On the real representations of the Poincare group

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Abstract

We do a study of the real representations of the Poincare group, motivated by the following: i) the classical electromagnetic field —from which the Poincare group was originally defined— transforms as a real representation of the Poincare group; ii) the localization of complex unitary representations of the Poincare group is incompatible with causality, Poincare covariance and energy positivity, while the complex representation corresponding to the photon is not localizable.

We start by reviewing the map from the complex to the real irreducible representations—finite-dimensional or unitary—of a Lie group on a Hilbert space.

Then we show that all the finite-dimensional real representations of the identity component of the Lorentz group are also representations of the full Lorentz group, in contrast with many complex representations.

We finally study the unitary irreducible representations of the Poincare group with discrete spin or helicity and show that: for each pair of complex representations with positive/negative energy, there is one real representation; the localization, compatible with causality and Poincare covariance, exists for representations with discrete spin or helicity.

1. Introduction

1.1. Motivation

Henri Poincaré defined the Poincare group as the set of transformations that leave invariant the Maxwell equations for the classical electromagnetic field. The classical electromagnetic field transforms as a real representation of the Poincare group.

The complex representations of the Poincare group were systematically studied[1–7] and used in the definition of quantum fields[8]. These studies were very important in the evolution of the role of symmetry in the Quantum Theory[9], which is based on complex Hilbert spaces[10–12].

We could not find in the literature a systematic study on the real representations of the Poincare group — even though representation theory[13, 14] and Quantum Theory [15–20] on real Hilbert spaces were investigated before — as it seems to be a common assumption that all fields of all modern theories must be quantum fields and therefore, somehow, every consistent representation must be complex. However, due to the existence of a map between real and complex representations, the motivation for this study is independent of the validity of such assumption.

The reasons motivating this study are:

1) The real representations of the Poincare group play a main role in the classical electromagnetism and general relativity[21]. It is reasonable to think that the real representations of the Poincare group will still play an important role in the most modern theories based on the classical electromagnetism and general relativity. As an example, the self-adjoint quantum fields — such as the Higgs boson, Majorana fermion or quantum electromagnetic field — transform as real representations under the action of the Poincare group.

2) The parity — included in the full Poincare group — and charge-parity transformations are not symmetries of the Electroweak interactions[22]. It is not clear why the charge-parity is an apparent symmetry of the Strong interactions[23] or how to explain the matter-antimatter asymmetry[24] through the charge-parity violation. We will show that that all the finite-dimensional real representations of the restricted Lorentz group are also representations of the parity; also that there are linear and angular momenta spaces for the real representations of the Poincare group, therefore independent of the charge and matter-antimatter properties. These results may be useful in future studies of the parity and charge-parity violations.

3) The localization of complex unitary representations of the Poincare group is incompatible with causality, Poincare covariance and energy positivity[25–27], while the complex representation corresponding to the photon is not localizable[3, 28, 29]. We will show that the localization of the real irreducible unitary representations of the Poincare group, compatible with causality and Poincare covariance, exists for representations with discrete spin or helicity. These results may clarify that the localization problems in complex representations come from the representation of the charge and matter-antimatter properties in relativistic Quantum Mechanics—which has always been problematic, remember the Dirac sea[30].

1.2. On the map from the complex to the real irreducible representations of a group

Many representations of a group—such as the finite-dimensional representations of semisimple Lie groups[31] or the unitary representations of separable locally compact groups[32]—are direct sums (or integrals) of irreducible representations.

The study of irreducible representations on complex Hilbert spaces is in general easier than on real Hilbert spaces, because the field of complex numbers is the algebraic closure — where any polynomial equation has a root — of the field of real numbers. Given a real Hilbert space, we can always obtain a complex Hilbert space through complexification — extension of the scalar multiplication to include multiplication by complex numbers.

Yet, given an irreducible representation on a real Hilbert space V, the representation on the complex Hilbert space resulting from the complexification of V may be reducible, because there is a 2-dimensional real representation of the field of complex numbers. Therefore, the complex irreducible representations do not generalize the real irreducible representations in the same way that the complex numbers generalize the real numbers.

There is a well studied map, one-to-one or two-to-one and surjective up to equivalence, from the complex to the real linear finite-dimensional irreducible representations of a real

Lie algebra[13, 33]. In Section 2, we review a similar map from the complex to the real irreducible representations—finite-dimensional or unitary—of a Lie group on a Hilbert space. This section follows closely[13], with the addition that we will also use the Schur's lemma for unitary representations on a complex Hilbert space[34].

Related studies can be found in the references [14–17, 35].

1.3. Finite-dimensional representations of the Lorentz group

The Poincare group, also called inhomogeneous Lorentz group, is the semi-direct product of the translations and Lorentz Lie groups[31]. Whether or not the Lorentz and Poincare groups include the parity and time reversal transformations depends on the context and authors. To be clear, we use the prefixes full/restricted when including/excluding parity and time reversal transformations. The Pin(3,1)/SL(2,C) groups are double covers of the full/restricted Lorentz group. The semi-direct product of the translations with the Pin(3,1)/SL(2,C) groups is called IPin(3,1)/ISL(2,C) Lie group — the letter (I) stands for inhomogeneous.

A projective representation of the Poincare group on a complex/real Hilbert space is an homomorphism, defined up to a complex phase/sign, from the group to the automorphisms of the Hilbert space. Since the IPin(3,1) group is a double cover of the full Poincare group, their projective representations are the same[36]. All finite-dimensional projective representations of a simply connected group, such as SL(2,C), are usual representations[6]. Both SL(2,C) and Pin(3,1) are semi-simple Lie groups, and so all its finite-dimensional representations are direct sums of irreducible representations[31]. Therefore, the study of the finite-dimensional projective representations of the restricted Lorentz group reduces to the study of the finite-dimensional irreducible representations of SL(2,C).

The Dirac spinor is an element of a 4 dimensional complex vector space, while the Majorana spinor is an element of a 4 dimensional real vector space[37–40]. The complex finite-dimensional irreducible representations of SL(2,C) can be written as linear combinations of tensor products of Dirac spinors.

In Section 3 we will review the Pin(3,1) and SL(2,C) semi-simple Lie groups and its relation with the Majorana, Dirac and Pauli matrices. We will obtain all the real finite-dimensional irreducible representations of SL(2,C) as linear combinations of tensor products of Majorana spinors, using the map from Section 2. Then we will check that all these real representations are also projective representations of the full Lorentz group, in contrast with the complex representations which are not all projective representations of the full Lorentz group.

1.4. Unitary representations of the Poincare group

According to Wigner's theorem, the most general transformations, leaving invariant the modulus of the internal product of a Hilbert space, are: unitary or anti-unitary operators, defined up to a complex phase, for a complex Hilbert; unitary, defined up to a signal, for a real Hilbert[3, 41]. This motivates the study of the (anti-)unitary projective representations of the full Poincare group.

All (anti-)unitary projective representations of ISL(2,C) are, up to isomorphisms, well defined unitary representations, because ISL(2,C) is simply connected[6]. Both ISL(2,C) and IPin(3,1) are separable locally compact groups and so all its (anti-)unitary projective representations are direct integrals of irreducible representations[32]. Therefore, the study of the (anti-)unitary projective representations of the restricted Poincare group reduces to the study of the unitary irreducible representations of ISL(2,C).

The spinor fields, space-time dependent spinors, are solutions of the free Dirac equation[42]. The real/complex Bargmann-Wigner fields[43, 44], space-time dependent linear combinations of tensor products of Majorana/Dirac spinors, are solutions of the free Dirac equation in each tensor index. The complex unitary irreducible projective representations of the Poincare group with discrete spin or helicity can be written as complex Bargmann-Wigner fields.

In Section 4, we will obtain all the real unitary irreducible projective representations of the Poincare group, with discrete spin or helicity, as real Bargmann-Wigner fields, using the map from Section 2. For each pair of complex representations with positive/negative energy, there is one real representation. We will define the Majorana-Fourier and Majorana-Hankel unitary transforms of the real Bargmann-Wigner fields, relating the coordinate space with the linear and angular momenta spaces. We will show that the localization of the real irreducible unitary representations of the full Poincare group, compatible with causality and Poincare covariance, exists for representations with discrete spin or helicity; the localization of the complex irreducible (anti-)unitary representations of the full Poincare group, compatible with causality and Poincare covariance, exists for massless representations with discrete helicity.

The free Dirac equation is diagonal in the Newton-Wigner representation[28], related to the Dirac representation through a Foldy-Wouthuysen transformation[45, 46] of Dirac spinor fields. The Majorana-Fourier transform, when applied on Dirac spinor fields, is related with the Newton-Wigner representation and the Foldy-Wouthuysen transformation. In the context of Clifford Algebras, there are studies on the geometric square roots of -1 [19, 47] and on the generalizations of the Fourier transform[48], with applications to image processing[49].

2. On the map from the complex to the real irreducible representations of a group

2.1. Representations on real and complex Hilbert spaces

Definition 2.1. A representation (M_G, V) of a Lie group G[50] on a real or complex Hilbert space V is defined by:

1) the representation space V, which is an Hilbert space;

2) the representation group homomorphism $M : G \to B(V)$ from the group elements to the bounded automorphisms with a bounded inverse, such that the map $M' : G \times V \to V$ defined by $M'(g, v) \equiv M(g)v$ is continuous.

Definition 2.2. Let V_n, with $n \in \{1, 2\}$, be two Hilbert spaces. The representations $(M_{n,G}, V_n)$ of a group G on the Hilbert spaces V_n are equivalent iff there is a linear bijection $\alpha : V_1 \to V_2$ such that for all $g \in G$, $\alpha \circ M_{1,G}(g) = M_{2,G}(g) \circ \alpha$.

Definition 2.3. Consider a representation (M_G, V). An endomorphism of (M_G, V) is an endomorphism $S : V \to V$ commuting with $M_G(g)$, for all $g \in G$.

