Mir Publishers, 1983.

INDEX

N

Q

R

S

Z

ABOUT THE AUTHORS

Dr.W.B.Vasantha Kandasamy is an Associate Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 646 research papers. She has guided over 100 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 82nd book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

She can be contacted at vasanthakandasamy@gmail.com
Web Site: http://mat.iitm.ac.in/home/wbv/public_html/
or http://www.vasantha.in

Dr. Florentin Smarandache is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2012 New Mexico-Arizona and 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at smarand@unm.edu

In this book the authors introduce a new notion of subset matrix semirings and subset polynomials semirings. Solving subset polynomial equations is an interesting exercise. Open problems about the solution set of subset polynomials are proposed.

ISBN 9781599732237

90000 >

9 781599 732237

US $35.00

Educational Publisher Inc.