On the real representations of the Poincare group

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Abstract

DRAFT VERSION

We study the real representations of the Poincare group and its relation with the complex representations. The classical electromagnetic field — from which the Poincare group was originally defined — is a real representation of the Poincare group.

We show that there is a map from the complex to the real irreducible representations of a Lie group on a Hilbert space — the map is known in the finite-dimensional representations of a real Lie algebra.

We show that all the finite-dimensional real representations of the restricted Lorentz group are also representations of the full Lorentz group, in contrast with many complex representations.

We study the unitary irreducible representations of the Poincare group with discrete spin and show that for each pair of complex representations with positive/negative energy, there is one real representation; we show that there are unitary transformations, defining linear and angular momenta spaces which are common for the real and complex representations.

Contents

1. Introduction

1.1. Motivation

Henri Poincaré defined the Poincare group as the set of transformations that leave invariant the Maxwell equations for the classical electromagnetic field. The classical electromagnetic field is a real representation of the Poincare group.

The complex representations of the Poincare group were systematically studied[1–6] and used in the definition of quantum fields[7, 8]. These studies were very important in the evolution of the role of symmetry in the Quantum Theory[9], which is based on complex Hilbert spaces[10].

We could not find in the literature a systematic study on the real representations of the Poincare group — even though representation theory[11, 12] and Quantum Theory [13–19] on real Hilbert spaces were investigated before — as it seems to be a common assumption that all fields of all modern theories must be quantum fields and therefore, somehow, every consistent representation must be complex. However, due to the existence of a map between real and complex representations, the motivation for this study is independent of the validity of such assumption.

The reasons motivating this study are:

1) The real representations of the Poincare group play a main role in the classical electromagnetism and general relativity[20]. It is reasonable to think that the real representations of the Poincare group will still play an important role in the most modern theories based on the classical electromagnetism and general relativity. As an example, the self-adjoint quantum fields — such as the Higgs boson, Majorana fermion or quantum electromagnetic field — transform as real representations under the action of the Poincare group[8].

2) The parity — included in the full Poincare group — and charge-parity transformations are not symmetries of the Electroweak interactions[21]. It is not clear why the charge-parity is an apparent symmetry of the Strong interactions[22] or how to explain the matter-antimatter asymmetry[23] through the charge-parity violation. We will show that that all the finite-dimensional real representations of the restricted Lorentz group are also representations of the parity; and that there are linear and angular momenta spaces which are common for the real and complex representations of the Poincare group, therefore independent of the charge and matter-antimatter properties. These results may be useful in future studies of the parity and charge-parity violations.

1.2. On the map from the complex to the real irreducible representations of a group

Many representations of a group— such as the finite-dimensional representations of semisimple Lie groups[24] or the unitary representations of separable locally compact groups[25] — are direct sums (or integrals) of irreducible representations.

The study of irreducible representations on complex Hilbert spaces is in general easier than on real Hilbert spaces, because the field of complex numbers is the algebraic closure — where any polynomial equation has a root — of the field of real numbers. Given a real Hilbert space, we can always obtain a complex Hilbert space through complexification — extension of the scalar multiplication to include multiplication by complex numbers.

Yet, given an irreducible representation on a real Hilbert space V, the representation on the complex Hilbert space resulting from the complexification of V may be reducible, because there is a 2-dimensional real representation of the field of complex numbers. Therefore, the complex representations are not a generalization of the real representations, in the same way that the complex numbers are a generalization of the real numbers.

There is a well studied map, one-to-one or two-to-one and surjective up to equivalence, from the complex to the real linear finite-dimensional irreducible representations of a real Lie algebra[11, 26]. In Section 2, we show that there is a similar map from the complex to the real irreducible representations of a Lie group on a Hilbert space. The proof is similar to the known proofs[11] but, instead of the second form of Schur's lemma, it uses the first form of Schur's lemma, valid for representations on infinite-dimensional Hilbert spaces.

Related studies, but for less general types of groups or representations, can be found in the references[12–15, 27].

1.3. Finite-dimensional representations of the Lorentz group

The Poincare group, also called inhomogeneous Lorentz group, is the semi-direct product of the translations and Lorentz Lie groups[24]. Whether or not the Lorentz and Poincare groups include the parity and time reversal transformations depends on the context and authors. To be clear, we use the prefixes full/restricted when including/excluding parity and time reversal transformations. The Pin(3,1)/SL(2,C) groups are double covers of the full/restricted Lorentz group. The semi-direct product of the translations with the Pin(3,1)/SL(2,C) groups is called IPin(3,1)/ISL(2,C) Lie group — the letter (I) stands for inhomogeneous.

A projective representation of the Poincare group on a complex/real Hilbert space is an homomorphism, defined up to a complex phase/sign, from the group to the automorphisms of the Hilbert space. Since the IPin(3,1) group is a double cover of the full Poincare group, their projective representations are the same[28]. All finite-dimensional projective representations of a simply connected group, such as SL(2,C), are well defined representations[5]. Both SL(2,C) and Pin(3,1) are semi-simple Lie groups, and so all its finite-dimensional representations are direct sums of irreducible representations[24]. Therefore, the study of the finite-dimensional projective representations of the restricted Lorentz group reduces to the study of the finite-dimensional irreducible representations of SL(2,C).

The Dirac spinor is an element of a 4 dimensional complex vector space, while the Majorana spinor is an element of a 4 dimensional real vector space[29–32]. The complex finite-dimensional irreducible representations of SL(2,C) can be written as linear combinations of tensor products of Dirac spinors.

In Section 3 we will review the Pin(3,1) and SL(2,C) semi-simple Lie groups and its relation with the Majorana, Dirac and Pauli matrices. We will obtain all the real finite-dimensional irreducible representations of SL(2,C) as linear combinations of tensor products of Majorana spinors, using the map from Section 2. Then we will check that all these real representations are also projective representations of the full Lorentz group, in contrast with the complex representations which are not all projective representations of the full Lorentz group.

1.4. Unitary representations of the Poincare group

According to Wigner's theorem, the most general transformations, leaving invariant the modulus of the internal product of a Hilbert space, are: unitary or anti-unitary operators, defined up to a complex phase, for a complex Hilbert; unitary, defined up to a signal, for a real Hilbert[33]. This motivates the study of the (anti-)unitary projective representations of the full Poincare group.

All (anti-)unitary projective representations of ISL(2,C) are, up to isomorphisms, well defined unitary representations, because ISL(2,C) is simply connected[5]. Both ISL(2,C) and IPin(3,1) are separable locally compact groups and so all its (anti-)unitary projective representations are direct integrals of irreducible representations[25]. Therefore, the study of the (anti-)unitary projective representations of the restricted Poincare group reduces to the study of the unitary irreducible representations of ISL(2,C).

The spinor fields, space-time dependent spinors, are solutions of the free Dirac equation[34]. The real/complex Bargmann-Wigner fields[35, 36], space-time dependent linear combinations of tensor products of Majorana/Dirac spinors, are solutions of the free Dirac equation in each tensor index. The complex unitary irreducible projective representations of the Poincare group with discrete spin can be written as complex Bargmann-Wigner fields.

In Section 4, we will obtain all the real unitary irreducible projective representations of the Poincare group, with discrete spin, as real Bargmann-Wigner fields, using the map from Section 2. For each pair of complex representations with positive/negative energy, there is one real representation. We will define the Majorana-Fourier and Majorana-Hankel unitary transforms of the real or complex Bargmann-Wigner fields. Then we relate the Majorana transforms to the linear and angular momenta of a representation of the Poincare group.

