The orthogonal real representations of the Poincare group

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Abstract

DRAFT VERSION

The Majorana spinor is an element of a 4 dimensional real vector space. The Majorana spinor representations of the Rotation and Lorentz groups are irreducible. The spinor fields are space-time dependent spinors, solutions of the free Dirac equation.

We define the Majorana-Fourier transform and relate it to the linear momentum of a spin one-half Poincare group representation. We show that the projective representation of the Poincare group on the Majorana spinor field is orthogonal and irreducible. Using the Bargmann-Wigner equations, we study all orthogonal irreducible projective real representations of the Poincare group, with finite or null mass and discrete spin.

Keywords: Majorana spinors, unitary operator, hilbert space

1. Introduction

The Poincare group, also called inhomogeneous Lorentz group, has a real Lie algebra [1]. The irreducibility of a representation of a real Lie algebra may depend on whether the representation space is a real or complex Hilbert space. In a physicists language, the complex Hilbert spaces have twice the number of degrees of freedom of the real ones.

The Poincare group is the semi-direct product of the translations and Lorentz groups. Whether or not the Lorentz and Poincare groups include the parity and time reversal transformations depends on the context and authors. To be clear, we use the prefixes full/restricted when including/excluding parity and time reversal transformations. A projective representation of the Poincare group on a Hilbert space is an homomorphism, defined up to a complex phase, from the group to the automorphisms of the Hilbert space. The representations of the Pin(3,1) group are projective representations of the full Lorentz group[2], while the representations of the SL(2,C) subgroup are projective representations of the restricted Lorentz subgroup.

The unitary projective representations of the Poincare group on complex Hilbert spaces were studied by many authors, including Wigner [3–8]. Since Quantum Mechanics is based on complex Hilbert spaces [9], these studies were very important in the evolution of the role of symmetry in the Quantum theory[10]. Although Quantum Theory in real Hilbert spaces was investigated before [11–18], to our knowledge, the orthogonal projective representations of the Poincare group on real Hilbert spaces were not studied.

The Dirac spinor is an element of a 4 dimensional complex vector space, while the Majorana spinor is an element of a 4 dimensional real vector space [19]. The Majorana

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spinor representation of both SL(2,C) and Pin(3,1) is irreducible [20]. The spinor fields, space-time dependent spinors, are solutions of the free Dirac equation [21–24]. The Hilbert space of Dirac spinor fields is complex, while the Hilbert space of Majorana spinor fields is real.

To study a system of many neutral particles with spin one-half, Majorana spinor fields are extended with second quantization operators and are called Majorana quantum fields or Majorana fermions [25–27]. There are important applications of the Majorana quantum field in theories trying to explain phenomena in neutrino physics, dark matter searches, the fractional quantum Hall effect and superconductivity [28]. Note that Majorana quantum fields are related to but are different from the Majorana spinor fields.

In the context of Clifford Algebras, there are studies on the geometric square roots of -1 [16–18] and on the generalizations of the Fourier transform [29], with applications to image processing[30].

The Bargmann-Wigner generalize the Dirac equation for arbitrary spins[31, 32].

Our goal is to study the projective unitary representations of the Poincare group on real Hilbert spaces. We define the Majorana-Fourier transform and relate it to the linear momentum of a spin one-half Poincare group representation. Using the Bargmann-Wigner equations, we study all orthogonal irreducible projective real representations of the Poincare group, with finite or null mass and discrete spin.

2. Majorana, Dirac and Pauli Matrices and Spinors

Definition 2.1. $\mathbb{F}^{m\times n}$ is the vector space of $m \times n$ matrices whose entries are elements of the field $\mathbb{F}$.

In the next remark we state the Pauli's fundamental theorem of gamma matrices. The proof can be found in [33].

Remark 2.2. *Let A^μ, B^μ, $\mu \in \{0,1,2,3\}$, be two sets of 4×4 complex matrices verifying:*

$$A^\mu A^\nu + A^\nu A^\mu = -2\eta^{\mu\nu} \tag{2.1}$$

$$B^\mu B^\nu + B^\nu B^\mu = -2\eta^{\mu\nu} \tag{2.2}$$

Where $\eta^{\mu\nu} \equiv diag(+1,-1,-1-1)$ is the Minkowski metric.

1) There is a complex matrix S, with $|detS| = 1$, such that $B^\mu = SA^\mu S^{-1}$, for all $\mu \in \{0,1,2,3\}$. S is unique up to a complex phase.