Definition 2.4. Consider a representation (M_G, V). An isomorphism of (M_G, V) is a bijective operator $S : V \to V$ commuting with $M_G(g)$, for all $g \in G$.

Definition 2.5. Let W be a linear subspace of V. (M_G, W) is a (topological) subrepresentation of (M_G, V) iff W is closed and invariant under the group action, that is, for all $w \in W$: $(M(g)w) \in W$, for all $g \in G$.

Definition 2.6. A representation (M_G, V) is (topologically) irreducible iff their only sub-representations are the non-proper or trivial sub-representations: (M_G, V) and $(M_G, \{0\})$, where $\{0\}$ is the null space. An irreducible representation is called irrep.

Definition 2.7. Consider a representation (M_G, V) on a complex Hilbert space. A C-conjugation operator of (M_G, V) is an involution of V commuting with $M_G(g)$, for all $g \in G$.

Definition 2.8. Consider a representation (M_G, W) on a real Hilbert space. A R-imaginary operator of (M_G, W), J, is an isomorphism of (M_G, W) verifying $J^2 = -1$.

Definition 2.9. Consider an irreducible representation (M_G, V) on a complex Hilbert space.

The representation is C-real iff there is a C-conjugation operator. The subset of C-real irreducible representations is $R_G(\mathbb{C})$.

The representation is C-pseudoreal iff there is no C-conjugation operator but there is an anti-isomorphism of (M_G, V). The subset of C-pseudoreal irreducible representations is $P_G(\mathbb{C})$.

The representation is C-complex iff there is there is no anti-isomorphism of (M_G, V). The subset of C-complex irreducible representations is $C_G(\mathbb{C})$.

Definition 2.10. Consider a representation (M_G, W) on a real Hilbert space. The representation (M_G, W^c) is the complexification of the representation (M_G, W), defined as $W^c \equiv \mathbb{C} \otimes W$, with the multiplication by scalars such that $a(bw) \equiv (ab)w$ for $a, b \in \mathbb{C}$ and $w \in W$. The internal product of W^c is defined as:

$$< v_r + iv_i, u_r + iu_i >_c \equiv < v_r, u_r > + < v_i, u_i > +i < v_r, u_i > -i < v_i, u_r >$$

for $u_r, u_i, v_r, v_i \in W$ and $< v_r, u_r >$ is the internal product of W.

Definition 2.11. Consider a representation (M_G, V) on a complex Hilbert space. The representation (M_G, V^r) is the realification of the representation (M_G, V), defined as $V^r \equiv V$ is a real Hilbert space with the multiplication by scalars restricted to reals such that $a(v) \equiv (a + i0)v$ for $a \in \mathbb{R}$ and $v \in V$. The internal product of V^r is defined as

$$< v, u >_r \equiv \frac{< v, u > + < u, v >}{2}$$

for $u, v \in V$ and $< v, u >$ is the internal product of V.

2.2. The map from the complex to the real representations

Definition 2.12. Consider the representation (M_G, W) on a real Hilbert space and let (M_G, W^c) be its complexification.

(M_G, W) is R-real iff (M_G, W^c) is C-real irreducible. The set of R-real irreducible representations is $R_G(\mathbb{R})$.

(M_G, W) is R-pseudoreal iff (M_G, V) is C-pseudoreal irreducible, with $W^c = V \oplus \bar{V}$. The set of R-pseudoreal irreducible representations is $P_G(\mathbb{R})$.

(M_G, W) is R-complex iff (M_G, V) is C-complex irreducible, with $W^c = V \oplus \bar{V}$. The set of R-complex irreducible representations is $C_G(\mathbb{R})$.

Proposition 2.13. *Any irreducible real representation is R-real or R-pseudoreal or R-complex.*

Proof. Consider an irreducible representation (M_G, W) on a real Hilbert space. There is a C-conjugation operator of (M_G, W^c), θ, defined by $\theta(u + iv) \equiv (u - iv)$ for $u, v \in W$, verifying $(W^c)_\theta = W$.

Let (M_G, X^c) be a proper non-trivial subrepresentation of (M_G, W^c). Then θ is a C-conjugation operator of the subrepresentations (M_G, Y^c) and (M_G, Z^c), where $Y^c \equiv \{u+\theta v : u, v \in X^c\}$ and $Z^c \equiv \{u : u, \theta u \in X^c\}$. Therefore, $Y^c = \{u + iv : u, v \in Y\}$ and $Z^c = \{u + iv : u, v \in Z\}$, where $Y \equiv \{\frac{1+\theta}{2} u : u \in Y^c\}$ and $Z \equiv \{\frac{1+\theta}{2} u : u \in Z^c\}$, are invariant closed subspaces of W. If $Y = \{0\}$ then $Z = \{0\}$ and $Y^c = X^c = \{0\}$, in contradiction with X^c being non-trivial. If $Z = W$ then $Y = W$ and $Z^c = X^c = W^c$, in contradiction with X^c being proper. Therefore $Z = \{0\}$ and $Y = W$, which implies $Z^c = \{0\}$ and $Y^c = W^c$.

So, (M_G, W) is equivalent to $(M_G, (X^c)^r)$, due to the existence of the bijective linear map $\alpha : (X^c)^r \to W$, $\alpha(u) = u + \theta u$, $\alpha^{-1}(u + \theta u) = u$, for $u \in (X^c)^r$. Suppose that there is a C-conjugation operator of (M_G, X^c), θ'. Then $(M_G, W_\pm)$ is a proper non-trivial subrepresentation of (M_G, W), where $W_\pm \equiv \{\frac{1\pm\theta'}{2} w : w \in W\}$, in contradiction with (M_G, W) being irreducible. $\square$

Proposition 2.14. *Any real representation which is R-real or R-pseudoreal or R-complex is irreducible.*

Proof. Consider an irreducible representation on a complex Hilbert space (M_G, V). There is a R-imaginary operator J of the representation (M_G, V^r), defined by $J(u) \equiv iu$, for $u \in V^r$.

Let (M_G, X^r) be a proper non-trivial subrepresentation of (M_G, V^r). Then J is an R-imaginary operator of (M_G, Y^r) and (M_G^r, Z^r), where $Y^r \equiv \{u + Jv : u, v \in X^r\}$ and $Z^r \equiv \{u : u, Ju \in X^r\}$. Then (M_G, Y) and (M_G, Z) are subrepresentations of (M_G, V), where the complex Hilbert spaces $Y \equiv Y^r$ and $Z \equiv Z^r$ have the scalar multiplication such that $(a+ib)(y) = ay+bJy$, for $a, b \in \mathbb{R}$ and $y \in Y$ or $y \in Z$. If $Y = \{0\}$, then $Z = X^r = \{0\}$ which is in contradiction with X^r being non-trivial. If $Z = V$, then $Y = V$ and $X^r = V^r$ which is in contradiction with X^r being non-trivial. So $Z = \{0\}$ and $Y = V$, which implies that $V = (X^r)^c$.

Then there is a C-conjugation operator of (M_G, V), θ, defined by $\theta(u + iv) \equiv u - iv$, for $u, v \in X^r$. We have $X^r = V_\theta$. Suppose there is a R-imaginary operator of (M_G, V_θ), J'.

Then $(M_G, V_\pm)$, where $V_\pm \equiv \{\frac{1 \pm iJ'}{2} v : v \in V\}$, are proper non-trivial subrepresentations of (M_G, V), in contradiction with (M_G, V) being irreducible.

Therefore, if (M_G, V) is C-real, then (M_G, V_θ) is R-real irreducible. If (M_G, V) is C-pseudoreal or C-complex, then (M_G, V_θ^r) is R-pseudoreal or R-complex, irreducible. □

2.3. Finite-dimensional representations

Lemma 2.15 (Schur's lemma for finite-dimensional representations[34])**.** *Consider an irreducible finite-dimensional representation (M_G, V) of a Lie group G on a complex Hilbert space V. If the representation (M_G, V) is irreducible then any endomorphism S of (M_G, V) is a complex scalar.*

Lemma 2.16 (Schur's lemma)**.** *If the real representation (M_G, V) is irreducible then the equivariant endomorphisms of (M_G, V) are either automorphisms or the null map.*

Proof. Let T be an equivariant endomorphism of (M_G, V). Then (M_G, N_T) and (M_G, I_T) are subrepresentations, where $N_T \equiv \{v \in V : Tv = 0\}$ and $I_T \equiv \{Tv : v \in V\}$. If (M_G, V) is irreducible, then either $N_T = V$ or $N_T = \{0\}$. If $N_T = V$ then T is the null map; if $N_T = \{0\}$, then T is injective. If $I_T = 0$ then T is the null map; if $I_T = V$, then T is surjective. Therefore T is either an automorphism or the null map. □

Lemma 2.17. *Consider an irreducible finite-dimensional representation (M_G, V) on a complex Hilbert space. A C-conjugation operator of (M_G, V), if it exists, is unique up to a complex phase.*

Proof. Let θ_1,θ_2 be two anti-isomorphisms of (M_G, V). The product $(\theta_2\theta_1)$ is an isomorphism of (M_G, V); since (M_G, V) is irreducible, $(\theta_2\theta_1) = re^{i\phi}$; with $\phi, r \in \mathbb{R}$, $r > 0$.