The free Dirac equation is diagonal in the Newton-Wigner representation[37], related to the Dirac representation through a Foldy-Wouthuysen transformation[38, 39] of Dirac spinor fields. The Majorana-Fourier transform, when applied on Dirac spinor fields, is related with the Newton-Wigner representation and the Foldy-Wouthuysen transformation. In the context of Clifford Algebras, there are studies on the geometric square roots of -1 [17, 18, 40] and on the generalizations of the Fourier transform[41], with applications to image processing[42].

2. On the map from the complex to the real irreducible representations of a group

2.1. Representations on real and complex Hilbert spaces

Definition 2.1. A representation (M_G, V) of a Lie group G[43] on a real or complex Hilbert space V is defined by:

1) the representation space V, which is an Hilbert space;

2) the representation group homomorphism $M : G \to B(V)$ from the group elements to the bounded automorphisms with a bounded inverse, such that the map $M' : G \times V \to V$ defined by $M'(g, v) \equiv M(g)v$ is continuous.

Definition 2.2. Let V_n, with $n \in \{1,2\}$, be two Hilbert spaces. The representations $(M_{n,G}, V_n)$ of a group G on the Hilbert spaces V_n are equivalent iff there is a linear bijection $\alpha : V_1 \to V_2$ such that for all $g \in G$, $\alpha \circ M_{1,G}(g) = M_{2,G}(g) \circ \alpha$.

Definition 2.3. Consider a representation (M_G, V). An equivariant endomorphism of (M_G, V) is an endomorphism of V commuting with $M_G(g)$, for all $g \in G$.

Definition 2.4. Let W be a closed linear subspace of V. (M_G, W) is a (topological) sub-representation of (M_G, V) iff W is invariant under the group action, that is, for all $w \in W$: $(M(g)w) \in W$, for all $g \in G$.

Definition 2.5. A representation (M_G, V) is (topologically) irreducible iff their only sub-representations are the non-proper or trivial sub-representations: (M_G, V) and $(M_G, \{0\})$, where $\{0\}$ is the null space. An irreducible representation is called irrep.

Lemma 2.6 (Schur's lemma)**.** *If the representation (M_G, V) is irreducible then the equivariant bounded endomorphisms of (M_G, V) are either automorphisms or the null map.*

Proof. Let T be an equivariant endomorphism of (M_G, V). Then (M_G, V_T) is a subrepresentations, where $V_T \equiv \{v \in V : Tv = 0\}$. The kernel of a bounded linear operator is a closed subspace. If (M_G, V) is irreducible, then either $V_T = V$ or $V_T = \{0\}$. If $V_T = V$ then T is the null map. If $V_T = \{0\}$, since T is linear, $Tu = Tv$ implies $u = v$ for $u, v \in V$; therefore T is one-to-one and hence an automorphism. □

Definition 2.7. Consider a representation (M_G, V) on a complex Hilbert space. A C-conjugation operator of (M_G, V) is an anti-linear bounded involution of V commuting with $M_G(g)$, for all $g \in G$.

Lemma 2.8. *Consider an irreducible representation (M_G, V) on a complex Hilbert space. A C-conjugation operator of (M_G, V), if it exists, is unique up to an isomorphism.*

Proof. Let θ_1,θ_2 be two C-conjugation operators of (M_G, V). Since (M_G, V) is irreducible, the equivariant endomorphism $T \equiv (1 + \theta_2\theta_1)$ is either an automorphism or the null map.

If T is the null map, then $\theta_2 = -\theta_1 = \alpha\theta_1\alpha^{-1}$; where the equivariant automorphism of (M_G, V), α, is defined by $\alpha v \equiv iv$ for $v \in V$.

If T is an automorphism, then $\theta_2 = T\theta_1 T^{-1}$. □

Definition 2.9. Consider a representation (M_G, V) on a complex Hilbert space.

The representation is C-real iff there is an C-conjugation operator. The subset of C-real irreducible representations is $R_G(\mathbb{C})$.

The representation is C-pseudoreal iff there is no C-conjugation operator but there is an equivariant anti-automorphism of (M_G, V) the representation is equivalent to its complex conjugate. The subset of C-pseudoreal irreducible representations is $P_G(\mathbb{C})$.

The representation is C-complex iff there is there is no equivariant anti-automorphism of (M_G, V). The subset of C-complex irreducible representations is $C_G(\mathbb{C})$.

Definition 2.10. Consider a representation (M_G, W) on a real Hilbert space. A R-imaginary operator of (M_G, W), J, is an equivariant bounded automorphism of (M_G, W) verifying $J^2 = -1$.

Lemma 2.11. *Consider an irreducible representation (M_G, W) on a real Hilbert space. A R-imaginary operator of (M_G, W), if it exists, is unique up an isomorphism or a sign.*

Proof. Let J_1,J_2 be two R-imaginary operators of (M_G, W). Since (M_G, W) is irreducible, the equivariant endomorphism $T \equiv (1 - J_2 J_1)$ is either an automorphism or the null map. If T is the null map, then $J_2 = -J_1$. If T is an automorphism, then $J_2 = T J_1 T^{-1}$. □

Definition 2.12. Consider the irreducible representation (M_G, W) on a real Hilbert space.

The representation is R-real iff there is no R-imaginary operator. The subset of R-real irreducible representations is $R_G(\mathbb{R})$.

The representation is R-pseudoreal iff there is a R-imaginary operator, unique up to an isomorphism. The subset of R-pseudoreal irreducible representations is $P_G(\mathbb{R})$.

The representation is R-complex iff there is a R-imaginary operator, non-unique up to an isomorphism. The subset of R-complex irreducible representations is $C_G(\mathbb{R})$.

Definition 2.13. Consider a representation (M_G, W) on a real Hilbert space. The representation (M_G, W^c) is the complexification of the representation (M_G, W), defined as $W^c \equiv \mathbb{C} \otimes W$, with the multiplication by scalars such that $a(bw) \equiv (ab)w$ for $a, b \in \mathbb{C}$ and $w \in W$. The internal product of W^c is defined as:

$$< v_r + i v_i, u_r + i u_i >_c \equiv < v_r, u_r > + < v_i, u_i > + i < v_r, u_i > - i < v_i, u_r >$$

for $u_r, u_i, v_r, v_i \in W$ and $< v_r, u_r >$ is the internal product of W.

Definition 2.14. Consider a representation (M_G, V) on a complex Hilbert space. The representation (M_G, V^r) is the realification of the representation (M_G, V), defined as $V^r \equiv V$ is a real Hilbert space with the multiplication by scalars restricted to reals such that $a(v) \equiv (a + i0)v$ for $a \in \mathbb{R}$ and $v \in V$. The internal product of V^r is defined as

$$< v, u >_r \equiv \frac{< v, u > + < u, v >}{2}$$

for $u, v \in V$ and $< v, u >$ is the internal product of V.

2.2. The map from complex to real representations

Definition 2.15. Consider a group G. The map $\mathcal{M}$ is defined (up to equivalence) as:

$$\mathcal{M}(M_G, V) \equiv (M_G, V_\theta)$$

where (M_G, V) is a C-real irreducible representation on a complex Hilbert space V, θ is the C-conjugation operator (unique up to equivalence) of (M_G, V) and $V_\theta \equiv \{\frac{1+\theta}{2} v : v \in V\}$.

$$\mathcal{M}(M_G, V) \equiv (M_G, V^r)$$

where (M_G, V) is a C-pseudoreal or C-complex irreducible representation on a complex Hilbert space V; (M_G, V^r) is the realification of (M_G, V).

Proposition 2.16. *Consider a group G. $\mathcal{M}$ is a map from $R_G(\mathbb{C})$ to $R_G(\mathbb{R})$ and from $P_G(\mathbb{C}) \cup C_G(\mathbb{C})$ to $P_G(\mathbb{R}) \cup C_G(\mathbb{R})$.*

Proof. Consider an irreducible representation on a complex Hilbert space (M_G, V). There is a R-imaginary operator J of the representation (M_G, V^r), defined by $J(u) \equiv iu$, for $u \in V^r$.