2) If A^μ and B^μ are all unitary, then S is unitary.

Proposition 2.3. *Let α^μ, β^μ, $\mu \in \{0,1,2,3\}$, be two sets of 4×4 real matrices verifying:*

$$\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = -2\eta^{\mu\nu} \tag{2.3}$$

$$\beta^\mu \beta^\nu + \beta^\nu \beta^\mu = -2\eta^{\mu\nu} \tag{2.4}$$

Then there is a real matrix S, with $|detS| = 1$, such that $\beta^\mu = S\alpha^\mu S^{-1}$, for all $\mu \in \{0,1,2,3\}$. S is unique up to a signal.

Proof. From remark 2.2, we know that there is a complex matrix T, unique up to a complex phase, such that $\beta^\mu = T\alpha^\mu T^{-1}$.

Conjugating the previous equation, we get $\beta^\mu = T^*\alpha^\mu T^{*-1}$. Then $T^* = e^{i2\theta}T$ for some real number θ. Therefore $S \equiv e^{i\theta}T$ is a real matrix, unique up to a signal. $\square$

Definition 2.4. The Majorana matrices, $i\gamma^\mu$, $\mu \in \{0,1,2,3\}$, are 4×4 complex unitary matrices verifying:

$$(i\gamma^\mu)(i\gamma^\nu) + (i\gamma^\nu)(i\gamma^\mu) = -2\eta^{\mu\nu} \tag{2.5}$$

The Dirac matrices are $\gamma^\mu \equiv -i(i\gamma^\mu)$.

In the Majorana bases, the Majorana matrices are 4×4 real orthogonal matrices. An example of the Majorana matrices in a particular Majorana basis is:

$$\begin{aligned}
i\gamma^1 &= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
i\gamma^0 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad = -\gamma^0\gamma^1\gamma^2\gamma^3
\end{aligned} \tag{2.6}$$

Definition 2.5. The Dirac spinor is a 4×1 complex column matrix, $\mathbb{C}^{4\times 1}$.

The space of Dirac spinors is a 4 dimensional complex vector space.

Definition 2.6. Let S be an invertible matrix such that $Si\gamma^\mu S^{-1}$ is real, for $\mu = 0,1,2,3$.
The set of Majorana spinors, $Pinor$, is the set of Dirac spinors verifying the Majorana condition:

$$Pinor \equiv \{u \in \mathbb{C}^{4\times 1} : S^* u^* = Su\} \tag{2.7}$$

Where * denotes complex conjugation.

The set of Majorana spinors is a 4 dimensional real vector space. Note that the linear combinations of Majorana spinors with complex scalars do not verify the Majorana condition. The Majorana spinor, in the Majorana bases, is a 4×1 real column matrix.
There are 16 linear independent products of Majorana matrices. These form a basis of the real vector space of endomorphisms of Majorana spinors, $End(Pinor)$. In the Majorana bases, $End(Pinor)$ is the vector space of 4×4 real matrices.

Definition 2.7. The Pauli matrices σ^k, $k \in \{1,2,3\}$ are 2×2 hermitian, unitary, anti-commuting, complex matrices. The Pauli spinor is a 2×1 complex column matrix. The space of Pauli spinors is denoted by $Pauli$.

The space of Pauli spinors, $Pauli$, is a 2 dimensional complex vector space and a 4 dimensional real vector space.

3. Majorana spinor representation of the Lorentz group

Remark 3.1. *The Lorentz group, $O(1,3) \equiv \{\lambda \in \mathbb{R}^{4\times 4} : \lambda^T \eta\lambda = \eta\}$, is the set of real matrices that leave the metric, $\eta = diag(1,-1,-1,-1)$, invariant.*
The proper orthochronous Lorentz subgroup is defined by $SO^+(1,3) \equiv \{\lambda \in O(1,3) : det(\lambda) = 1, {\lambda^0}_0 > 0\}$. It is a normal subgroup. The discrete Lorentz subgroup of parity and time-reversal is $\Delta \equiv \{1, \eta, -\eta, -1\}$.
The Lorentz group is the semi-direct product of the previous subgroups, $O(1,3) = \Delta \ltimes SO^+(1,3)$.