We have $1 = (\theta_2)^2 = r^2 e^{i\phi}\theta_1 e^{i\phi}\theta_1 = r^2$. Therefore $\theta_2 = \alpha\theta_1\alpha^{-1}$; where $\alpha \equiv e^{i\frac{\phi}{2}}$ is a complex phase. □

Proposition 2.18. *Two R-real irreducible finite-dimensional representations are isomorphic iff their complexifications are isomorphic.*

Proof. Let (M_G, V) and (N_G, W) be C-real irreducible representations, with θ_M and θ_N the respective C-conjugation operators. If there is an isomorphism $\alpha : V \to W$ such that $\alpha M_G(g) = N_G(g)\alpha$ for all $g \in G$, then $\vartheta \equiv \alpha\theta_M\alpha^{-1}$ is an anti-isomorphism of (N_G, W). Since it is unique up to a phase, then $\theta_N = e^{i\phi}\vartheta$. Therefore $e^{i\frac{\phi}{2}}\alpha$ is an isomorphism between (M_G, V_{θ_M}) and (N_G, W_{θ_N}), where $V_{\theta_M} \equiv \{(1 + \theta_M)v : v \in V\}$. □

Proposition 2.19. *Two C-complex or C-pseudoreal irreducible finite-dimensional representations are isomorphic or anti-isomorphic iff their realifications are isomorphic.*

Proof. Let (M_G, V) and (N_G, W) be R-complex or R-pseudoreal irreducible representations, with J_M and J_N the respective R-imaginary operators. If there is an isomorphism $\alpha : V \to W$ such that $\alpha M_G(g) = N_G(g)\alpha$ for all $g \in G$, then $K \equiv \alpha J_M \alpha^{-1}$ is a R-imaginary operator of (N_G, W). When considering (N_G, W_{J_N}) and (M_G, V_{J_M}), where $W_{J_N} \equiv \{(1 - iJ_N)w : w \in W\}$, we get that $(1 - J_N K)(1 - K J_N) = c$ as an operator of W_{J_N}. If $c = 0$ then $K = -J_N$ and α defines an anti-isomorphism between (M_G, V_{J_M}) and (N_G, W_{J_N}). If $c \neq 0$ then $(1 - J_N K)\alpha$ is an isomorphism between (M_G, V_{J_M}) and (N_G, W_{J_N}). □

Proposition 2.20. *The space of endomorphisms of a R-real irreducible representation is isomorphic to* $\mathbb{R}$.

Proof. Let (M_G, V) be a C-real irreducible representation, with θ the C-conjugation operator. If there is an endomorphism $\alpha : V \to V$ such that $\alpha M_G(g) = M_G(g)\alpha$ for all $g \in G$, we know from Schur lemma that $\alpha = re^{i\varphi}$. Then the endomorphism of V_θ is a real number. □

Proposition 2.21. *The space of endomorphisms of a R-complex irreducible representation is isomorphic to* $\mathbb{C}$.

Proof. Let (M_G, V) be a R-complex irreducible representation, with J the R-imaginary operator and consider the complex irreducible representation (M_G, V_J), where $V_J \equiv \{(1 - iJ)v : v \in V\}$. If there is a non-null endomorphism α of (M_G, V), then $K \equiv (1 + J\alpha J)$ is an endomorphism of (M_G, V). If K is an automorphism then V_J is equivalent to $\overline{V}_J$ which would imply that (M_G, V) is R-pseudoreal. Then $K = 0$, α is an endomorphism of (M_G, V_J) and hence $\alpha = re^{J\theta}$. □

Proposition 2.22. *The space of endomorphisms of a R-pseudoreal irreducible representation is isomorphic to* $\mathbb{H}$ *(quaternions).*

Proof. Let (M_G, V) be a R-pseudoreal irreducible representation, with J the R-imaginary operator and consider the complex irreducible representation (M_G, V_J), where $V_J \equiv \{(1 - iJ)v : v \in V\}$. Let K_0 be an automorphism of (M_G, V) anti-commuting with J, then $K_0^2 = re^{J\theta}$ and $K_0 re^{J\theta} = K_0(K_0^2) = (K_0^2)K_0 = re^{J\theta}K_0$, therefore $K^2 = -1$, where $K \equiv K_0/\sqrt{r}$. If there is a non-null endomorphism α of (M_G, V), then $S \equiv (1 - J\alpha J)/2$ and $T \equiv (1 + J\alpha J)/2$ are endomorphisms of (M_G, V). If T is an automorphism then KT is an automorphism of (M_G, V_J) and hence $T = Kc + KJd$. If T is null then $c = d = 0$. If S is an automorphism then S is an automorphism of (M_G, V_J) and hence $S = a + Jb$. If S is null then $a = b = 0$. Therefore $\alpha = S + T = a + Jb + Kc + KJd$, which is isomorphic to the quaternions. □

Definition 2.23. A finite-dimensional representation is completely reducible iff it can be expressed as a direct sum of irreducible representations.

Remark 2.24 (Weyl theorem)**.** *All finite-dimensional representations of a semi-simple Lie group (such as SL(2,C)) are completely reducible.*

2.4. Unitary representations

Remark 2.25. *Let* H_n*, with* $n \in \{1, 2\}$*, be two Hilbert spaces with internal products* $<,>$: $H_n \times H_n \to \mathbb{F}$*,*$(\mathbb{F} = \mathbb{R}, \mathbb{C})$*. A linear operator* $U : H_1 \to H_2$ *is unitary iff:*

1) it is surjective;

2) for all $x \in H_1$*,* $< U(x), U(x) >=< x, x >$*.*

The inverse operator $U^{-1} : H_2 \to H_1$ *is defined by:*

$$< x, U^{-1}y >=< Ux, y >, \ x \in H_1, y \in H_2$$

Definition 2.26. Consider a representation (M_G, V). An isometry of (M_G, V) is a unitary isomorphism of (M_G, V).

Proposition 2.27. *Let H_n, with $n \in \{1, 2\}$, be two complex Hilbert spaces and H_n^r its complexification. The following two statements are equivalent:*
1) The operator $U : H_1 \to H_2$ is unitary;
2) The operator $U^r : H_1^r \to H_2^r$ is unitary, where $U^r(h) \equiv U(h)$, for $h \in H_1$.

Proof. Since $< h, h >=< h, h >_r$ and $U^r(h) = U(h)$, for $h \in H_1$, we get the result. □

Lemma 2.28 (Schur's lemma for unitary representations[34])**.** *Consider an irreducible unitary representation (M_G, V) of a Lie group G on a complex Hilbert space V. If the representation (M_G, V) is irreducible then any normal operator N of (M_G, V) is a scalar.*

Lemma 2.29. *Consider an irreducible unitary representation (M_G, V) on a complex Hilbert space. An anti-isometry of (M_G, V), if it exists, is unique up to a complex phase.*

Proof. Let θ_1,θ_2 be two anti-isometries of (M_G, V). The product $(\theta_2\theta_1)$ is an isometry of (M_G, V); since (M_G, V) is irreducible, $(\theta_2\theta_1) = e^{i\phi}$; with $\phi \in \mathbb{R}$.

Therefore $\theta_2 = \alpha\theta_1\alpha^{-1}$; where $\alpha \equiv e^{i\frac{\phi}{2}}$ is a complex phase. □

Proposition 2.30. *Two R-real irreducible unitary representations are isometric iff their complexifications are isometric.*

Proof. Let (M_G, V) and (N_G, W) be C-real irreducible representations, with θ_M and θ_N the respective C-conjugation operators. If there is an isometry $\alpha : V \to W$ such that $\alpha M_G(g) = N_G(g)\alpha$ for all $g \in G$, then $\vartheta \equiv \alpha\theta_M\alpha^{-1}$ is an anti-isometry of (N_G, W). Since it is unique up to a phase, then $\theta_N = e^{i\phi}\vartheta$. Therefore $e^{i\frac{\phi}{2}}\alpha$ is an isometry between (M_G, V_θ) and (N_G, W_θ), where $V_{\theta_M} \equiv \{(1 + \theta_M)v : v \in V\}$. □

Proposition 2.31. *Two C-complex or C-pseudoreal irreducible unitary representations are isomorphic or anti-isomorphic iff their realifications are isomorphic.*

Proof. Let (M_G, V) and (N_G, W) be R-complex or R-pseudoreal irreducible representations, with J_M and J_N the respective R-imaginary operators. If there is an isometry $\alpha : V \to W$ such that $\alpha M_G(g) = N_G(g)\alpha$ for all $g \in G$, then $K \equiv \alpha J_M\alpha^{-1}$ is a R-imaginary operator of (N_G, W). When considering (N_G, W_{J_N}) and (M_G, V_{J_M}), where $W_{J_N} \equiv \{(1-iJ_N)w : w \in W\}$, we get that $(1 - J_NK)(1 - KJ_N) = r$ as an operator of W_{J_N}, where r is a non-negative null real scalar. If $c = 0$ then $K = -J_N$ and α defines an anti-isometry between (M_G, V_{J_M}) and (N_G, W_{J_N}). If $c \neq 0$ then $(1 - J_NK)\alpha c^{-\frac{1}{2}}$ is an isometry between (M_G, V_{J_M}) and (N_G, W_{J_N}). □

Proposition 2.32. *The space of endomorphisms of a R-real irreducible representation is isomorphic to $\mathbb{R}$.*

Proof. Let (M_G, V) be a C-real irreducible representation, with θ the C-conjugation operator. If there is an endomorphism $\alpha : V \to V$ such that $\alpha M_G(g) = M_G(g)\alpha$ for all $g \in G$, we know from Schur lemma that $\alpha = re^{i\varphi}$. Then the endomorphism of V_θ is a real number. □

Proposition 2.33. *The space of endomorphisms of a R-complex irreducible representation is isomorphic to* $\mathbb{C}$*.*

Proof. Let (M_G, V) be a R-complex irreducible representation, with J the R-imaginary operator. If there is an endomorphism α of (M_G, V), then $KK^\dagger$ is an endomorphism of the C-complex irreducible representation (M_G, V_J), where $K \equiv (\alpha + J\alpha J)$ and $V_J \equiv \{(1 - iJ)v : v \in V\}$. If $KK^\dagger = r > 0$, then $\frac{K}{\sqrt{r}}$ is unitary and V_J is equivalent to $\overline{V}_J$ which would imply that (M_G, V) is C-pseudoreal. Therefore $K = 0$ and hence α is an endomorphism of (M_G, V_J), so $\alpha = re^{J\theta}$. □

Proposition 2.34. *The space of endomorphisms of a R-pseudoreal irreducible representation is isomorphic to* $\mathbb{H}$ *(quaternions).*

Proof. Let (M_G, V) be a R-pseudoreal irreducible representation, with J the R-imaginary operator. If there is an endomorphism α of (M_G, V), then $SS^\dagger$ and $TT^\dagger$ are a self-adjoint endomorphisms of the C-complex irreducible representation (M_G, V_J), where $S \equiv (\alpha - J\alpha J)/2$, $T \equiv (\alpha + J\alpha J)/2$ and $V_J \equiv \{(1 - iJ)v : v \in V\}$. Let K be an unitary operator of (M_G, V) and anti-commuting with J, then $K^2 = e^{J\theta}$ and $Ke^{J\theta} = K(K^2) = (K^2)K = e^{J\theta}K$, therefore $K^2 = -1$. If $TT^\dagger = t > 0$, then $\frac{T}{\sqrt{t}}$ is unitary and anti-commutes with J, TK is a normal endomorphism of (M_G, V_J) and therefore $T = Kc + KJd$; if $TT^\dagger = 0$ then $c = d = 0$. If $SS^\dagger = s > 0$, then $\frac{S}{\sqrt{s}}$ is unitary and commutes with J, S is a normal endomorphism of (M_G, V_J) and therefore $S = a + Jb$; if $SS^\dagger = 0$ then $a = b = 0$.