Let (M_G, X^r) be a proper non-trivial subrepresentation of (M_G, V^r). Then J is an R-imaginary operator of (M_G, Y^r) and (M_G^r, Z^r), where $Y^r \equiv \{u + Jv : u, v \in X^r\}$ and $Z^r \equiv \{u : u, Ju \in X^r\}$. Then (M_G, Y) and (M_G, Z) are subrepresentations of (M_G, V), where the complex Hilbert spaces $Y \equiv Y^r$ and $Z \equiv Z^r$ have the scalar multiplication such that $(a+ib)(y) = ay + bJy$, for $a, b \in \mathbb{R}$ and $y \in Y$ or $y \in Z$. If $Y = \{0\}$, then $Z = X^r = \{0\}$ which is in contradiction with X^r being non-trivial. If $Z = V$, then $Y = V$ and $X^r = V^r$ which is in contradiction with X^r being non-trivial. So $Z = \{0\}$ and $Y = V$, which implies that $V = (X^r)^c$.

Then there is a C-conjugation operator of (M_G, V), θ, defined by $\theta(u + iv) \equiv u - iv$, for $u, v \in X^r$. We have $X^r = V_\theta$. Suppose there is a R-imaginary operator of (M_G, V_θ), J'. Then $(M_G, V_\pm)$, where $V_\pm \equiv \{\frac{1 \pm iJ'}{2} v : v \in V\}$, are proper non-trivial subrepresentations of (M_G, V), in contradiction with (M_G, V) being irreducible.

Therefore, if (M_G, V) is C-real, then (M_G, V_θ) is R-real irreducible. If (M_G, V) is C-pseudoreal or C-complex, then (M_G, V_θ^r) is R-pseudoreal or R-complex, irreducible. □

Proposition 2.17. *Consider a group G. $\mathcal{M}$ is a surjective map from $R_G(\mathbb{C})$ to $R_G(\mathbb{R})$ and from $P_G(\mathbb{C}) \cup C_G(\mathbb{C})$ to $P_G(\mathbb{R}) \cup C_G(\mathbb{R})$.*

Proof. Consider an irreducible representation (M_G, W) on a real Hilbert space. There is a C-conjugation operator of (M_G, W^c), θ, defined by $\theta(u + iv) \equiv (u - iv)$ for $u, v \in W$, verifying $(W^c)_\theta = W$.

Let (M_G, X^c) be a proper non-trivial subrepresentation of (M_G, W^c). Then θ is a C-conjugation operator of the subrepresentations (M_G, Y^c) and (M_G, Z^c), where $Y^c \equiv \{u + \theta v : u, v \in X^c\}$ and $Z^c \equiv \{u : u, \theta u \in X^c\}$. Therefore, $Y^c = \{u + iv : u, v \in Y\}$ and $Z^c = \{u + iv : u, v \in Z\}$, where $Y \equiv \{\frac{1+\theta}{2} u : u \in Y^c\}$ and $Z \equiv \{\frac{1+\theta}{2} u : u \in Z^c\}$ are invariant subspaces of W. If $Y = \{0\}$ then $Z = \{0\}$ and $Y^c = X^c = \{0\}$, in contradiction with X^c being non-trivial. If $Z = W$ then $Y = W$ and $Z^c = X^c = W^c$, in contradiction with X^c being proper. Therefore $Z = \{0\}$ and $Y = W$, which implies $Z^c = \{0\}$ and $Y^c = W^c$. So, (M_G, W) is equivalent to $(M_G, (X^c)^r)$, due to the existence of the bijective linear map $\alpha : (X^c)^r \to W$, $\alpha(u) = u + \theta u$, $\alpha^{-1}(u + \theta u) = u$, for $u \in (X^c)^r$.

There is a R-imaginary operator of (M_G, W), J, defined as $J(u + \theta u) \equiv (iu - i\theta u)$ for $u \in X^c$. We can check that $X^c = W_J \equiv \{\frac{1 - iJ}{2} w : w \in W^c\}$ and $\theta(W_J) = W_{-J}$. Suppose that there is a C-conjugation operator of (M_G, W_J), θ'. Then θ' anti-commutes with J and $(M_G, W_\pm)$ is a proper non-trivial subrepresentation of (M_G, W), where $W_\pm \equiv \{\frac{1 \pm \theta}{2} w : w \in W\}$, in contradiction with (M_G, W) being irreducible.

Therefore, if (M_G, W) is R-real, then (M_G, W^c) is C-real irreducible. We have for $V \equiv W^c$, $\mathcal{M}(M_G, V) = (M_G, V_\theta)$ is equivalent to (M_G, W).

If (M_G, W) is R-pseudoreal or R-complex, then (M_G, W_J) is C-pseudoreal or C-complex, irreducible. We have that for $V \equiv W_J$, $\mathcal{M}(M_G, V) = (M_G^r, V^r)$ is equivalent to (M_G, W). □

Proposition 2.18. *Consider a group G. Up to equivalence, $\mathcal{M}$ defines a one-to-one surjective map from $R_G(\mathbb{C})$ to $R_G(\mathbb{R})$ and from $P_G(\mathbb{C})$ to $P_G(\mathbb{R})$; and a two-to-one surjective map from $C_G(\mathbb{C})$ to $C_G(\mathbb{R})$.*

Proof. Consider an irreducible representation (M_G, W) on a real vector space. There is a C-conjugation operator of (M_G, W^c), θ.

If (M_G, W) is R-real, then (M_G, W^c) is C-real irreducible. Therefore the correspondence $V_\theta \equiv W$ is, up to isomorphisms, uniquely determined.

If (M_G, W) is R-pseudoreal or R-complex, then (M_G, W_J) is C-pseudoreal or C-complex irreducible. The correspondence $V \equiv W_J$ only depends on J, the R-imaginary operator of (M_G, W). J is unique up a sign and isomorphisms. There is an equivariant anti-automorphism of (M_G, V), S, iff there is an equivariant automorphism of (M_G, V^r). S exists iff the R-imaginary operator of (M_G, V^r), J, is unique up to equivalence. □

2.3. Unitary and completely reducible representations

Definition 2.19. A representation is completely reducible iff it can be expressed as a direct sum (or direct integral) of irreducible representations.

Remark 2.20 (Weyl theorem)**.** *All finite-dimensional representations of a semi-simple Lie group (such as SL(2,C)) are completely reducible.*

Remark 2.21. *Let H_n, with $n \in \{1, 2\}$, be two Hilbert spaces with internal products $<,>$: $H_n \times H_n \to \mathbb{F}$,($\mathbb{F} = \mathbb{R}, \mathbb{C}$). A linear operator $U : H_1 \to H_2$ is unitary iff:*

1) it is surjective;

2) for all $x \in H_1$, $< U(x), U(x) >=< x, x >$.

Remark 2.22. *Given two real Hilbert spaces H_1, H_2 and an unitary operator $U : H_1 \to H_2$, the inverse operator $U^{-1} : H_2 \to H_1$ is defined by:*

$$< x, U^{-1}y >=< Ux, y >, \ x \in H_1, y \in H_2$$

Proposition 2.23. *Let H_n, with $n \in \{1, 2\}$, be two complex Hilbert spaces and H_n^r its complexification. The following two statements are equivalent:*

1) The operator $U : H_1 \to H_2$ is unitary;

2) The operator $U^r : H_1^r \to H_2^r$ is unitary, where $U^r(h) \equiv U(h)$, for $h \in H_1$.