Definition 3.2. The set Maj is the 4 dimensional real space of the linear combinations of the Majorana matrices, $i\gamma^\mu$:

$$Maj \equiv \{a_\mu i\gamma^\mu : a_\mu \in \mathbb{R},\ \mu \in \{0,1,2,3\}\} \tag{3.1}$$

Definition 3.3. $Pin(3,1)$ [2] is the group of endomorphisms of Majorana spinors that leave the space Maj invariant, that is:

$$Pin(3,1) \equiv \Big\{S \in End(Pinor) :\ |detS| = 1,\ S^{-1}(i\gamma^\mu)S \in Maj,\ \mu \in \{0,1,2,3\}\Big\} \tag{3.2}$$

Proposition 3.4. *The map $\Lambda : Pin(3,1) \to O(1,3)$ defined by:*

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu \equiv S^{-1}(i\gamma^\mu)S \tag{3.3}$$

is two-to-one and surjective. It defines a group homomorphism.

Proof. 1) Let $S \in Pin(3,1)$. Since the Majorana matrices are a basis of the real vector space Maj, there is an unique real matrix $\Lambda(S)$ such that:

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu = S^{-1}(i\gamma^\mu)S \tag{3.4}$$

Therefore, Λ is a map with domain $Pin(3,1)$. Now we can check that $\Lambda(S) \in O(1,3)$:

$$(\Lambda(S))^\mu{}_\alpha \eta^{\alpha\beta}(\Lambda(S))^\nu{}_\beta = -\frac{1}{2}(\Lambda(S))^\mu{}_\alpha\{i\gamma^\alpha, i\gamma^\beta\}(\Lambda(S))^\nu{}_\beta = \tag{3.5}$$

$$= -\frac{1}{2}S\{i\gamma^\mu, i\gamma^\nu\}S^{-1} = S\eta^{\mu\nu}S^{-1} = \eta^{\mu\nu} \tag{3.6}$$

We have proved that Λ is a map from $Pin(3,1)$ to $O(1,3)$.

2) Since any $\lambda \in O(1,3)$ conserve the metric η, the matrices $\alpha^\mu \equiv \lambda^\mu{}_\nu i\gamma^\nu$ verify:

$$\{\alpha^\mu, \alpha^\nu\} = -2\lambda^\mu{}_\alpha \eta^{\alpha\beta}\lambda^\nu{}_\beta = -2\eta^{\mu\nu} \tag{3.7}$$

In a basis where the Majorana matrices are real, from Proposition 2.3 there is a real invertible matrix S_λ, with $|detS_\Lambda| = 1$, such that $\lambda^\mu{}_\nu i\gamma^\nu = S_\lambda^{-1}(i\gamma^\mu)S_\lambda$. The matrix S_Λ is unique up to a sign. So, $\pm S_\lambda \in Pin(3,1)$ and we proved that the map $\Lambda : Pin(3,1) \to O(1,3)$ is two-to-one and surjective.

3) The map defines a group homomorphism because:

$$\Lambda^\mu{}_\nu(S_1)\Lambda^\nu{}_\rho(S_2)i\gamma^\rho = \Lambda^\mu{}_\nu S_2^{-1}i\gamma^\nu S_2 \tag{3.8}$$

$$= S_2^{-1}S_1^{-1}i\gamma^\mu S_1 S_2 = \Lambda^\mu{}_\rho(S_1S_2)i\gamma^\rho \tag{3.9}$$

□

Remark 3.5. *The group $SL(2,\mathbb{C}) = \{e^{\theta^j i\sigma^j + b^j\sigma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$ is simply connected. Its projective representations are equivalent to its ordinary representations[8].*

There is a two-to-one, surjective map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$, defined by:

$$\Upsilon^\mu{}_\nu(T)\sigma^\nu \equiv T^\dagger\sigma^\mu T \tag{3.10}$$

Where $T \in SL(2,\mathbb{C})$, $\sigma^0 = 1$ and σ^j, $j \in \{1,2,3\}$ are the Pauli matrices.