Therefore $\alpha = S + T = a + Jb + Kc + KJd$, which is isomorphic to the quaternions. □

Definition 2.35. A unitary representation is completely reducible iff it can be expressed as a direct integral of irreducible representations.

Remark 2.36. *All unitary representations of a separable locally compact group (such as the Poincare group) are completely reducible.*

3. Finite-dimensional representations of the Lorentz group

3.1. Majorana, Dirac and Pauli Matrices and Spinors

Definition 3.1. $\mathbb{F}^{m\times n}$ is the vector space of $m \times n$ matrices whose entries are elements of the field $\mathbb{F}$.

In the next remark we state the Pauli's fundamental theorem of gamma matrices. The proof can be found in the reference[51].

Remark 3.2 (Pauli's fundamental theorem)**.** *Let* A^μ*,* B^μ*,* $\mu \in \{0, 1, 2, 3\}$*, be two sets of* 4×4 *complex matrices verifying:*

$$A^\mu A^\nu + A^\nu A^\mu = -2\eta^{\mu\nu} \tag{1}$$

$$B^\mu B^\nu + B^\nu B^\mu = -2\eta^{\mu\nu} \tag{2}$$

Where $\eta^{\mu\nu} \equiv diag(+1, -1, -1 - 1)$ is the Minkowski metric.

1) There is an invertible complex matrix S such that $B^\mu = SA^\mu S^{-1}$, for all $\mu \in \{0, 1, 2, 3\}$. S is unique up to a non-null scalar.

2) If A^μ and B^μ are all unitary, then S is unitary.

Proposition 3.3. *Let α^μ, β^μ, $\mu \in \{0, 1, 2, 3\}$, be two sets of 4×4 real matrices verifying:*

$$\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = -2\eta^{\mu\nu} \tag{3}$$

$$\beta^\mu \beta^\nu + \beta^\nu \beta^\mu = -2\eta^{\mu\nu} \tag{4}$$

Then there is a real matrix S, with $|detS| = 1$, such that $\beta^\mu = S\alpha^\mu S^{-1}$, for all $\mu \in \{0, 1, 2, 3\}$. S is unique up to a signal.

Proof. From remark 3.2, we know that there is an invertible matrix T', unique up to a non-null scalar, such that $\beta^\mu = T'\alpha^\mu T'^{-1}$. Then $T \equiv T'/|det(T')|$ has $|detT| = 1$ and it is unique up to a complex phase.

Conjugating the previous equation, we get $\beta^\mu = T^*\alpha^\mu T^{*-1}$. Then $T^* = e^{i2\theta}T$ for some real number θ. Therefore $S \equiv e^{i\theta}T$ is a real matrix, with $|detS| = 1$, unique up to a signal. □

Definition 3.4. The Majorana matrices, $i\gamma^\mu$, $\mu \in \{0, 1, 2, 3\}$, are 4×4 complex unitary matrices verifying:

$$(i\gamma^\mu)(i\gamma^\nu) + (i\gamma^\nu)(i\gamma^\mu) = -2\eta^{\mu\nu} \tag{5}$$

The Dirac matrices are $\gamma^\mu \equiv -i(i\gamma^\mu)$.

In the Majorana bases, the Majorana matrices are 4×4 real orthogonal matrices. An example of the Majorana matrices in a particular Majorana basis is:

$$\begin{aligned}
i\gamma^1 &= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
i\gamma^0 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad = -\gamma^0\gamma^1\gamma^2\gamma^3
\end{aligned} \tag{6}$$

In reference [52] it is proved that the set of five anti-commuting 4×4 real matrices is unique up to isomorphisms. So, for instance, with 4×4 real matrices it is not possible to obtain the euclidean signature for the metric.

Definition 3.5. The Dirac spinor is a 4×1 complex column matrix, $\mathbb{C}^{4\times 1}$.

The space of Dirac spinors is a 4 dimensional complex vector space.

Lemma 3.6. *The charge conjugation operator Θ, is an anti-linear involution commuting with the Majorana matrices $i\gamma^\mu$. It is unique up to a complex phase.*

Proof. In the Majorana bases, the complex conjugation is a charge conjugation operator. Let Θ and Θ' be two charge conjugation operators operators. Then, $\Theta\Theta'$ is a complex invertible matrix commuting with $i\gamma^\mu$, therefore, from Pauli's fundamental theorem, $\Theta\Theta' = c$, where c is a non-null complex scalar. Therefore $\Theta' = c^*\Theta$ and from $\Theta'\Theta' = 1$, we get that $c^*c = 1$. $\square$

Definition 3.7. Let Θ be a charge conjugation operator.

The set of Majorana spinors, $Pinor$, is the set of Dirac spinors verifying the Majorana condition (defined up to a complex phase):

$$Pinor \equiv \{u \in \mathbb{C}^{4\times 1} : \Theta u = u\} \tag{7}$$

The set of Majorana spinors is a 4 dimensional real vector space. Note that the linear combinations of Majorana spinors with complex scalars do not verify the Majorana condition.

There are 16 linear independent products of Majorana matrices. These form a basis of the real vector space of endomorphisms of Majorana spinors, $End(Pinor)$. In the Majorana bases, $End(Pinor)$ is the vector space of 4×4 real matrices.

Definition 3.8. The Pauli matrices $\sigma^k,\ k \in \{1, 2, 3\}$ are 2×2 hermitian, unitary, anti-commuting, complex matrices. The Pauli spinor is a 2×1 complex column matrix. The space of Pauli spinors is denoted by $Pauli$.

The space of Pauli spinors, $Pauli$, is a 2 dimensional complex vector space and a 4 dimensional real vector space. The realification of the space of Pauli spinors is isomorphic to the space of Majorana spinors.

3.2. On the Lorentz, SL(2,C) and Pin(3,1) groups

Remark 3.9. *The Lorentz group, $O(1, 3) \equiv \{\lambda \in \mathbb{R}^{4\times 4} : \lambda^T \eta \lambda = \eta\}$, is the set of real matrices that leave the metric, $\eta = diag(1, -1, -1, -1)$, invariant.*

The proper orthochronous Lorentz subgroup is defined by $SO^+(1, 3) \equiv \{\lambda \in O(1, 3) : det(\lambda) = 1, {\lambda^0}_0 > 0\}$. It is a normal subgroup. The discrete Lorentz subgroup of parity and time-reversal is $\Delta \equiv \{1, \eta, -\eta, -1\}$.

The Lorentz group is the semi-direct product of the previous subgroups, $O(1, 3) = \Delta \ltimes SO^+(1, 3)$.

Definition 3.10. The set Maj is the 4 dimensional real space of the linear combinations of the Majorana matrices, $i\gamma^\mu$:

$$Maj \equiv \{a_\mu i\gamma^\mu : a_\mu \in \mathbb{R},\ \mu \in \{0, 1, 2, 3\}\} \tag{8}$$

Definition 3.11. $Pin(3, 1)$ [36] is the group of endomorphisms of Majorana spinors that leave the space Maj invariant, that is:

$$Pin(3, 1) \equiv \Big\{S \in End(Pinor) :\ |detS| = 1,\ S^{-1}(i\gamma^\mu)S \in Maj,\ \mu \in \{0, 1, 2, 3\}\Big\} \tag{9}$$

Proposition 3.12. *The map $\Lambda : Pin(3,1) \to O(1,3)$ defined by:*

$$(\Lambda(S))^{\mu}{}_{\nu} i\gamma^{\nu} \equiv S^{-1}(i\gamma^{\mu})S \tag{10}$$

is two-to-one and surjective. It defines a group homomorphism.

Proof. 1) Let $S \in Pin(3,1)$. Since the Majorana matrices are a basis of the real vector space Maj, there is an unique real matrix $\Lambda(S)$ such that:

$$(\Lambda(S))^{\mu}{}_{\nu} i\gamma^{\nu} = S^{-1}(i\gamma^{\mu})S \tag{11}$$

Therefore, Λ is a map with domain $Pin(3,1)$. Now we can check that $\Lambda(S) \in O(1,3)$:

$$(\Lambda(S))^{\mu}{}_{\alpha}\eta^{\alpha\beta}(\Lambda(S))^{\nu}{}_{\beta} = -\frac{1}{2}(\Lambda(S))^{\mu}{}_{\alpha}\{i\gamma^{\alpha}, i\gamma^{\beta}\}(\Lambda(S))^{\nu}{}_{\beta} = \tag{12}$$

$$= -\frac{1}{2}S\{i\gamma^{\mu}, i\gamma^{\nu}\}S^{-1} = S\eta^{\mu\nu}S^{-1} = \eta^{\mu\nu} \tag{13}$$

We have proved that Λ is a map from $Pin(3,1)$ to $O(1,3)$.