Proof. Since $< h, h >=< h, h >_r$ and $U^r(h) = U(h)$, for $h \in H_1$, we get the result. □

Remark 2.24. *All unitary representations of a separable locally compact group (such as the Poincare group) are completely reducible.*

3. Finite-dimensional representations of the Lorentz group

3.1. Majorana, Dirac and Pauli Matrices and Spinors

Definition 3.1. $\mathbb{F}^{m\times n}$ is the vector space of $m\times n$ matrices whose entries are elements of the field $\mathbb{F}$.

In the next remark we state the Pauli's fundamental theorem of gamma matrices. The proof can be found in the reference[44].

Remark 3.2 (Pauli's fundamental theorem)**.** *Let A^μ, B^μ, $\mu\in\{0,1,2,3\}$, be two sets of 4×4 complex matrices verifying:*

$$A^\mu A^\nu + A^\nu A^\mu = -2\eta^{\mu\nu} \tag{1}$$

$$B^\mu B^\nu + B^\nu B^\mu = -2\eta^{\mu\nu} \tag{2}$$

Where $\eta^{\mu\nu}\equiv diag(+1,-1,-1-1)$ is the Minkowski metric.

1) There is an invertible complex matrix S such that $B^\mu = SA^\mu S^{-1}$, for all $\mu\in\{0,1,2,3\}$. S is unique up to a non-null scalar.

2) If A^μ and B^μ are all unitary, then S is unitary.

Proposition 3.3. *Let α^μ, β^μ, $\mu\in\{0,1,2,3\}$, be two sets of 4×4 real matrices verifying:*

$$\alpha^\mu\alpha^\nu + \alpha^\nu\alpha^\mu = -2\eta^{\mu\nu} \tag{3}$$

$$\beta^\mu\beta^\nu + \beta^\nu\beta^\mu = -2\eta^{\mu\nu} \tag{4}$$

Then there is a real matrix S, with $|detS| = 1$, such that $\beta^\mu = S\alpha^\mu S^{-1}$, for all $\mu\in\{0,1,2,3\}$. S is unique up to a signal.

Proof. From remark 3.2, we know that there is an invertible matrix T', unique up to a non-null scalar, such that $\beta^\mu = T'\alpha^\mu T'^{-1}$. Then $T\equiv T'/|det(T')|$ has $|detT| = 1$ and it is unique up to a complex phase.

Conjugating the previous equation, we get $\beta^\mu = T^*\alpha^\mu T^{*-1}$. Then $T^* = e^{i2\theta}T$ for some real number θ. Therefore $S\equiv e^{i\theta}T$ is a real matrix, with $|detS| = 1$, unique up to a signal. $\square$

Definition 3.4. The Majorana matrices, $i\gamma^\mu$, $\mu\in\{0,1,2,3\}$, are 4×4 complex unitary matrices verifying:

$$(i\gamma^\mu)(i\gamma^\nu) + (i\gamma^\nu)(i\gamma^\mu) = -2\eta^{\mu\nu} \tag{5}$$

The Dirac matrices are $\gamma^\mu \equiv -i(i\gamma^\mu)$.

In the Majorana bases, the Majorana matrices are 4×4 real orthogonal matrices. An example of the Majorana matrices in a particular Majorana basis is:

$$\begin{aligned}
&i\gamma^1 = \begin{bmatrix} +1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & +1\end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0\\ 0 & 0 & 0 & +1\\ +1 & 0 & 0 & 0\\ 0 & +1 & 0 & 0\end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0\\ +1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\end{bmatrix}\\
&i\gamma^0 = \begin{bmatrix} 0 & 0 & +1 & 0\\ 0 & 0 & 0 & +1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0\\ +1 & 0 & 0 & 0\\ 0 & 0 & 0 & +1\\ 0 & 0 & -1 & 0\end{bmatrix} \quad = -\gamma^0\gamma^1\gamma^2\gamma^3
\end{aligned} \tag{6}$$

In reference [45] it is proved that the set of five anti-commuting 4×4 real matrices is unique up to isomorphisms. So, for instance, with 4×4 real matrices it is not possible to obtain the euclidean signature for the metric.

Definition 3.5. The Dirac spinor is a 4×1 complex column matrix, $\mathbb{C}^{4\times 1}$.

The space of Dirac spinors is a 4 dimensional complex vector space.

Lemma 3.6. *The charge conjugation operator Θ, is an anti-linear involution commuting with the Majorana matrices $i\gamma^\mu$. It is unique up to a complex phase.*

Proof. In the Majorana bases, the complex conjugation is a charge conjugation operator. Let Θ and Θ' be two charge conjugation operators operators. Then, $\Theta\Theta'$ is a complex invertible matrix commuting with $i\gamma^\mu$, therefore, from Pauli's fundamental theorem, $\Theta\Theta' = c$, where c is a non-null complex scalar. Therefore $\Theta' = c^*\Theta$ and from $\Theta'\Theta' = 1$, we get that $c^*c = 1$. $\square$

Definition 3.7. Let Θ be a charge conjugation operator.

The set of Majorana spinors, $Pinor$, is the set of Dirac spinors verifying the Majorana condition (defined up to a complex phase):

$$Pinor \equiv \{u \in \mathbb{C}^{4\times 1} : \Theta u = u\} \tag{7}$$

The set of Majorana spinors is a 4 dimensional real vector space. Note that the linear combinations of Majorana spinors with complex scalars do not verify the Majorana condition.

There are 16 linear independent products of Majorana matrices. These form a basis of the real vector space of endomorphisms of Majorana spinors, $End(Pinor)$. In the Majorana bases, $End(Pinor)$ is the vector space of 4×4 real matrices.

Definition 3.8. The Pauli matrices σ^k, $k \in \{1, 2, 3\}$ are 2×2 hermitian, unitary, anti-commuting, complex matrices. The Pauli spinor is a 2×1 complex column matrix. The space of Pauli spinors is denoted by $Pauli$.

The space of Pauli spinors, $Pauli$, is a 2 dimensional complex vector space and a 4 dimensional real vector space. The realification of the space of Pauli spinors is isomorphic to the space of Majorana spinors.

3.2. On the Lorentz, SL(2,C) and Pin(3,1) groups

Remark 3.9. *The Lorentz group, $O(1,3) \equiv \{\lambda \in \mathbb{R}^{4\times 4} : \lambda^T \eta \lambda = \eta\}$, is the set of real matrices that leave the metric, $\eta = diag(1, -1, -1, -1)$, invariant.*

The proper orthochronous Lorentz subgroup is defined by $SO^+(1,3) \equiv \{\lambda \in O(1,3) : det(\lambda) = 1, {\lambda^0}_0 > 0\}$. It is a normal subgroup. The discrete Lorentz subgroup of parity and time-reversal is $\Delta \equiv \{1, \eta, -\eta, -1\}$.

The Lorentz group is the semi-direct product of the previous subgroups, $O(1,3) = \Delta \ltimes SO^+(1,3)$.

Definition 3.10. The set Maj is the 4 dimensional real space of the linear combinations of the Majorana matrices, $i\gamma^\mu$:

$$Maj \equiv \{a_\mu i\gamma^\mu : a_\mu \in \mathbb{R},\ \mu \in \{0,1,2,3\}\} \tag{8}$$

Definition 3.11. $Pin(3,1)$ [28] is the group of endomorphisms of Majorana spinors that leave the space Maj invariant, that is:

$$Pin(3,1) \equiv \Big\{S \in End(Pinor) :\ |detS| = 1,\ S^{-1}(i\gamma^\mu)S \in Maj,\ \mu \in \{0,1,2,3\}\Big\} \tag{9}$$

Proposition 3.12. *The map $\Lambda : Pin(3,1) \to O(1,3)$ defined by:*

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu \equiv S^{-1}(i\gamma^\mu)S \tag{10}$$

is two-to-one and surjective. It defines a group homomorphism.