Lemma 3.6. *Consider that* $\{M_+, M_-, i\gamma^5 M_+, i\gamma^5 M_-\}$ *and* $\{P_+, P_-, iP_+, iP_-\}$ *are orthonormal basis of the 4 dimensional real vector spaces Pinor and Pauli, respectively, verifying:*

$$\gamma^0\gamma^3 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm \tag{3.11}$$

The isomorphism $\Sigma : Pauli \to Pinor$ *is defined by:*

$$\Sigma(P_+) = M_+,\ \Sigma(iP_+) = i\gamma^5 M_+ \tag{3.12}$$

$$\Sigma(P_-) = M_-,\ \Sigma(iP_-) = i\gamma^5 M_- \tag{3.13}$$

The group $Spin^+(3,1) \equiv \{\Sigma \circ A \circ \Sigma^{-1} : A \in SL(2,\mathbb{C})\}$ *is a subgroup of* $Pin(1,3)$. *For all* $S \in Spin^+(1,3)$, $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$.

Proof. From remark 3.5, $Spin^+(3,1) = \{e^{\theta^j i\gamma^5\gamma^0\gamma^j + b^j\gamma^0\gamma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$. Then, for all $T \in SL(2,C)$:

$$-i\gamma^0 \Sigma \circ T^\dagger \circ \Sigma^{-1} i\gamma^0 = \Sigma \circ T^{-1} \circ \Sigma^{-1} \tag{3.14}$$

Now, the map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$ is given by:

$$\Upsilon^\mu{}_\nu(T) i\gamma^\nu = (\Sigma \circ T^{-1} \circ \Sigma^{-1}) i\gamma^\mu (\Sigma \circ T \circ \Sigma^{-1}) \tag{3.15}$$

Then, all $S \in Spin^+(3,1)$ leaves the space Maj invariant:

$$S^{-1} i\gamma^\mu S = \Upsilon^\mu{}_\nu(\Sigma^{-1} \circ S \circ \Sigma) i\gamma^\nu \in Maj \tag{3.16}$$

Since all the products of Majorana matrices, except the identity, are traceless, then $det(S) = 1$. So, $Spin^+(3,1)$ is a subgroup of $Pin(1,3)$ and $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$. $\square$

Definition 3.7. The discrete Pin subgroup $\Omega \subset Pin(3,1)$ is:

$$\Omega \equiv \{\pm 1, \pm i\gamma^0, \pm\gamma^0\gamma^5, \pm i\gamma^5\} \tag{3.17}$$

The previous lemma implies that $Spin^+(1,3)$ is a double cover of $SO^+(3,1)$. We can check that for all $\omega \in \Omega$, $\Lambda(\pm\omega) \in \Delta$. That is, the discrete Pin subgroup is the double cover of the discrete Lorentz subgroup. Therefore, $Pin(3,1) = \Omega \ltimes Spin^+(1,3)$

Since there is a two-to-one surjective group homomorphism, $Pin(3,1)$ is a double cover of $O(1,3)$, $Spin^+(3,1)$ is a double cover of $SO^+(1,3)$ and $Spin^+(1,3) \cap SU(4)$ is a double cover of $SO(3)$. We can check that $Spin^+(1,3) \cap SU(4)$ is isomorphic to $SU(2)$.

4. Hilbert spaces of Majorana and Pauli spinor fields

Definition 4.1. The complex Hilbert space of Pauli spinors, *Pauli*, has the internal product:

$$< \phi, \psi >= \phi^\dagger \psi;\ \phi, \psi \in Pauli \tag{4.1}$$

Definition 4.2. The real Hilbert space of Majorana spinors, *Pinor*, has the internal product:

$$< \Phi, \Psi >= \Phi^\dagger \Psi;\ \Phi, \Psi \in Pinor \tag{4.2}$$

Definition 4.3. Consider that $\{M_+, M_-, i\gamma^0 M_+, i\gamma^0 M_-\}$ and $\{P_+, P_-, iP_+, iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces *Pinor* and *Pauli*, respectively, verifying:

$$\gamma^3\gamma^5 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm \tag{4.3}$$

Let H be a real Hilbert space. For all $h \in H$, the bijective linear map $\Theta_H : Pauli \otimes_{\mathbb{R}} H \to Pinor \otimes_{\mathbb{R}} H$ is defined by:

$$\Theta_H(h \otimes_{\mathbb{R}} P_+) = h \otimes_{\mathbb{R}} M_+,\ \Theta_H(h \otimes_{\mathbb{R}} iP_+) = h \otimes_{\mathbb{R}} i\gamma^0 M_+ \tag{4.4}$$

$$\Theta_H(h \otimes_{\mathbb{R}} P_-) = h \otimes_{\mathbb{R}} M_-,\ \Theta_H(h \otimes_{\mathbb{R}} iP_-) = h \otimes_{\mathbb{R}} i\gamma^0 M_- \tag{4.5}$$

Definition 4.4. Let H_n, with $n \in \{1, 2\}$, be two real Hilbert spaces and $U : Pauli \otimes_{\mathbb{R}} H_1 \to Pauli \otimes_{\mathbb{R}} H_2$ be an operator. The operator $U^\Theta : \Theta_{H_2} \circ U \circ \Theta^{-1}_{H_1} : Pinor \otimes_{\mathbb{R}} H_1 \to Pinor \otimes_{\mathbb{R}} H_2$ is defined as $U^\Theta \equiv \Theta_{H_2} \circ U \circ \Theta^{-1}_{H_1}$.

