The Projective Line as a Meridian

by Kelly McKennon

1. Introduction

(1.1) Abstract We investigate that mathematical idea which in algebra is known as a *cross ratio*, in one-dimensional geometry as a *projective line*, in two-dimensional geometry as a *circle*, and in three-dimensional geometry as a *regulus*. We view each of these in its natural habitat, and show how each is an avatar of one Platonic object, which object we term a *meridian*.

(1.2) Cross-Ratio The cross ratio looms large in the development of projective geometry. It was known to Pappus of Alexandria back in the first half of the fourth century, and was used by Karl von Staudt in 1857 to present the first entirely synthetic treatment of the subject. Von Staudt[1] introduced the notion of a *Wurf* or *throw* : this was a pair of ordered pairs of points on a line. Throws were separated into equivalence classes by projectivities of the line, in relation to the situation of the line in a plane.

In Section 2 we follow a somewhat similar course, the main difference being that we regard the line as a set by itself without the influence of a surrounding plane, setting out four postulates to which the equivalence classes of throws must be subject. This approach not only induces a "projective" structure on the set, but also provides a particular model of a meridian, with from four to six distinguished points.

(1.3) Projective Line A "projective line" $\mathcal{M}$ over a field $\mathcal{F}$ is often defined as the family of lines through the origin of a two dimensional vector space $\mathcal{V}$ over $\mathcal{F}$. The "projective structure" of such a projective line is induced by a set of so-called homogeneous coordinates: $\mathcal{V}$ is given a coordinate system, and the homogeneous coordinates of any (line) element is the set of inherited coordinates of all the points on that line distinct from the origin of $\mathcal{V}$: they are related of course by all having the same ratio.

An equivalent "synthetic" definition is to consider a line $\mathcal{N}$ embedded in a projective plane and then to use "complete quadrilaterals" to define addition and multiplication. Given any "throw" $\{[A,B],[C,D]\}$ in the sense of von Staudt, and any fifth point E, there exist many complete quadrilaterals for which each of the pairs of the throws lie on the intersections of opposing lines of the quadrilateral, and such that one of the other lines passes through the fifth point.

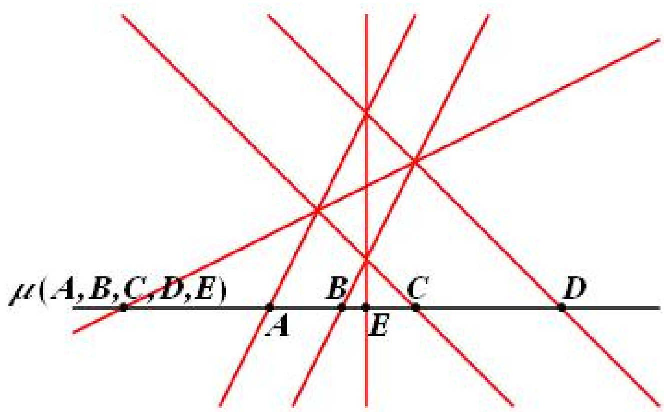

Fig. 1: The Quinary Operator on a Line in the Real Projective Plane.

However, for each of these complete quadrilaterals the remaining line cuts $\mathcal{N}$ at the same point. This defines a quinary operator μ on the points of N. One fixes three distinct points of $\mathcal{N}$, calling them 0, 1 and ∞, and then places them in a certain way in three of the arguments of μ to obtain a binary operator. One of these ways defines addition, and another way defines multiplication: in such wise that the complement of ∞ in $\mathcal{N}$ becomes a field.

[1] *Cf.* [Von Staudt] pp. 166 *et seq.* and [Veblen & Young] §55.

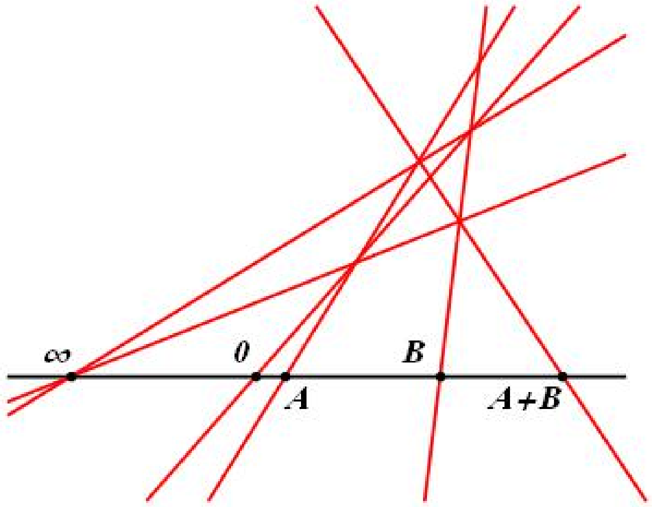

Fig. 2: Addition of Points on a Line in the Real Projective Plane.