Proof. 1) Let $S \in Pin(3,1)$. Since the Majorana matrices are a basis of the real vector space Maj, there is an unique real matrix $\Lambda(S)$ such that:

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu = S^{-1}(i\gamma^\mu)S \tag{11}$$

Therefore, Λ is a map with domain $Pin(3,1)$. Now we can check that $\Lambda(S) \in O(1,3)$:

$$(\Lambda(S))^\mu{}_\alpha \eta^{\alpha\beta}(\Lambda(S))^\nu{}_\beta = -\frac{1}{2}(\Lambda(S))^\mu{}_\alpha\{i\gamma^\alpha, i\gamma^\beta\}(\Lambda(S))^\nu{}_\beta = \tag{12}$$

$$= -\frac{1}{2}S\{i\gamma^\mu, i\gamma^\nu\}S^{-1} = S\eta^{\mu\nu}S^{-1} = \eta^{\mu\nu} \tag{13}$$

We have proved that Λ is a map from $Pin(3,1)$ to $O(1,3)$.

2) Since any $\lambda \in O(1,3)$ conserve the metric η, the matrices $\alpha^\mu \equiv \lambda^\mu{}_\nu i\gamma^\nu$ verify:

$$\{\alpha^\mu, \alpha^\nu\} = -2\lambda^\mu{}_\alpha \eta^{\alpha\beta}\lambda^\nu{}_\beta = -2\eta^{\mu\nu} \tag{14}$$

In a basis where the Majorana matrices are real, from Proposition 3.3 there is a real invertible matrix S_λ, with $|detS_\Lambda| = 1$, such that $\lambda^\mu{}_\nu i\gamma^\nu = S_\lambda^{-1}(i\gamma^\mu)S_\lambda$. The matrix S_Λ is unique up to a sign. So, $\pm S_\lambda \in Pin(3,1)$ and we proved that the map $\Lambda : Pin(3,1) \to O(1,3)$ is two-to-one and surjective.

3) The map defines a group homomorphism because:

$$\Lambda^\mu{}_\nu(S_1)\Lambda^\nu{}_\rho(S_2)i\gamma^\rho = \Lambda^\mu{}_\nu S_2^{-1}i\gamma^\nu S_2 \tag{15}$$

$$= S_2^{-1}S_1^{-1}i\gamma^\mu S_1 S_2 = \Lambda^\mu{}_\rho(S_1S_2)i\gamma^\rho \tag{16}$$

□

Remark 3.13. *The group $SL(2,\mathbb{C}) = \{e^{\theta^j i\sigma^j + b^j\sigma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$ is simply connected. Its projective representations are equivalent to its ordinary representations[5].*

There is a two-to-one, surjective map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$, defined by:

$$\Upsilon^\mu{}_\nu(T)\sigma^\nu \equiv T^\dagger\sigma^\mu T \tag{17}$$

Where $T \in SL(2,\mathbb{C})$, $\sigma^0 = 1$ and σ^j, $j \in \{1,2,3\}$ are the Pauli matrices.

Lemma 3.14. *Consider that* $\{M_+, M_-, i\gamma^5 M_+, i\gamma^5 M_-\}$ *and* $\{P_+, P_-, iP_+, iP_-\}$ *are orthonormal basis of the 4 dimensional real vector spaces Pinor and Pauli, respectively, verifying:*

$$\gamma^0\gamma^3 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm \tag{18}$$

The isomorphism $\Sigma : Pauli \to Pinor$ *is defined by:*

$$\Sigma(P_+) = M_+,\ \Sigma(iP_+) = i\gamma^5 M_+ \tag{19}$$
$$\Sigma(P_-) = M_-,\ \Sigma(iP_-) = i\gamma^5 M_- \tag{20}$$

The group $Spin^+(3,1) \equiv \{\Sigma \circ A \circ \Sigma^{-1} : A \in SL(2,\mathbb{C})\}$ *is a subgroup of* $Pin(1,3)$*. For all* $S \in Spin^+(1,3)$*,* $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$*.*

Proof. From remark 3.13, $Spin^+(3,1) = \{e^{\theta^j i\gamma^5\gamma^0\gamma^j + b^j\gamma^0\gamma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$. Then, for all $T \in SL(2,C)$:

$$-i\gamma^0 \Sigma \circ T^\dagger \circ \Sigma^{-1} i\gamma^0 = \Sigma \circ T^{-1} \circ \Sigma^{-1} \tag{21}$$

Now, the map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$ is given by:

$$\Upsilon^\mu{}_\nu(T) i\gamma^\nu = (\Sigma \circ T^{-1} \circ \Sigma^{-1}) i\gamma^\mu (\Sigma \circ T \circ \Sigma^{-1}) \tag{22}$$

Then, all $S \in Spin^+(3,1)$ leaves the space Maj invariant:

$$S^{-1} i\gamma^\mu S = \Upsilon^\mu{}_\nu(\Sigma^{-1} \circ S \circ \Sigma) i\gamma^\nu \in Maj \tag{23}$$

Since all the products of Majorana matrices, except the identity, are traceless, then $det(S) = 1$. So, $Spin^+(3,1)$ is a subgroup of $Pin(1,3)$ and $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$. □

Definition 3.15. The discrete Pin subgroup $\Omega \subset Pin(3,1)$ is:

$$\Omega \equiv \{\pm 1, \pm i\gamma^0, \pm\gamma^0\gamma^5, \pm i\gamma^5\} \tag{24}$$

The previous lemma and the fact that Λ is continuous, implies that $Spin^+(1,3)$ is a double cover of $SO^+(3,1)$. We can check that for all $\omega \in \Omega$, $\Lambda(\pm\omega) \in \Delta$. That is, the discrete Pin subgroup is the double cover of the discrete Lorentz subgroup. Therefore, $Pin(3,1) = \Omega \ltimes Spin^+(1,3)$

Since there is a two-to-one continuous surjective group homomorphism, $Pin(3,1)$ is a double cover of $O(1,3)$, $Spin^+(3,1)$ is a double cover of $SO^+(1,3)$ and $Spin^+(1,3) \cap SU(4)$ is a double cover of $SO(3)$. We can check that $Spin^+(1,3) \cap SU(4)$ is equivalent to $SU(2)$.

3.3. Finite-dimensional representations of SL(2,C)

Remark 3.16. *Since SL(2,C) is a semisimple Lie group, all its finite-dimensional (real or complex) representations are direct sums of irreducible representations.*

Remark 3.17. *The finite-dimensional complex irreducible representations of SL(2,C) are labeled by (m,n), where $2m, 2n$ are natural numbers. Up to equivalence, the representation space $V_{(m,n)}$ is the tensor product of the complex vector spaces V_m^+ and V_n^-, where $V_m^\pm$ is a symmetric tensor with $2m$ Dirac spinor indexes, such that ${\gamma^5}_k v = \pm v$, where $v \in V_m^\pm$ and ${\gamma^5}_k$ is the Dirac matrix γ^5 acting on the k-th index of v.*

The group homomorphism consists in applying the same matrix of $Spin^+(1,3)$, correspondent to the $SL(2,C)$ group element we are representing, to each index of v. $V_{(0,0)}$ is equivalent to $\mathbb{C}$ and the image of the group homomorphism is the identity.

These are also projective representations of the time reversal transformation, but, for $m \neq n$, not of the parity transformation, that is, under the parity transformation, $(V_m^+ \otimes V_n^-) \to (V_m^- \otimes V_n^+)$ and under the time reversal transformation $(V_m^+ \otimes V_n^-) \to (V_m^+ \otimes V_n^-)$.