Remark 4.5. *Let H_n, with $n \in \{1, 2\}$, be two Hilbert spaces with internal products $<,>: H_n \times H_n \to \mathbb{F}$,($\mathbb{F} = \mathbb{R}, \mathbb{C}$). A linear operator $U : H_1 \to H_2$ is unitary iff:*
1) it is surjective;
2) for all $x \in H_1$, $< U(x), U(x) >=< x, x >$.

Remark 4.6. *Given two real Hilbert spaces H_1, H_2 and an unitary operator $U : H_1 \to H_2$, the inverse operator $U^{-1} : H_2 \to H_1$ is defined by:*

$$< x, U^{-1}y >=< Ux, y >,\ x \in H_1, y \in H_2 \tag{4.6}$$

Proposition 4.7. *Let H_n, with $n \in \{1, 2\}$, be two real Hilbert spaces. The following two statements are equivalent:*
1) The operator $U : Pauli \otimes_{\mathbb{R}} H_1 \to Pauli \otimes_{\mathbb{R}} H_2$ is unitary;
2) The operator $U^\Theta : Pinor \otimes_{\mathbb{R}} H_1 \to Pinor \otimes_{\mathbb{R}} H_2$ is orthogonal.

Proof. Because Θ_{H_n} is bijective, U is surjective iff $\Theta_{H_2} \circ U \circ \Theta^{-1}_{H_1}$ is surjective.
For all $g \in Pauli \otimes_{\mathbb{R}} H_1$, we have:

$$< g, g >=< \Theta_{H_1}(g), \Theta_{H_1}(g) > \tag{4.7}$$

$$< U(g), U(g) >=< \Theta_{H_2}(U(g)), \Theta_{H_2}(U(g)) > \tag{4.8}$$

Since Θ_{H_n} is bijective, we get that the following two statements are equivalent:
1) for all $g \in Pauli \otimes_{\mathbb{R}} H_1$, $< g, g >=< U(g), U(g) >$;
2) for all $g' \in Pinor \otimes_{\mathbb{R}} H_1$, $< g', g' >=< \Theta_{H_2}(U(\Theta^{-1}_{H_1}(g'))), \Theta_{H_2}(U(\Theta^{-1}_{H_1}(g'))) >$. □

Definition 4.8. The space of Majorana spinor fields over a set S, $Pinor(S) \equiv Pinor \otimes_{\mathbb{R}} L^2(S)$, is the real Hilbert space of Majorana spinors whose entries, in a Majorana basis, are real Lebesgue square integrable functions of S.

Definition 4.9. The space of Pauli spinor fields over a set S, $Pauli(S) \equiv Pauli \otimes_{\mathbb{R}} L^2(S)$ is the complex Hilbert space of Pauli spinors whose components are complex Lebesgue square integrable functions of S.

5. Linear Momentum of Majorana spinor fields

Definition 5.1. $L^2(\mathbb{R}^n)$ is the real Hilbert space of real functions of n real variables whose square is Lebesgue integrable in $\mathbb{R}^n$. The internal product is:

$$< f, g >\equiv \int d^n x f(x)g(x),\ \ f, g \in L^2(\mathbb{R}^n) \tag{5.1}$$

Remark 5.2. *The Pauli-Fourier Transform $\mathcal{F}_P : Pauli(\mathbb{R}^n) \to Pauli(\mathbb{R}^n)$ is an unitary operator defined by:*

$$\mathcal{F}_P\{\psi\}(\vec{p}) \equiv \int d^n\vec{x} \frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^n}}\psi(\vec{x}),\ \ \psi \in Pauli(\mathbb{R}^n) \tag{5.2}$$

Where the domain of the integral is $\mathbb{R}^n$.