2) Since any $\lambda \in O(1,3)$ conserve the metric η, the matrices $\alpha^{\mu} \equiv \lambda^{\mu}{}_{\nu} i\gamma^{\nu}$ verify:

$$\{\alpha^{\mu}, \alpha^{\nu}\} = -2\lambda^{\mu}{}_{\alpha}\eta^{\alpha\beta}\lambda^{\nu}{}_{\beta} = -2\eta^{\mu\nu} \tag{14}$$

In a basis where the Majorana matrices are real, from Proposition 3.3 there is a real invertible matrix S_{λ}, with $|detS_{\Lambda}| = 1$, such that $\lambda^{\mu}{}_{\nu} i\gamma^{\nu} = S_{\lambda}^{-1}(i\gamma^{\mu})S_{\lambda}$. The matrix S_{Λ} is unique up to a sign. So, $\pm S_{\lambda} \in Pin(3,1)$ and we proved that the map $\Lambda : Pin(3,1) \to O(1,3)$ is two-to-one and surjective.

3) The map defines a group homomorphism because:

$$\Lambda^{\mu}{}_{\nu}(S_1)\Lambda^{\nu}{}_{\rho}(S_2) i\gamma^{\rho} = \Lambda^{\mu}{}_{\nu} S_2^{-1} i\gamma^{\nu} S_2 \tag{15}$$

$$= S_2^{-1}S_1^{-1} i\gamma^{\mu} S_1 S_2 = \Lambda^{\mu}{}_{\rho}(S_1 S_2) i\gamma^{\rho} \tag{16}$$

□

Remark 3.13. *The group $SL(2,\mathbb{C}) = \{e^{\theta^j i\sigma^j + b^j\sigma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$ is simply connected. Its projective representations are equivalent to its ordinary representations[6].*

There is a two-to-one, surjective map $\Upsilon : SL(2,\mathbb{C}) \to SO^{+}(1,3)$, defined by:

$$\Upsilon^{\mu}{}_{\nu}(T)\sigma^{\nu} \equiv T^{\dagger}\sigma^{\mu}T \tag{17}$$

Where $T \in SL(2,\mathbb{C})$, $\sigma^0 = 1$ and σ^j, $j \in \{1,2,3\}$ are the Pauli matrices.

Lemma 3.14. *Consider that $\{M_+, M_-, i\gamma^5 M_+, i\gamma^5 M_-\}$ and $\{P_+, P_-, iP_+, iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces $Pinor$ and $Pauli$, respectively, verifying:*

$$\gamma^0\gamma^3 M_{\pm} = \pm M_{\pm},\ \sigma^3 P_{\pm} = \pm P_{\pm} \tag{18}$$

The isomorphism $\Sigma : Pauli \to Pinor$ is defined by:

$$\Sigma(P_+) = M_+,\ \Sigma(iP_+) = i\gamma^5 M_+ \tag{19}$$
$$\Sigma(P_-) = M_-,\ \Sigma(iP_-) = i\gamma^5 M_- \tag{20}$$

The group $Spin^+(3,1) \equiv \{\Sigma \circ A \circ \Sigma^{-1} : A \in SL(2,\mathbb{C})\}$ is a subgroup of $Pin(1,3)$. For all $S \in Spin^+(1,3)$, $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$.

Proof. From remark 3.13, $Spin^+(3,1) = \{e^{\theta^j i\gamma^5\gamma^0\gamma^j + b^j\gamma^0\gamma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$. Then, for all $T \in SL(2,C)$:

$$-i\gamma^0\Sigma \circ T^\dagger \circ \Sigma^{-1} i\gamma^0 = \Sigma \circ T^{-1} \circ \Sigma^{-1} \tag{21}$$

Now, the map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$ is given by:

$$\Upsilon^\mu{}_\nu(T) i\gamma^\nu = (\Sigma \circ T^{-1} \circ \Sigma^{-1}) i\gamma^\mu (\Sigma \circ T \circ \Sigma^{-1}) \tag{22}$$

Then, all $S \in Spin^+(3,1)$ leaves the space Maj invariant:

$$S^{-1} i\gamma^\mu S = \Upsilon^\mu{}_\nu(\Sigma^{-1} \circ S \circ \Sigma) i\gamma^\nu \in Maj \tag{23}$$

Since all the products of Majorana matrices, except the identity, are traceless, then $det(S) = 1$. So, $Spin^+(3,1)$ is a subgroup of $Pin(1,3)$ and $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$. □

Definition 3.15. The discrete Pin subgroup $\Omega \subset Pin(3,1)$ is:

$$\Omega \equiv \{\pm 1, \pm i\gamma^0, \pm\gamma^0\gamma^5, \pm i\gamma^5\} \tag{24}$$

The previous lemma and the fact that Λ is continuous, implies that $Spin^+(1,3)$ is a double cover of $SO^+(3,1)$. We can check that for all $\omega \in \Omega$, $\Lambda(\pm\omega) \in \Delta$. That is, the discrete Pin subgroup is the double cover of the discrete Lorentz subgroup. Therefore, $Pin(3,1) = \Omega \ltimes Spin^+(1,3)$

Since there is a two-to-one continuous surjective group homomorphism, $Pin(3,1)$ is a double cover of $O(1,3)$, $Spin^+(3,1)$ is a double cover of $SO^+(1,3)$ and $Spin^+(1,3) \cap SU(4)$ is a double cover of $SO(3)$. We can check that $Spin^+(1,3) \cap SU(4)$ is equivalent to $SU(2)$.

3.3. Finite-dimensional representations of SL(2,C)

Remark 3.16. *Since SL(2,C) is a semisimple Lie group, all its finite-dimensional (real or complex) representations are direct sums of irreducible representations.*

Remark 3.17. *The finite-dimensional complex irreducible representations of SL(2,C) are labeled by (m,n), where $2m, 2n$ are natural numbers. Up to equivalence, the representation space $V_{(m,n)}$ is the tensor product of the complex vector spaces V_m^+ and V_n^-, where $V_m^\pm$ is a symmetric tensor with $2m$ Dirac spinor indexes, such that $\gamma^5{}_k v = \pm v$, where $v \in V_m^\pm$ and $\gamma^5{}_k$ is the Dirac matrix γ^5 acting on the k-th index of v.*

The group homomorphism consists in applying the same matrix of $Spin^+(1,3)$*, correspondent to the* $SL(2,C)$ *group element we are representing, to each index of* v*.* $V_{(0,0)}$ *is equivalent to* $\mathbb{C}$ *and the image of the group homomorphism is the identity.*

These are also projective representations of the time reversal transformation, but, for $m \neq n$*, not of the parity transformation, that is, under the parity transformation,* $(V_m^+ \otimes V_n^-) \to (V_m^- \otimes V_n^+)$ *and under the time reversal transformation* $(V_m^+ \otimes V_n^-) \to (V_m^+ \otimes V_n^-)$*.*

Lemma 3.18. *The finite-dimensional real irreducible representations of SL(2,C) are labeled by* (m, n)*, where* $2m, 2n$ *are natural numbers and* $m \geq n$*. Up to equivalence, the representation space* $W_{(m,n)}$ *is defined for* $m \neq n$ *as:*

$$W_{(m,n)} \equiv \{\frac{1+(i\gamma^5)_1 \otimes (i\gamma^5)_1}{2} w : w \in W_m \otimes W_n\}$$
$$W_{(m,m)} \equiv \{\frac{1+(i\gamma^5)_1 \otimes (i\gamma^5)_1}{2} w : w \in (W_m)^2\}$$

where W_m *is a symmetric tensor with* m *Majorana spinor indexes, such that* $(i\gamma^5)_1(i\gamma^5)_k w = -w$*, where* $w \in W_m$*;* $(i\gamma^5)_k$ *is the Majorana matrix* $i\gamma^5$ *acting on the k-th index of* w*;* $(W_m)^2$ *is the space of the linear combinations of the symmetrized tensor products* $(u \otimes v + v \otimes u)$*, for* $u, v \in W_m$*.*

The group homomorphism consists in applying the same matrix of $Spin^+(1,3)$*, correspondent to the* $SL(2,C)$ *group element we are representing, to each index of the tensor. In the* $(0,0)$ *case,* $W_{(0,0)}$ *is equivalent to* $\mathbb{R}$ *and the image of the group homomorphism is the identity.*

These are also projective representations of the full Lorentz group, that is, under the parity or time reversal transformations, $(W_{m,n} \to W_{m,n})$*.*

Proof. For $m \neq n$ the complex irreducible representations of SL(2,C) are C-complex. The complexification of $W_{(m,n)}$ verifies $W^c_{(m,n)} = (V_m^+ \otimes V_n^-) \oplus (V_m^- \otimes V_n^+)$.

For $m = n$ the complex irreducible representations of SL(2,C) are C-real. In a Majorana basis, the C-conjugation operator of $V_{(m,m)}$, θ, is defined as $\theta(u \otimes v) \equiv v^* \otimes u^*$, where $u \in V_m^+$ and $v \in V_m^-$. We can check that there is a bijection $\alpha : W_{(m,m)} \to (V_{(m,m)})_\theta$, defined by $\alpha(w) \equiv \frac{1-i(i\gamma^5)_1 \otimes 1}{2} w$; $\alpha^{-1}(v) \equiv v + v^*$, for $w \in W_{(m,m)}$, $v \in (V_{(m,m)})_\theta$.

Using the map from Section 2, we can check that the representations $W_{(m,n)}$, with $m \geq n$, are the unique finite-dimensional real irreducible representations of SL(2,C), up to isomorphisms.

We can check that $W^c_{(m,n)}$ is equivalent to $W^c_{(n,m)}$, therefore, invariant under the parity or time reversal transformations. □

As examples of real irreducible representations of $SL(2,C)$ we have for $(1/2, 0)$ the Majorana spinor, for $(1/2, 1/2)$ the linear combinations of the matrices $\{1, \gamma^0\vec{\gamma}\}$, for $(1, 0)$ the linear combinations of the matrices $\{i\vec{\gamma}, \vec{\gamma}\gamma^5\}$. The group homomorphism is defined as $M(S)(u) \equiv Su$ and $M(S)(A) \equiv SAS^\dagger$, for $S \in Spin^+(1,3)$, $u \in Pinor$, $A \in \{1, \vec{\gamma}\gamma^0\}$ or $A \in \{i\vec{\gamma}, \vec{\gamma}\gamma^5\}$.