Lemma 3.18. *The finite-dimensional real irreducible representations of SL(2,C) are labeled by (m,n), where $2m, 2n$ are natural numbers and $m \geq n$. Up to equivalence, the representation space $W_{(m,n)}$ is defined for $m \neq n$ as:*

$$W_{(m,n)} \equiv \{\frac{1+(i\gamma^5)_1 \otimes (i\gamma^5)_1}{2} w : w \in W_m \otimes W_n\}$$
$$W_{(m,m)} \equiv \{\frac{1+(i\gamma^5)_1 \otimes (i\gamma^5)_1}{2} w : w \in (W_m)^2\}$$

where W_m is a symmetric tensor with m Majorana spinor indexes, such that $(i\gamma^5)_1(i\gamma^5)_k w = -w$, where $w \in W_m$; $(i\gamma^5)_k$ is the Majorana matrix $i\gamma^5$ acting on the k-th index of w; $(W_m)^2$ is the space of the linear combinations of the symmetrized tensor products $(u \otimes v + v \otimes u)$, for $u, v \in W_m$.

The group homomorphism consists in applying the same matrix of $Spin^+(1,3)$, correspondent to the $SL(2,C)$ group element we are representing, to each index of the tensor. In the $(0,0)$ case, $W_{(0,0)}$ is equivalent to $\mathbb{R}$ and the image of the group homomorphism is the identity.

These are also projective representations of the full Lorentz group, that is, under the parity or time reversal transformations, $(W_{m,n} \to W_{m,n})$.

Proof. For $m \neq n$ the complex irreducible representations of SL(2,C) are C-complex. The complexification of $W_{(m,n)}$ verifies $W^c_{(m,n)} = (V_m^+ \otimes V_n^-) \oplus (V_m^- \otimes V_n^+)$.

For $m = n$ the complex irreducible representations of SL(2,C) are C-real. In a Majorana basis, the C-conjugation operator of $V_{(m,m)}$, θ, is defined as $\theta(u \otimes v) \equiv v^* \otimes u^*$, where $u \in V_m^+$ and $v \in V_m^-$. We can check that there is a bijection $\alpha : W_{(m,m)} \to (V_{(m,m)})_\theta$, defined by $\alpha(w) \equiv \frac{1-i(i\gamma^5)_1 \otimes 1}{2} w$; $\alpha^{-1}(v) \equiv v + v^*$, for $w \in W_{(m,m)}$, $v \in (V_{(m,m)})_\theta$.

Using the map from chapter 1, we can check that the representations $W_{(m,n)}$, with $m \geq n$, are the unique finite-dimensional real irreducible representations of SL(2,C), up to isomorphisms.

We can check that $W^c_{(m,n)}$ is equivalent to $W^c_{(n,m)}$, therefore, invariant under the parity or time reversal transformations. □

As examples of real irreducible representations of $SL(2,C)$ we have for $(1/2,0)$ the Majorana spinor, for $(1/2,1/2)$ the linear combinations of the matrices $\{1,\gamma^0\vec{\gamma}\}$, for $(1,0)$ the linear combinations of the matrices $\{i\vec{\gamma},\vec{\gamma}\gamma^5\}$. The group homomorphism is defined as $M(S)(u)\equiv Su$ and $M(S)(A)\equiv SAS^\dagger$, for $S\in Spin^+(1,3)$, $u\in Pinor$, $A\in\{1,\vec{\gamma}\gamma^0\}$ or $A\in\{i\vec{\gamma},\vec{\gamma}\gamma^5\}$.

We can check that the domain of M can be extended to $Pin(1,3)$, leaving the considered vector spaces invariant. For $m=n$, we can define the "pseudo-representation" $W'_{(m,m)}\equiv\{((i\gamma^5)_1\otimes 1)w : w\in W_{(m,m)}\}$ which is equivalent to $W_{(m,m)}$ as an $SL(2,C)$ representation, but under parity transforms with the opposite sign. As an example, the "pseudo-representation" $(1/2,1/2)$ is defined as the linear combinations of the matrices $\{i\gamma^5, i\gamma^5\vec{\gamma}\gamma^0\}$.

4. Unitary representations of the Poincare group

4.1. Hilbert spaces of Pauli and Majorana spinor fields

Definition 4.1. Consider that $\{M_+,M_-,i\gamma^0M_+,i\gamma^0M_-\}$ and $\{P_+,P_-,iP_+,iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces $Pinor$ and $Pauli$, respectively, verifying:

$$\gamma^3\gamma^5M_\pm=\pm M_\pm,\ \sigma^3P_\pm=\pm P_\pm$$

Let H be a real Hilbert space. For all $h\in H$, the bijective linear map $\Theta_H : Pauli\otimes_{\mathbb{R}}H\to Pinor\otimes_{\mathbb{R}}H$ is defined by:

$$\Theta_H(h\otimes_{\mathbb{R}}P_+)=h\otimes_{\mathbb{R}}M_+,\ \Theta_H(h\otimes_{\mathbb{R}}iP_+)=h\otimes_{\mathbb{R}}i\gamma^0M_+$$
$$\Theta_H(h\otimes_{\mathbb{R}}P_-)=h\otimes_{\mathbb{R}}M_-,\ \Theta_H(h\otimes_{\mathbb{R}}iP_-)=h\otimes_{\mathbb{R}}i\gamma^0M_-$$

Definition 4.2. Let H_n, with $n\in\{1,2\}$, be two real Hilbert spaces and $U : Pauli\otimes_{\mathbb{R}}H_1\to Pauli\otimes_{\mathbb{R}}H_2$ be an operator. The operator $U^\Theta : Pinor\otimes_{\mathbb{R}}H_1\to Pinor\otimes_{\mathbb{R}}H_2$ is defined as $U^\Theta\equiv\Theta_{H_2}\circ U\circ\Theta^{-1}_{H_1}$.

The space of Majorana spinors is isomorphic to the realification of the space of Pauli spinors.

Definition 4.3. The real Hilbert space $Pinor(\mathbb{X})\equiv Pinor\otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in $Pinor$.

Definition 4.4. The complex Hilbert space $Pauli(\mathbb{X})\equiv Pauli\otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in $Pauli$.

Remark 4.5. *The Fourier Transform $\mathcal{F}_P : Pauli(\mathbb{R}^3)\to Pauli(\mathbb{R}^3)$ is an unitary operator defined by:*

$$\mathcal{F}_P\{\psi\}(\vec{p})\equiv\int d^n\vec{x}\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^n}}\psi(\vec{x}),\ \psi\in Pauli(\mathbb{R}^3)$$

Where the domain of the integral is $\mathbb{R}^3$.

Remark 4.6. *The inverse Fourier transform verifies:*

$$-\vec{\partial}^2\ \mathcal{F}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{F}_P^{-1} \circ R)\{\psi\}(\vec{x})$$
$$i\vec{\partial}_k\ \mathcal{F}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{F}_P^{-1} \circ R'_k)\{\psi\}(\vec{x})$$

Where $\psi \in Pauli(\mathbb{R}^3)$ and $R, R'_k : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{R}^3)$, with $k \in \{1, 2, 3\}$, are linear maps defined by:

$$R\{\psi\}(\vec{p}) \equiv (\vec{p})^2\psi(\vec{p})$$
$$R'_k\{\psi\}(\vec{p}) \equiv \vec{p}_k\ \psi(\vec{p})$$

Definition 4.7. Let $\vec{x} \in \mathbb{R}^3$. The spherical coordinates parametrization is:

$$\vec{x} = r(\sin(\theta)\sin(\varphi)\vec{e_1} + \sin(\theta)\sin(\varphi)\vec{e_2} + \cos(\theta)\vec{e_3})$$

where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a fixed orthonormal basis of $\mathbb{R}^3$ and $r \in [0, +\infty[$, $\theta \in [0, \pi]$, $\varphi \in [-\pi, \pi]$.