Definition 5.3. The Majorana-Fourier Transform $\mathcal{F}_M : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is an operator defined by:

$$\mathcal{F}_M\{\Psi\}(\vec{p}) \equiv \int d^3\vec{x}\ \frac{e^{-i\gamma^0\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}}\Psi(\vec{x}),\ \ \Psi \in Pinor(\mathbb{R}^3) \tag{5.3}$$

Where the domain of the integral is $\mathbb{R}^3$, $m \geq 0$, $E_p \equiv \sqrt{\vec{p}^2 + m^2}$ and $\not{p} = E_p\gamma^0 - \vec{p}\cdot\vec{\gamma}$.

Proposition 5.4. *The Majorana-Fourier Transform is an unitary operator.*

Proof. The Majorana-Fourier Transform can be written as:

$$\mathcal{F}_M\{\Psi\}(\vec{p}) \equiv \sqrt{\frac{E_p + m}{2E_p}}\Big(\int d^3\vec{x}\ \frac{e^{-i\gamma^0\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}\Psi(\vec{x})\Big) \tag{5.4}$$

$$-\sqrt{\frac{E_p - m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|}\Big(\int d^3\vec{x}\ \frac{e^{+i\gamma^0\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}\Psi(\vec{x})\Big) \tag{5.5}$$

So, one gets:

$$\mathcal{F}_M\{\Psi\} = S \circ \mathcal{F}_P^{\Theta}\{\Psi\} \tag{5.6}$$

Where $S : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} S\{\Psi\}(+\vec{p}) \\ S\{\Psi\}(-\vec{p}) \end{bmatrix} \equiv \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & -\sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ \sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} \begin{bmatrix} \Psi(+\vec{p}) \\ \Psi(-\vec{p}) \end{bmatrix} \tag{5.7}$$

We can check that the 2×2 matrix appearing in the equation above is orthogonal. Therefore S is an unitary operator. Since $\mathcal{F}_P^{\Theta}$ is also unitary, $\mathcal{F}_M$ is unitary. □

Proposition 5.5. *The inverse Majorana-Fourier Transform verifies:*

$$(\gamma^0\vec{\gamma}\cdot\vec{\partial} + i\gamma^0 m)\mathcal{F}_M^{-1}\{\Psi\}(\vec{x}) = (\mathcal{F}_M^{-1} \circ R)\{\Psi\}(\vec{x}) \tag{5.8}$$

Where $\Psi \in Pinor(\mathbb{R}^3)$ and $R : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is a bijective linear map defined by $R\{\Psi\}(\vec{p}) = i\gamma^0 E_p\Psi(\vec{p})$.

Proof. We have $\mathcal{F}_M^{-1} = (\mathcal{F}_P^{\Theta})^{-1} \circ S^{-1}$. Then:

$$(\gamma^0 \vec{\gamma} \cdot \vec{\partial} + i\gamma^0 m)(\mathcal{F}_P^{\Theta})^{-1}\{\Psi\}(\vec{x}) = ((\mathcal{F}_P^{\Theta})^{-1} \circ Q)\{\Psi\}(\vec{x}) \tag{5.9}$$

Where $Q : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} Q\{\Psi\}(+\vec{p}) \\ Q\{\Psi\}(-\vec{p}) \end{bmatrix} \equiv \begin{bmatrix} i\gamma^0 m & i\vec{p}\cdot\vec{\gamma} \\ -i\vec{p}\cdot\vec{\gamma} & i\gamma^0 m \end{bmatrix} \begin{bmatrix} \Psi(+\vec{p}) \\ \Psi(-\vec{p}) \end{bmatrix} \tag{5.10}$$

Now we show that $Q \circ S^{-1} = S^{-1} \circ R$:

$$\begin{bmatrix} i\gamma^0 m & i\vec{p}\cdot\vec{\gamma} \\ -i\vec{p}\cdot\vec{\gamma} & i\gamma^0 m \end{bmatrix} \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ -\sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} = \tag{5.11}$$

$$= \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ -\sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} \begin{bmatrix} i\gamma^0 E_p & 0 \\ 0 & i\gamma^0 E_p \end{bmatrix} \tag{5.12}$$

□

6. Semisimple representations of groups over real and complex Hilbert spaces

Definition 6.1. A representation (M_G, V) of a Lie group G on an Hilbert space V is defined by:

1) the representation space V, which is an Hilbert space;

2) the representation group homomorphism $M : G \to B(V)$ from the group elements to the bounded automorphisms with a bounded inverse, such that the map $M' : G \times V \to V$ defined by $M'(g, v) \equiv M(g)v$ is continuous.