We can check that the domain of M can be extended to $Pin(1,3)$, leaving the considered vector spaces invariant. For $m = n$, we can define the "pseudo-representation" $W'_{(m,m)} \equiv \{((i\gamma^5)_1 \otimes 1)w : w \in W_{(m,m)}\}$ which is equivalent to $W_{(m,m)}$ as an $SL(2,C)$ representation, but under parity transforms with the opposite sign. As an example, the "pseudo-representation" $(1/2,1/2)$ is defined as the linear combinations of the matrices $\{i\gamma^5, i\gamma^5\vec{\gamma}\gamma^0\}$.

4. Unitary representations of the Poincare group

4.1. Bargmann-Wigner fields

Definition 4.1. Consider that $\{M_+, M_-, i\gamma^0 M_+, i\gamma^0 M_-\}$ and $\{P_+, P_-, iP_+, iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces *Pinor* and *Pauli*, respectively, verifying:

$$\gamma^3\gamma^5 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm$$

Let H be a real Hilbert space. For all $h \in H$, the bijective linear map $\Theta_H : Pauli \otimes_{\mathbb{R}} H \to Pinor \otimes_{\mathbb{R}} H$ is defined by:

$$\begin{aligned}\Theta_H(h \otimes_{\mathbb{R}} P_+) = h \otimes_{\mathbb{R}} M_+,\ \Theta_H(h \otimes_{\mathbb{R}} iP_+) = h \otimes_{\mathbb{R}} i\gamma^0 M_+\\ \Theta_H(h \otimes_{\mathbb{R}} P_-) = h \otimes_{\mathbb{R}} M_-,\ \Theta_H(h \otimes_{\mathbb{R}} iP_-) = h \otimes_{\mathbb{R}} i\gamma^0 M_-\end{aligned}$$

Definition 4.2. Let H_n, with $n \in \{1,2\}$, be two real Hilbert spaces and $U : Pauli \otimes_{\mathbb{R}} H_1 \to Pauli \otimes_{\mathbb{R}} H_2$ be an operator. The operator $U^\Theta : Pinor \otimes_{\mathbb{R}} H_1 \to Pinor \otimes_{\mathbb{R}} H_2$ is defined as $U^\Theta \equiv \Theta_{H_2} \circ U \circ \Theta_{H_1}^{-1}$.

The space of Majorana spinors is isomorphic to the realification of the space of Pauli spinors.

Definition 4.3. The real Hilbert space $Pinor(\mathbb{X}) \equiv Pinor \otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in *Pinor*.

Definition 4.4. The complex Hilbert space $Pauli(\mathbb{X}) \equiv Pauli \otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in *Pauli*.

Remark 4.5. *The Fourier Transform $\mathcal{F}_P : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{R}^3)$ is an unitary operator defined by:*

$$\mathcal{F}_P\{\psi\}(\vec{p}) \equiv \int d^n\vec{x} \frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^n}} \psi(\vec{x}),\ \psi \in Pauli(\mathbb{R}^3)$$

Where the domain of the integral is $\mathbb{R}^3$.

Remark 4.6. *The inverse Fourier transform verifies:*

$$\begin{aligned}-\vec{\partial}^2\ \mathcal{F}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{F}_P^{-1} \circ R)\{\psi\}(\vec{x})\\ i\vec{\partial}_k\ \mathcal{F}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{F}_P^{-1} \circ R'_k)\{\psi\}(\vec{x})\end{aligned}$$

Where $\psi \in Pauli(\mathbb{R}^3)$ and $R, R'_k : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{R}^3)$, with $k \in \{1,2,3\}$, are linear maps defined by:

$$R\{\psi\}(\vec{p}) \equiv (\vec{p})^2\psi(\vec{p})$$
$$R'_k\{\psi\}(\vec{p}) \equiv \vec{p}_k\ \psi(\vec{p})$$

Definition 4.7. Let $\vec{x} \in \mathbb{R}^3$. The spherical coordinates parametrization is:

$$\vec{x} = r(\sin(\theta)\sin(\varphi)\vec{e_1} + \sin(\theta)\sin(\varphi)\vec{e_2} + \cos(\theta)\vec{e_3})$$

where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a fixed orthonormal basis of $\mathbb{R}^3$ and $r \in [0, +\infty[$, $\theta \in [0, \pi]$, $\varphi \in [-\pi, \pi]$.

Definition 4.8. Let

$$\mathbb{S}^3 \equiv \{(p, l, \mu) : p \in \mathbb{R}_{\geq 0}; l, \mu \in \mathbb{Z}; l \geq 0; -l \leq \mu \leq l\}$$

The Hilbert space $L^2(\mathbb{S}^3)$ is the real Hilbert space of real Lebesgue square integrable functions of $\mathbb{S}^3$. The internal product is:

$$< f, g >= \sum_{l=0}^{+\infty}\sum_{\mu=-l}^{l-1}\int_0^{+\infty} dp f(p, l, \mu)g(p, l, \mu),\ \ f, g \in L^2(\mathbb{S}^3)$$

Definition 4.9. The Spherical transform $\mathcal{H}_P : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}_P\{\psi\}(p, l, \mu) \equiv \int r^2 dr d(\cos\theta) d\varphi \frac{2p}{\sqrt{2\pi}} j_l(pr) Y_{l\mu}(\theta, \varphi)\psi(r, \theta, \varphi),\ \ \psi \in Pauli(\mathbb{R}^3)$$

The domain of the integral is $\mathbb{R}^3$. The spherical Bessel function of the first kind j_l [53], the spherical harmonics $Y_{l\mu}$[54] and the associated Legendre functions of the first kind $P_{l\mu}$ are:

$$\begin{aligned} j_l(r) \equiv& r^l\Big(-\frac{1}{r}\frac{d}{dr}\Big)^l\frac{\sin r}{r} \\ Y_{l\mu}(\theta, \varphi) \equiv& \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^\mu(\cos\theta)e^{i\mu\varphi} \\ P_l^\mu(\xi) \equiv& \frac{(-1)^\mu}{2^l l!}(1-\xi^2)^{\mu/2}\frac{\mathrm{d}^{l+\mu}}{\mathrm{d}\xi^{l+\mu}}(\xi^2-1)^l \end{aligned}$$

Remark 4.10. *Due to the properties of spherical harmonics and Bessel functions, the Spherical transform is an unitary operator. The inverse Spherical transform verifies:*

$$-\vec{\partial}^2\ \mathcal{H}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_P^{-1} \circ R)\{\psi\}(\vec{x})$$
$$(-x^1 i\partial_2 + x^2 i\partial_1)\ \mathcal{H}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_P^{-1} \circ R')\{\psi\}(\vec{x})$$

Where $\psi \in Pauli(\mathbb{S}^3)$ and $R, R' : Pauli(\mathbb{S}^3) \to Pauli(\mathbb{S}^3)$ are linear maps defined by:

$$R\{\psi\}(p, l, \mu) \equiv p^2\psi(p, l, \mu)$$
$$R'\{\psi\}(p, l, \mu) \equiv \mu\ \psi(p, l, \mu)$$

Definition 4.11. The real vector space $Pinor_j$, with $2j$ a positive integer, is the space of linear combinations of the tensor products of $2j$ Majorana spinors, symmetric on the spinor indexes. The real vector space $Pinor_0$ is the space of linear combinations of the tensor products of 2 Majorana spinors, anti-symmetric on the spinor indexes.

Definition 4.12. The real Hilbert space $Pinor_j(\mathbb{X}) \equiv Pinor_j \otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in $Pinor_j$.

Definition 4.13. The Hilbert space $Pinor_{j,n}$, with $(j-\nu)$ an integer and $-j \leq n \leq j$ is defined as:

$$Pinor_{j,n} \equiv \{\Psi \in Pinor_j : \sum_{k=1}^{k=2j} (\gamma^0)_1 \Big(\gamma^0\gamma^3\gamma^5\Big)_k \Psi = 2n\Psi\}$$

Where $\Big(\gamma^3\gamma^5\Big)_k$ is the matrix $\gamma^3\gamma^5$ acting on the Majorana index k. Given

Definition 4.14. The Spherical transform $\mathcal{H}'_P : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}'_P\{\psi\}(p,l,J,\nu) \equiv \sum_{\mu=-l}^{l}\sum_{n=-j}^{j} < l\mu j n|J\nu > \Big(\mathcal{H}_P^{\Theta}\Big)_1 \{\psi\}(p,l,\mu,n),\ \psi \in Pinor_j(\mathbb{R}^3)$$

$< l\mu jn|J\nu >$ are the Clebsh-Gordon coefficients and $\psi(p,l,\mu,n) \in Pinor_{j,n}$ such that $\psi(p,l,\mu) = \sum_{n=-j}^{j}\psi(p,l,\mu,n)$. $(j-n)$, $(J-\nu)$ and $(J-j)$ are integers, with $-J \leq \nu \leq J$ and $|j-l| \leq J \leq j+l$. $\Big(\mathcal{H}_P^{\Theta}\Big)_1$ is the realification of the transform $\mathcal{H}_P$, with the imaginary number replaced by the matrix $i\gamma^0$ acting on the first Majorana index of ψ.

Proposition 4.15. *Consider a unitary operator $U : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{X})$ such that $U \circ H^2 = E^2 \circ U$, where*

$$iH\{\Psi\}(\vec{x}) \equiv \Big(\gamma^0\vec{\partial\!\!\!/} + i\gamma^0 m\Big)_k \Psi(\vec{x})$$

the Majorana matrices act on some Majorana index k; $E^2\{\Phi\}(X) \equiv E^2(X)\Phi(X)$ with $E(X) \geq m \geq 0$ a real number.