Definition 4.8. Let

$$\mathbb{S}^3 \equiv \{(p, l, \mu) : p \in \mathbb{R}_{\geq 0}; l, \mu \in \mathbb{Z}; l \geq 0; -l \leq \mu \leq l\}$$

The Hilbert space $L^2(\mathbb{S}^3)$ is the real Hilbert space of real Lebesgue square integrable functions of $\mathbb{S}^3$. The internal product is:

$$< f, g >= \sum_{l=0}^{+\infty} \sum_{\mu=-l}^{l-1} \int_0^{+\infty} dp f(p, l, \mu) g(p, l, \mu),\ \ f, g \in L^2(\mathbb{S}^3)$$

Definition 4.9. The Spherical transform $\mathcal{H}_P : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}_P\{\psi\}(p, l, \mu) \equiv \int r^2 dr d(\cos\theta) d\varphi \frac{2p}{\sqrt{2\pi}} j_l(pr) Y_{l\mu}(\theta, \varphi)\psi(r, \theta, \varphi),\ \ \psi \in Pauli(\mathbb{R}^3)$$

The domain of the integral is $\mathbb{R}^3$. The spherical Bessel function of the first kind j_l [46], the spherical harmonics $Y_{l\mu}$[47] and the associated Legendre functions of the first kind $P_{l\mu}$ are:

$$j_l(r) \equiv r^l \Big(-\frac{1}{r}\frac{d}{dr}\Big)^l \frac{\sin r}{r}$$
$$Y_{l\mu}(\theta, \varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^\mu(\cos\theta) e^{i\mu\varphi}$$
$$P_l^\mu(\xi) \equiv \frac{(-1)^\mu}{2^l l!}(1-\xi^2)^{\mu/2} \frac{\mathrm{d}^{l+\mu}}{\mathrm{d}\xi^{l+\mu}}(\xi^2-1)^l$$

Remark 4.10. *Due to the properties of spherical harmonics and Bessel functions, the Spherical transform is an unitary operator. The inverse Spherical transform verifies:*

$$-\vec{\partial}^2\ \mathcal{H}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_P^{-1}\circ R)\{\psi\}(\vec{x})$$
$$(-x^1 i\partial_2 + x^2 i\partial_1)\ \mathcal{H}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_P^{-1}\circ R')\{\psi\}(\vec{x})$$

Where $\psi \in Pauli(\mathbb{S}^3)$ *and* $R, R' : Pauli(\mathbb{S}^3) \to Pauli(\mathbb{S}^3)$ *are linear maps defined by:*

$$R\{\psi\}(p,l,\mu) \equiv p^2\psi(p,l,\mu)$$
$$R'\{\psi\}(p,l,\mu) \equiv \mu\ \psi(p,l,\mu)$$

Definition 4.11. The real vector space $Pinor_j$, with $2j$ a positive integer, is the space of linear combinations of the tensor products of $2j$ Majorana spinors, symmetric on the spinor indexes. The real vector space $Pinor_0$ is the space of linear combinations of the tensor products of 2 Majorana spinors, anti-symmetric on the spinor indexes.

Definition 4.12. The real Hilbert space $Pinor_j(\mathbb{X}) \equiv Pinor_j \otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in $Pinor_j$.

Definition 4.13. The Hilbert space $Pinor_{j,n}$, with $(j-\nu)$ an integer and $-j \leq n \leq j$ is defined as:

$$Pinor_{j,n} \equiv \{\Psi \in Pinor_j : \sum_{k=1}^{k=2j}\left(\gamma^3\gamma^5\right)_k \Psi = 2n\Psi\}$$

Where $\left(\gamma^3\gamma^5\right)_k$ is the matrix $\gamma^3\gamma^5$ acting on the Majorana index k. Given

Definition 4.14. The Spherical transform $\mathcal{H}_M : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}_M\{\psi\}(p,l,J,\nu) \equiv \sum_{\mu=-l}^{l}\sum_{n=-j}^{j} < l\mu jn|J\nu > \left(\mathcal{H}_P^{\Theta}\right)_1\{\psi\}(p,l,\mu,n),\ \psi \in Pinor_j(\mathbb{R}^3)$$

$< l\mu jn|J\nu >$ are the Clebsh-Gordon coefficients and $\psi(p,l,\mu,n) \in Pinor_{j,n}$ such that $\psi(p,l,\mu) = \sum_{n=-j}^{j}\psi(p,l,\mu,n)$. $(j-n)$, $(J-\nu)$ and $(J-j)$ are integers, with $-J \leq \nu \leq J$ and $|j-l| \leq J \leq j+l$. $\left(\mathcal{H}_P^{\Theta}\right)_1$ is the realification of the transform $\mathcal{H}_P$, with the imaginary number replaced by the matrix $i\gamma^0$ acting on the first Majorana index of ψ.

Proposition 4.15. *Consider a unitary operator* $U : Pinor_j(X) \to Pinor_j(\mathbb{R}^3)$*, defined by* $U\{\Psi\}(\vec{x}) \equiv \int_{\mathbb{X}} dX U(\vec{x},X)\Psi(X)$ *and such that* $H^2 \circ U = U \circ E^2$*, where*

$$iH\{\Psi\}(\vec{x}) \equiv \left(\gamma^0\vec{\not\partial} + i\gamma^0 m\right)_k \Psi(\vec{x})$$

the Majorana matrices act on some Majorana index k; $E^2\{\Phi\}(X) \equiv E^2(X)\Phi(X)$ with $E(X) \geq m \geq 0$ a real number.

Then the operator $U' : Pinor(X) \to Pinor(\mathbb{R}^3)$ is unitary, where U' is defined by:

$$U'\{\Psi\}(\vec{x}) \equiv \int_{\mathbb{X}} dX \frac{E(X) + H(\vec{x})\gamma^0}{\sqrt{E(X)+m}\sqrt{2E(X)}} U(\vec{x}, X)\Psi(X)$$

Proof. We have that

$$< U'\{\Psi\}, U'\{\Psi\} >= \int d^3\vec{x} dX dY$$

$$\Psi^\dagger(Y)U^\dagger(\vec{x}, X)\frac{E(Y) + \gamma^0 H(\vec{x})}{\sqrt{E(Y)+m}\sqrt{2E(Y)}}\frac{E(X) + H(\vec{x})\gamma^0}{\sqrt{E(X)+m}\sqrt{2E(X)}}U(\vec{x}, X)\Psi(X)$$

From the symmetry of $H^2(\vec{x})$, in the integral we can set $E^2(X) = E^2(Y)$ and hence $E(X) = E(Y)$. Since we have:

$$\frac{E(X) + \gamma^0 H(\vec{x})}{\sqrt{E(X)+m}\sqrt{2E(X)}}\frac{E(X) + H(\vec{x})\gamma^0}{\sqrt{E(X)+m}\sqrt{2E(X)}} = 1$$

And U is unitary, we get $< U'\{\Psi\}, U'\{\Psi\} >=< \Psi, \Psi >$.

We also have that

$$< U^{-1}\{\Psi\}, U^{-1}\{\Psi\} >= \int dX d^3\vec{x} d^3\vec{y}$$

$$\Psi^\dagger(\vec{y})\frac{E(X) + H(\vec{y})\gamma^0}{\sqrt{E(X)+m}\sqrt{2E(X)}}U(\vec{y}, X)U^\dagger(\vec{x}, X)\frac{E(X) + \gamma^0 H(\vec{x})}{\sqrt{E(X)+m}\sqrt{2E(X)}}\Psi(\vec{x})$$

Since U is unitary, we get $< U'^{-1}\{\Psi\}, U'^{-1}\{\Psi\} >=< \Psi, \Psi >$. Therefore, U' is unitary. □

Definition 4.16. The complex Hilbert space $Dirac_j(\mathbb{X}) \equiv Pinor_j(\mathbb{X}) \otimes \mathbb{C}$ is the complexification of $Pinor_j(\mathbb{X})$.