Definition 6.2. Let W be a subspace of V. (M_G, W) is a subrepresentation of (M_G, V) if W is invariant under the group action, that is, for all $w \in W$: $(M(g)w) \in W$, for all $g \in G$.

Definition 6.3. $W^\perp$ is the orthogonal complement of the subspace W of the vector space V if:

1) all $v \in V$ can be expressed as $v = w + x$, where $w \in W$ and $x \in W^\perp$;

2) if $w \in W$ and $x \in W^\perp$, then $< x, w >= 0$.

Definition 6.4. The representation (M_G, V) is semi-simple if for all subrepresentation (M_G, W) of (M_G, V) , $(M_G, W^\perp)$ is also a subrepresentation of (M_G, V), where $W^\perp$ is the orthogonal complement of the subspace W.

Lemma 6.5. *Consider a representation (M_G, V) of a group G. For all $g \in G$, if there is $h \in G$ such that for all $x, w \in W$: $< x, M(h)w >=< xM(g), w >$, then the representation (M_G, V) is semi-simple.*

Proof. Let (M_G, W) be a subrepresentation of (M_G, V). $W^\perp$ is the orthogonal complement of W.

Assume that for all $g \in G$, there is $h \in G$ such that for all $x \in W^\perp$, $w \in W$: $< M(g)x, w >=< x, M(h)w >$.

Since W is invariant then $w' \equiv (M(h)w) \in W$.

Since $x \in W^\perp$ and $w' \in W$, then $< x, w' >= 0$.

This implies that if x is in the orthogonal complement of W ($x \in W^\perp$), also $M(g)x$ is in the orthogonal complement of W ($M(g)x \in W^\perp$), for all $g \in G$. □

Definition 6.6. A representation (M_G, V) is irreducible if their only sub-representations are the trivial sub-representations: (M_G, V) and $(M_G, \{0\})$, where $\{0\}$ is the null space.

Lemma 6.7. *Consider a semi-simple representation (M_G, V) of a group G. The set of hermitian linear involutions of V that commutes with $M(g)$, for all $g \in G$, is $\{+1, -1\}$, iff the representation (M_G, V) is irreducible (1 is the identity automorphism).*

Proof. Let (M_G, W) and $(M_G, W^\perp)$ be sub-representations of (M_G, V), where $W^\perp$, the orthogonal complement of W.

There is an automorphism $P : V \to V$, such that, for $w, w' \in W$, $x, x' \in W^\perp$, $P(w + x) = (w - x)$. $P^2 = 1$ and P is hermitian:

$$< w' + x', P(w + x) >=< w', w > - < x', x >=< P(w' + x'), w + x > \tag{6.1}$$

Let $w' \equiv M(g)w \in W$ and $x' \equiv M(g)x \in W^\perp$:

$$M(\Lambda)P(w + x) = M(\Lambda)(w - x) = (w' - x') \tag{6.2}$$

$$PM(\Lambda)(w + x) = P(w' + x') = (w' - x') \tag{6.3}$$

Which implies that P commutes with $M(g)$ for all $g \in G$.

$P = +1$ iff $W = V$:

$$+(w + x) = P(w + x) = (w - x) \Leftrightarrow x = 0 \tag{6.4}$$

$P = -1$ iff W is the null space:

$$-(w + x) = P(w + x) = (w - x) \Leftrightarrow w = 0 \tag{6.5}$$

□

Definition 6.8. A group G is semi-simple iff all its finite-dimensional representations are semisimple.

Definition 6.9. Consider the real semi-simple irreducible representation (M_G, W). If there is not a skew-hermitian automorphism J that squares to -1 and commutes with the representation, then the real representation is called absolutely real. If such automorphism J exists then the real representation is called absolutely complex.

Definition 6.10. Consider the complex semi-simple irreducible representation (M_G, W). If there is an anti-linear involution θ which commutes with the representation, then the complex representation is called absolutely real. If such involution does not exist then the complex representation is called absolutely complex.

Proposition 6.11. *There is a one-to-one surjective map M, up to isomorphisms, from the real to the complex irreducible representations of a semisimple group that are absolutely real.*

Proof. Consider an irreducible real representation (M_G, W) which is absolutely real. Then the complexification $(M_G, W \otimes \mathbb{C})$ is irreducible because the only automorphism commuting with the representation is proportional to the identity. This complex representation commutes with the complex conjugation and hence is absolutely real.