Then the operator $U' : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{X})$ is unitary, where U' is defined by:

$$U' \equiv \frac{E + UH\gamma^0U^\dagger}{\sqrt{E+m}\sqrt{2E}}$$

Proof. Note that since $E^2 = U^\dagger H^2 U$, $E = \sqrt{E^2}$ commutes with $UH\gamma^0U^\dagger$. We have that

$$(U')^\dagger(U') = \frac{E + U\gamma^0HU^\dagger}{\sqrt{E+m}\sqrt{2E}}\frac{E + UH\gamma^0U^\dagger}{\sqrt{E+m}\sqrt{2E}} = 1$$

We also have that $(U')(U')^\dagger = 1$. Therefore, U' is unitary. $\square$

Definition 4.16. The Fourier-Majorana transform $\mathcal{F}_M : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{R}^3)$ is an unitary operator defined by:

$$\mathcal{F}_M\{\Psi\}(\vec{p}) \equiv \int d^3\vec{x} \Big(\frac{e^{-i\gamma^0\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}\Big)_1 \prod_{k=1}^{2j} \Big(\frac{E_p + H(\vec{x})\gamma^0}{\sqrt{E_p+m}\sqrt{2E_p}}\Big)_k \Psi(\vec{x}),\ \ \Psi \in Pinor_j(\mathbb{R}^3)$$

The matrices with the index k apply on the corresponding spinor index of Ψ.

Definition 4.17. The Hankel-Majorana transform $\mathcal{H}_M : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{S}^3)$ is an unitary operator defined by:

$$\mathcal{H}_M\{\Psi\}(p,l,J,\nu) \equiv \sum_{\mu=-l}^{l} \sum_{n=-j}^{j} < l\mu jn|J\nu > \int d^3\vec{x}$$
$$\Big(\frac{2p}{\sqrt{2\pi}} j_l(pr) Y_{l\mu}(\theta,\varphi)\Big)_1 \prod_{k=1}^{2j} \Big(\frac{E_p + H(\vec{x})\gamma^0}{\sqrt{E_p+m}\sqrt{2E_p}}\Big)_k \Psi(\vec{x},n)$$

The matrices with the index k apply on the corresponding spinor index of $\Psi \in Pinor_j(\mathbb{R}^3)$. $< l\mu jn|J\nu >$ are the Clebsh-Gordon coefficients and $\Psi(\vec{x},n) \in Pinor_{j,n}$ such that $\Psi(\vec{x}) = \sum_{n=-j}^{j} \Psi(\vec{x},n)$.

The inverse Fourier-Majorana transform verifies:

$$(iH(\vec{x}))_k\ \mathcal{F}_M^{-1}\{\psi\}(\vec{x}) = (\mathcal{F}_M^{-1} \circ R)\{\psi\}(\vec{x})$$
$$\vec{\partial}_l\ \mathcal{F}_M^{-1}\{\psi\}(\vec{x}) = (\mathcal{F}_M^{-1} \circ R')\{\psi\}(\vec{x})$$

Where $\psi \in Pinor_j(\mathbb{R}^3)$ and $R, R' : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{R}^3)$ are linear maps defined by:

$$R\{\psi\}(\vec{p}) \equiv (i\gamma^0)_k E_p \psi(\vec{p})$$
$$R'\{\psi\}(\vec{p}) \equiv (i\gamma^0)_1 \vec{p}_l\ \psi(\vec{p})$$

The inverse Hankel-Majorana transform verifies:

$$(iH(\vec{x}))_k\ \mathcal{H}_M^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_M^{-1} \circ R)\{\psi\}(\vec{x})$$
$$(-x^1\partial_2 + x^2\partial_1 + \sum_{k=1}^{2j}(i\gamma^0\gamma^3\gamma^5)_k)\ \mathcal{H}_M^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_M^{-1} \circ R')\{\psi\}(\vec{x})$$

Where $\psi \in Pinor_j(\mathbb{S}^3)$ and $R, R' : Pinor_j(\mathbb{S}^3) \to Pinor_j(\mathbb{S}^3)$ are linear maps defined by:

$$R\{\psi\}(p,l,J,\nu) \equiv (i\gamma^0)_k E_p \psi(p,l,J,\nu)$$
$$R'\{\psi\}(p,l,J,\nu) \equiv (i\gamma^0)_1 \nu\ \psi(p,l,J,\nu)$$

Definition 4.18. The space of (real) Bargmann-Wigner fields $BW_j(\mathbb{R}^3)$ is defined as:

$$BW_j \equiv \{\Psi \in Pinor_j(\mathbb{R}^3) : \left(e^{iH(\vec{x})t}\right)_k \Psi = \left(e^{iH(\vec{x})t}\right)_1 \Psi; 1 \leq k \leq 2j; t \in \mathbb{R}\}$$

Note that if the equality $e^{-iH_1 t}\Psi = e^{-iH_2 t}\Psi$ holds for all differentiable $\Psi \in H$ then for the continuous linear extension the equality holds for all $\Psi \in H$, by the bounded linear transform theorem.

Definition 4.19. The complex Hilbert space $Dirac_j(\mathbb{X}) \equiv Pinor_j(\mathbb{X}) \otimes \mathbb{C}$ is the complexification of $Pinor_j(\mathbb{X})$. The space of complex Bargmann-Wigner fields is the complexification of the space of real Bargmann-Wigner fields.

4.2. Real unitary representations of the Poincare group

Definition 4.20. The $IPin(3,1)$ group is defined as the semi-direct product $Pin(3,1) \ltimes \mathbb{R}^4$, with the group's product defined as $(A,a)(B,b) = (AB, a + \Lambda(A)b)$, for $A, B \in Pin(3,1)$ and $a, b \in \mathbb{R}^4$ and $\Lambda(A)$ is the Lorentz transformation corresponding to A.

The $ISL(2,C)$ group is isomorphic to the subgroup of $IPin(3,1)$, obtained when $Pin(3,1)$ is restricted to $Spin^+(1,3)$. The full/restricted Poincare group is the representation of the $IPin(3,1)/ISL(2,C)$ group on Lorentz vectors, defined as $\{(\Lambda(A), a) : A \in Pin(3,1), a \in \mathbb{R}^4\}$.

Definition 4.21. Given a Lorentz vector l, the little group G_l is the subgroup of $SL(2,C)$ such that for all $g \in G_l$, $g\not{l} = \not{l}g$.

Proposition 4.22. *Given a Lorentz vector l, consider a set of matrices $\alpha_k \in SL(2,C)$ verifying $\alpha_k \not{l} = \not{k}\alpha_k$. Let $H_k \equiv \{\alpha^{-1}_{\Lambda_S(k)} S \alpha_k : S \in SL(2,C)\}$. Then $H_k = G_l$.*

Proof. We can check that $H_k \subset G_l$. For any $s \in G_l$, there is $S = \alpha_{\Lambda_S(k)} s \alpha_k^{-1}$ such that $s \in H_k$. □

For $i\not{l} = i\gamma^0$, we can set $\alpha_p = \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2m}}$ and $G_l = SU(2)$. For $i\not{l} = (i\gamma^0 + i\gamma^3)$, we can set $\alpha_p = R_p B_v$, where the boost velocity is $v = \frac{E_p^2 - 1}{E_p^2 + 1}$ along z and $R_p = e^{-\gamma^2\gamma^1\theta/2} e^{-\gamma^1\gamma^3\phi/2}$ is a rotation from the z axis to the axis $\frac{\vec{p}}{E_p} = (\sin\phi\cos\theta\gamma_1 + \sin\phi\sin\theta\gamma_2 + \cos\phi\gamma_3)$; $G_l = SE(2)$

$$SE(2) = \{(1 + i\gamma^5(\gamma^1 a + \gamma^2 b)(\gamma^0 + \gamma^3))e^{i\gamma^0\gamma^3\gamma^5\theta} : a, b, \theta \in \mathbb{R}\}. \tag{25}$$

Remark 4.23. *The complex irreducible projective representations of the Poincare group with finite mass split into positive and negative energy representations, which are complex conjugate of each other. They are labeled by one number j, with $2j$ being a natural number. The positive energy representation spaces V_j are, up to isomorphisms, written as a symmetric tensor product of Dirac spinor fields defined on the 3-momentum space, verifying $(\gamma^0)_k \Psi_j(\vec{p}) = \Psi_j(\vec{p})$. The matrices with the index k apply in the corresponding spinor index of Ψ_j.*

The representation space V_0 is, up to isomorphisms, written in a Majorana basis as a complex scalar defined on the 3-momentum space.

The representation map is given by:

$$L_S\{\Psi\}(\vec{p}) = \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}} \prod_{k=1}^{2j} (\alpha^{-1}_{\Lambda(p)} S \alpha_p)_k \Psi(\vec{\Lambda}^{-1}(p))$$
$$T_a\{\Psi\}(\vec{p}) = e^{-ip\cdot a}\Psi(\vec{p})$$

Where $\alpha_p = \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2m}}$.

Proposition 4.24. *The real irreducible projective representations of the Poincare group with finite mass are labeled by one number j, with $2j$ being a natural number. The representation spaces W_j are, up to isomorphisms, written as a symmetric tensor product of Majorana spinor fields defined on the 3-momentum space, verifying $(i\gamma^0)_k\Psi_j(\vec{p}) = (i\gamma^0)_1\Psi_j(\vec{p})$. The matrices with the index k apply in the corresponding spinor index of Ψ_j.*

The representation space V_0 is, up to isomorphisms, written in a Majorana basis as a real scalar defined on the 3-momentum space, times the identity matrix of a Majorana spinor space.