4.2. On the Poincare, ISL(2,C) and IPin(3,1) groups

Definition 4.17. The $IPin(3,1)$ group is defined as the semi-direct product $Pin(3,1) \ltimes \mathbb{R}^4$, with the group's product defined as $(A,a)(B,b) = (AB, a + \Lambda(A)b)$, for $A, B \in Pin(3,1)$ and $a, b \in \mathbb{R}^4$ and $\Lambda(A)$ is the Lorentz transformation corresponding to A.

The $ISL(2,C)$ group is isomorphic to the subgroup of $IPin(3,1)$, obtained when $Pin(3,1)$ is restricted to $Spin^+(1,3)$. The full/restricted Poincare group is the representation of the $IPin(3,1)/ISL(2,C)$ group on Lorentz vectors, defined as $\{(\Lambda(A), a) : A \in Pin(3,1), a \in \mathbb{R}^4\}$.

4.3. Majorana spinor field representation of the Poincare group

Consider a Majorana spinor field $\Psi \in Pinor(\mathbb{R}^3)$. Let the Dirac Hamiltonian, H, be defined in the configuration space by:

$$iH\{\Psi\}(\vec{x}) \equiv (\gamma^0\vec{\gamma}\cdot\vec{\partial} + i\gamma^0 m)\Psi(\vec{x}),\ m \geq 0$$

In the Majorana-momentum space:

$$iH\{\Psi\}(\vec{p}) \equiv i\gamma^0 E_p \Psi(\vec{p})$$

The free Dirac equation is verified by:

$$(\partial_0 + iH)e^{-iHx^0}\{\Psi\} = 0$$

We can check that the free Dirac equation in the Majorana-momentum space is equal to the free Dirac equation in the Newton-Wigner representation[37], related to the Dirac representation through a Foldy-Wouthuysen transformation[38, 39] of Dirac spinor fields.

Definition 4.18. Given a Majorana spinor field $\Psi \in Pinor(\mathbb{R}^3)$, we define $\Psi(x) \equiv e^{-iHx^0}\{\Psi\}(\vec{x})$. The Majorana spinor field projective representation of the Poincare group is defined, up to a sign, as:

$$P(\Lambda_S, b)\{\Psi\}(x) \equiv \pm S\Psi(\Lambda_S^{-1}x + b)$$

Where $\Lambda_S \in O(1,3)$, $S \in Pin(3,1)$ is such that $\Lambda_{S\ \nu}^{\mu}\gamma^\nu = S\gamma^\mu S^{-1}$ and $b \in \mathbb{R}^4$.

The translations in space-time are given by $P(1,b)$. Doing a Majorana-Fourier transform, we get: $P(1,b)\{\Psi\}(x^0,\vec{p}) \equiv e^{-i\gamma^0 p\cdot b}\Psi(x^0,\vec{p})$, with $p^2 = m^2$. Therefore, p is related with the 4-momentum of the Poincare representation.

The rotations are defined by $P(R,0)$, where R is a rotation. Doing a Majorana-Hankel transform, we get for a rotation along z by an angle θ:

$$P(R,0)\{\Psi\}(x^0,p,l,\mu) \equiv e^{i\gamma^0(\mu+\frac{1}{2})\theta}\Psi(x^0,p,l,\mu)$$

Therefore, μ is related with the angular momentum of a spin one-half Poincare representation.

4.4. Finite mass case

Remark 4.19. *The complex irreducible projective representations of the Poincare group with finite mass split into positive and negative energy representations, which are complex conjugate of each other. They are labeled by one number j, with $2j$ being a natural number. The positive energy representation spaces V_j are, up to isomorphisms, written as:*

$$\Psi_j(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3}\prod_{k=1}^{2j}\Big(\frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}}\Big)_k e^{-ip\cdot x}\Psi_j(\vec{p})$$

where $p^0 = E_p$ and $\Psi_j(\vec{p})$ is a symmetric tensor product of Dirac spinor fields defined on the 3-momentum space, verifying $(\gamma^0)_k\Psi_j(\vec{p}) = \Psi_j(\vec{p})$. The matrices with the index k apply in the corresponding spinor index of Ψ_j

The representation space V_0 is, up to isomorphisms, written as:

$$\Psi_0(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{-ip\cdot x}(1+\gamma^0)(i\gamma^5)\Psi_0(\vec{p}) \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}}$$

where $p^0 = E_p$ and $\Psi_0(\vec{p})$ is a scalar defined on the 3-momentum space.

The representation map consists in applying the spin one-half representation map to every spinor index.

Proposition 4.20. *The real irreducible projective representations of the Poincare group with finite mass are labeled by one number j, with $2j$ being a natural number. The representation spaces W_j with $j > 0$ are, up to isomorphisms, written as:*

$$\Psi_j(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3} \prod_{k=1}^{2j} \Big(\frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}}\Big)_k \Big(e^{-i\gamma^0 p\cdot x}\Big)_1 \Psi_j(\vec{p})$$

where $p^0 = E_p$ and $\Psi_j(\vec{p})$ is a symmetric tensor product of Majorana spinor fields defined on the 3-momentum space, verifying $(i\gamma^0)_k\Psi_j(\vec{p}) = (i\gamma^0)_1\Psi_j(\vec{p})$. The matrices with the index k apply on the corresponding spinor index of Ψ_j.

The representation space W_0 is, up to isomorphisms, written as:

$$\Psi_0(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{-i\gamma^0 p\cdot x} i\gamma^5\Psi_0(\vec{p}) \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}}$$

where $p^0 = E_p$ and $\Psi_0(\vec{p})$ is a scalar defined on the 3-momentum space.

The representation map consists in applying the spin one-half representation map to every spinor index.

4.5. Null mass case

When the mass goes to zero, the representation spaces that we had are no longer irreducible, since the helicity becomes invariant under Lorentz transformations. This is independent of whether the representation is real or complex. The subspaces $V_j^\pm$ or $W_j^\pm$, where for all $\Psi_j \in V_j^\pm$ or $\Psi_j \in W_j^\pm$, $(\vec{\not{p}}\gamma^5)_k\Psi_j(\vec{p}) = \pm\Psi_j(\vec{p})$ for all the indexes k, are the irreducible representations for null mass and discrete helicity. These are not invariant under parity. There are also massless representations with continuous spin, which will not be studied.

5. Conclusion

The complex representations are not a generalization of the real representations, in the same way that the complex numbers are a generalization of the real numbers. There is a map, one-to-one or two-to-one and surjective up to equivalence, from the complex to the real irreducible representations of a Lie group on a Hilbert space.

All the real finite-dimensional projective representations of the restricted Lorentz group are also projective representations of the full Lorentz group, in contrast with the complex representations which are not all projective representations of the full Lorentz group.

We obtained all the real unitary irreducible projective representations of the Poincare group, with discrete spin, as real Bargmann-Wigner fields. For each pair of complex representations with positive/negative energy, there is one real representation. The Majorana-Fourier and Majorana-Hankel unitary transforms of the real or complex Bargmann-Wigner fields are related to the linear and angular momenta of a representation of the Poincare group, that is, there are linear and angular momenta spaces which are common for the real and complex representations of the Poincare group, therefore independent of the charge and matter-antimatter properties. This allows to define, with a common formalism, momenta spaces for the classical electromagnetic field or for the quantum Dirac field, for instance.

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