Consider an irreducible complex representation (M_G, V) which is absolutely real. Then there is an anti-linear involution θ which commutes with the representation. θ is unique up to a sign because for other θ', $\theta\theta'$ is a linear involution commuting with the representation and hence $\theta\theta' = \pm 1$. Consider the real vector spaces $V_\pm \equiv \{v \in V : \theta(v) = \pm v\}$. Then the two real representations $(M_G, V_\pm)$ are isomorphic to each other, irreducible and absolutely real because the only automorphisms commuting with the representation are proportional to the identity. □

Proposition 6.12. *There is an two-to-one surjective map M, up to isomorphisms, from the complex to the real irreducible representations of a semisimple group that are absolutely complex.*

Proof. Consider an irreducible complex representation (M_G, V) which is absolutely complex. Let $V' \equiv \{(u, v) : u, v \in V\}$ and for $u, v \in V$ let $M'_G(u, v) \equiv (M_G u, M^*_G v)$, where M^*_G. Then there is an anti-linear involution defined by $\theta(u, v) \equiv (v^*, u^*)$ which commutes with the representation (M'_G, V'). The skew-hermitian automorphism defined by $J(u, v) = (iu, -iv)$ commutes with θ and (M'_G, V'). Consider the real vector spaces $V'_\pm \equiv \{v \in V' : \theta(v) = \pm v\}$.

Now we need to show that the representations $(M'_G, V'_\pm)$ are irreducible. $V'_\pm$ are real Hilbert spaces because the anti-linear involution θ can be written as $UCU^\dagger$, where U is defined by $U(u, v) = \frac{1}{\sqrt{2}}(-iu + iv, u + v)$ and C by $C(u, v) = (u^*, v^*)$. If $(M'_G, V'_\pm)$ is reducible, then there is a linear involution P' not proportional to the identity which commutes with $U^\dagger M'_G U$ and with C. Hence the involution $P \equiv UP'U^\dagger$ commutes with M'_G and θ. The most general definition is $P(u, v) \equiv (Au + Bv, B^\dagger u + Dv)$ where A and D are hermitian. If P commutes with M'_G, then A and D are real numbers because (M_G, V) is irreducible and $BM^*_G = M_G B$. If P commutes with θ, then $B = B^T$ and $A = D$. If P is an involution then either $A^2 = 1$ and $B = 0$ or and $A = 0$ and $BB^\dagger = 1$. The first case is in contradiction with P not being proportional to the identity, the second case is in contradiction with the non existence of an anti-linear involution commuting with M_G such as $BCM_G = M_G BC$. Hence $(M'_G, V'_\pm)$ is irreducible.

Consider an irreducible real representation (M_G, W) which is absolutely complex. Let J be a skew-hermitian operator squaring to -1 and commuting with M_G. Consider the vector spaces $W_\pm \equiv \{w \in W \otimes \mathbb{C} : iJw = \pm w\}$ and the irreducible representations $(M_G, W_\pm)$. Suppose that there is an anti-linear involution θ commuting with M_G and iJ. Then $C\theta$ is a linear involution of W not proportional to the identity commuting with M_G, which is in contradiction with (M_G, W) being irreducible. The representations $(M_G, W_\pm)$ are isomorphic to a pair of representations which are complex conjugate of each other. □

7. Poincare group representations

The complex irreducible unitary representations of the Poincare group are absolutely complex. Applying the map defined before, we can obtain all real irreducible orthogonal representations. The representations of $SU(2)$ are given by symmetric tensor products of

Pauli spinors for non-null spins. We can transform them to symmetric tensor products of Majorana spinors by introducing projectors $\frac{1+\gamma_1^0\gamma_n^0}{2}$. Explicitly, we have for null spin:

$$\frac{1+\gamma^0\otimes\gamma^0}{2}(\Psi_\uparrow\otimes\Psi_\downarrow-\Psi_\downarrow\otimes\Psi_\uparrow) \tag{7.1}$$

For non-null spin j, we have $2j-1$ projectors and a symmetric tensor product of $2j$ Majorana spinors. Applying an inverse Fourier-Majorana transform, we obtain the real solutions of the Bargmann-Wigner equations.

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