The representation map is given by:

$$L_S\{\Psi\}(\vec{p}) = \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}} \prod_{k=1}^{2j} (\alpha^{-1}_{\Lambda(p)} S \alpha_p)_k \Psi(\vec{\Lambda}^{-1}(p))$$
$$T_a\{\Psi\}(\vec{p}) = e^{-i\gamma^0 p\cdot a}\Psi(\vec{p})$$

Remark 4.25. *The complex irreducible projective representations of the Poincare group with null mass split into positive and negative energy representations, which are complex conjugate of each other. They are labeled by one number j, with $2j$ being an integer number. The positive energy representation spaces V_j are, up to isomorphisms, written as a symmetric tensor product of Dirac spinor fields defined on the 3-momentum space, verifying $(\gamma^0)_k\Psi_j(\vec{p}) = \Psi_j(\vec{p})$ and $(\gamma^3\gamma^5)_k\Psi_j(\vec{p}) = \pm\Psi_j(\vec{p})$, with the plus sign if j is positive and the minus sign if j is negative.*

The representation space V_0 is, up to isomorphisms, written in a Majorana basis as a scalar defined on the 3-momentum space.

The representation map is given by:

$$L_S\{\Psi\}(\vec{p}) = \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}} \prod_{k=1}^{2j} (e^{i\gamma^0\gamma^3\gamma^5\theta})_k \Psi(\vec{\Lambda}^{-1}(p))$$
$$T_a\{\Psi\}(\vec{p}) = e^{-ip\cdot a}\Psi(\vec{p})$$

Where θ is the angle of the rotation of the little group $SE(2)$.

Remark 4.26. *The real irreducible projective representations of the Poincare group with null mass are labeled by one number j, with $2j$ being an integer number. The positive energy representation spaces V_j are, up to isomorphisms, written as a symmetric tensor product of Majorana spinor fields defined on the 3-momentum space, verifying $(i\gamma^0)_k\Psi_j(\vec{p}) = (i\gamma^0)_1\Psi_j(\vec{p})$ and $(\gamma^3\gamma^5)_k\Psi_j(\vec{p}) = \pm\Psi_j(\vec{p})$, with the plus sign if j is positive and the minus sign if j is negative.*

The representation space V_0 is, up to isomorphisms, written in a Majorana basis as the realification of the complex functions defined on the 3-momentum space, with the operator correspondent to the imaginary unit given by the matrix $i\gamma^0$ of a Majorana spinor space.

The representation map is given by:

$$\begin{aligned} L_S\{\Psi\}(\vec{p}) &= \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}}\prod_{k=1}^{2j}(e^{i\gamma^0\gamma^3\gamma^5\theta})_k\Psi(\vec{\Lambda}^{-1}(p)) \\ T_a\{\Psi\}(\vec{p}) &= e^{-i\gamma^0 p\cdot a}\Psi(\vec{p}) \end{aligned}$$

Where θ is the angle of the rotation of the little group $SE(2)$.

4.3. Localization

We can apply Fourier-Majorana transforms to the unitary representations, going from the momenta to the coordinate space.

For $m > 0$, given a Lorentz transformation:

$$\begin{aligned} L_S\{\Psi\}(x) = S\Psi(\Lambda^{-1}(x)) &= \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} S\frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{-i\gamma^0\Lambda(p)\cdot x}\Psi(\vec{p}) \\ &= \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} \frac{\not{\Lambda}(p)\gamma^0 + m}{\sqrt{\Lambda^0(p)+m}\sqrt{2\Lambda^0(p)}} e^{-i\gamma^0\Lambda(p)\cdot x} R\sqrt{\frac{\Lambda^0(p)}{E_p}}\Psi(\vec{p}) \\ &= \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} \frac{(\Lambda^{-1})^0(p)}{E_p}\frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{-i\gamma^0 p\cdot x} R\sqrt{\frac{E_p}{(\Lambda^{-1})^0(p)}}\Psi(\vec{\Lambda}^{-1}(p)) \end{aligned}$$

Then:

$$\mathcal{F}_M \circ L_S\{\Psi\}(x^0,\vec{p}) = e^{-i\gamma^0 E_p x^0} R\sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}}\Psi(\vec{\Lambda}^{-1}(p))$$

Where $R = (\alpha^{-1}_{\Lambda(p)}S\alpha_p)$.

For $m = 0$, given a Lorentz transformation:

$$\begin{aligned} L_S\{\Psi\}(x) = S\Psi(\Lambda^{-1}(x)) &= \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} S\frac{\not{p}\gamma^0}{E_p\sqrt{2}} R_p e^{-i\gamma^0\Lambda(p)\cdot x}\Psi(\vec{p}) \\ &= \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} \frac{(\Lambda^{-1})^0(p)}{E_p}\frac{\not{p}\gamma^0}{E_p\sqrt{2}} R_p e^{-i\gamma^0 p\cdot x} R\sqrt{\frac{E_p}{(\Lambda^{-1})^0(p)}}\Psi(\vec{\Lambda}^{-1}(p)) \end{aligned}$$

Then:

$$\mathcal{F}_M \circ L_S\{\Psi\}(x^0, \vec{p}) = e^{-i\gamma^0 E_p x^0} R \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}} \Psi(\vec{\Lambda}^{-1}(p))$$

Where $R = e^{i\gamma^5\gamma^3\gamma^0\theta}$.

Definition 4.27. Consider a measurable space (X, M), where M is a σ-algebra of subsets of X. A projection-valued-measure, π, is a map from M to the set of self-adjoint projections on a Hilbert space H such that $\pi(X)$ is the identity operator on H and the function $< \psi, \pi(A)\psi >$, with $A \in M$ is a measure on M, for all $\psi \in H$.

Definition 4.28. Suppose now that X is a representation of G. Then, a system of imprimitivity is a pair (U, π), where π is a projection valued measure and U an unitary representation of G on the Hilbert space H, such that $U(g)\pi(A)U^{-1}(g) = \pi(gA)$.

Remark 4.29 (Theorem 6.12 of [3])**.** *There is a one-to-one correspondence between the complex system of imprimitivity (U,P), based on* $\mathbb{R}^3$*, and the representations of* $SU(2)$*. The system (U,P) is equivalent to the system induced by the representation of* $SU(2)$*.*

Certainly, an unitary representation of the Poincare group is also an unitary representation of the Euclidean group. In order to be localizable, we want it to be a system of imprimitivity (U,P) based on $\mathbb{R}^3$ for the Euclidean group. In coordinate space, the characteristic function χ_A where A is a measurable subset of $\mathbb{R}^3$ is a projection valued measure of a system of imprimitivity. The equality $e^{-iH_1t}\chi_A\Psi = e^{-iH_2t}\chi_A\Psi$ holds almost everywhere, then, since e^{-iH_1t} is bounded, there is no problem in the boundary of the subset A which has null measure.

Then we obtain that all the irreducible real representations of the full Poincare group with discrete spin or helicity are localizable. For the complex representations, for $m = 0$, there is one projector γ^5(in coordinate space) which is a Casimir operator, therefore, for anti-unitary parity and time reversal operations, the irreducible complex (anti-)unitary representations of the full Poincare group with $m = 0$ and discrete helicity are localizable.

Remark 4.30 (Corollary 9.15 of [3])**.** *Let (V,P0) be a complex system of imprimitivity based on R3 for the group M. Then, an arbitrary projection valued measure P based on R3 has the property that (V,P) is also a system of imprimitivity for M if and only if there exists a unitary operator Y commuting with V such that P=Y P0 Y-1.*

Assuming the Dirac representation of the Poincare group, for a real irreducible representation of the Poincare group, since the representation is R-complex irreducible, $Y = e^{i\gamma^0\varphi}$. Upon complexification, $Y = e^{i\theta}e^{i\gamma^0\varphi}$. Now we can demand Y to be real and then to commute with the time inversion operator, and we get $Y = 1$.

The system of imprimitivity (U,P), based on $\mathbb{R}^3$, is Poincare covariant because for time $x^0 = 0$ at point $\vec{x} = 0$, we have for the Lorentz group $L\{\Psi\}(0) = S\Psi(0)$.

The system of imprimitivity (U,P), based on $\mathbb{R}^3$ is compatible with causality because the propagator $\Delta(x) = 0$ for $x^2 < 0$ (space-like x), where the propagator is defined for spin or helicity 1/2 as:

$$\Delta(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}} e^{-i\gamma^0 p\cdot x} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}} \tag{26}$$

And verifies:

$$\Psi(x) = \int d^3\vec{y}\Delta(x-y)\Psi(y) \tag{27}$$

To show it we just need to do a Lorentz transformation such that $x^0 = 0$ and then show that $\Delta((0,\vec{x})) = 0$ for $\vec{x} \neq 0$.

Therefore a localization for real unitary irreducible representations of the Poincare group with discrete spin and helicity, compatible with Poincare covariance and causality exists.

5. Conclusion

The complex irreducible representations are not a generalization of the real irreducible representations, in the same way that the complex numbers are a generalization of the real numbers. There is a map, one-to-one or two-to-one and surjective up to equivalence, from the complex to the real irreducible representations of a Lie group on a Hilbert space.

All the real finite-dimensional projective representations of the restricted Lorentz group are also projective representations of the full Lorentz group, in contrast with the complex representations which are not all projective representations of the full Lorentz group.

We obtained all the real unitary irreducible projective representations of the Poincare group, with discrete spin, as real Bargmann-Wigner fields. For each pair of complex representations with positive/negative energy, there is one real representation. The Majorana-Fourier and Majorana-Hankel unitary transforms of the real Bargmann-Wigner fields relate the coordinate space with the linear and angular momenta spaces. The localization of the real Bargmann-Wigner fields, compatible with causality and Poincare covariance, exists.

We might be interested in the position as an observable. Now the question is, given an irreducible representation of the Poincare group, should the position be invariant under a $U(1)$ symmetry? Unfortunately, everyone known to the author that studied this problem assumed that it should. But the answer a priori is no it should not, because the $U(1)$ is related with the gauge symmetry which is a local symmetry and it would be useful to have a well defined notion of localization before we start considering local symmetries. The localization problems[3, 25–29], only appear in real Hilbert spaces if we require that all the observables are invariant under a U(1) symmetry—usually associated with the charge—, this is related with the result in quantum field theory that causality requires the existence of anti-particles[6].

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