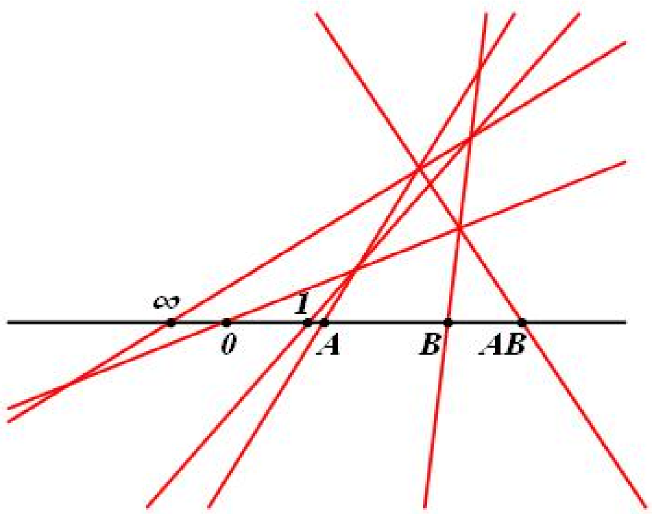

Fig. 3: Multiplication of Points on a Line in the Real Projective Plane.

Of course both the analytic and the synthetic methods require *a priori* a projective plane. It turns out that the projective plane may be dispensed with, and the quinary operator can be defined through a compact set of axioms. Section (4) follows such a program.

(1.4) Circle A circle $\mathcal{C}$ embedded in a plane is another model for a meridian, which in some respects is more illuminating than a straight line $\mathcal{N}$. The connection between the two is the so called "stereographic projection", where the line is aligned tangent to the circle, a point P is designated on the other side of the circle, and lines through P correspond points on $\mathcal{C}$ with points on $\mathcal{N}$ through intersection. This correspondence then transfers the quinary operator on $\mathcal{N}$ induced by complete quadrilaterals to a quinary operator on $\mathcal{C}$. However this quinary operator on $\mathcal{C}$ can be obtained directly and more simply. If one takes a throw on $\mathcal{C}$, each of the pairs of points of the throw determines a line. The two lines intersect in a point Q. If one draws a line through Q and a fifth point on $\mathcal{C}$, that line intersects $\mathcal{C}$ in exactly one other point (unless the line is tangent to $\mathcal{C}$). This other point is the value of the quinary operator.

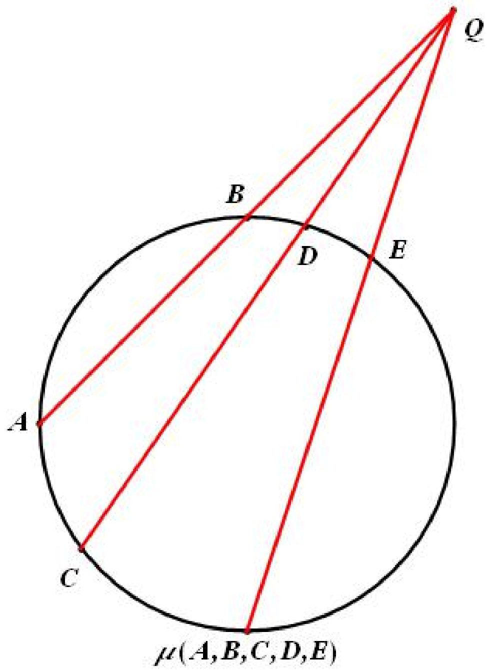

Fig. 4: The Quinary Operator on a Circle.

Given a circle $\mathcal{C}$ in the plane, there are a number of other curious operators induced on $\mathcal{C}$ by the plane. In fact each line $\mathcal{N}$ in the plane induces such a (trinary) operator κ as follows. Let $[A, B, C]$ be an ordered triple on $\mathcal{C}$, none of the points of which are on $\mathcal{N}$. The line through A and C intersects $\mathcal{N}$ at a single point K. The line through K and B intersects the circle at one other point[2]: this is by definition the value of κ at $[A, B, C]$ (we denote it in the figures below by $\lfloor A, B, C \rfloor$). If one fixes B and lets A and C vary, one obtains a binary operator which is, in fact, a group operator. When the line $\mathcal{N}$ intersects the circle $\mathcal{C}$ through two points, the resulting group is isomorphic with the multiplicative group of non-zero real numbers. When the line $\mathcal{N}$ is tangent to the circle $\mathcal{C}$, the resulting group is isomorphic with the additive group of real numbers. When the line $\mathcal{N}$ does not intersect $\mathcal{C}$, the resulting group is isomorphic to the group of complex numbers of modulus one.

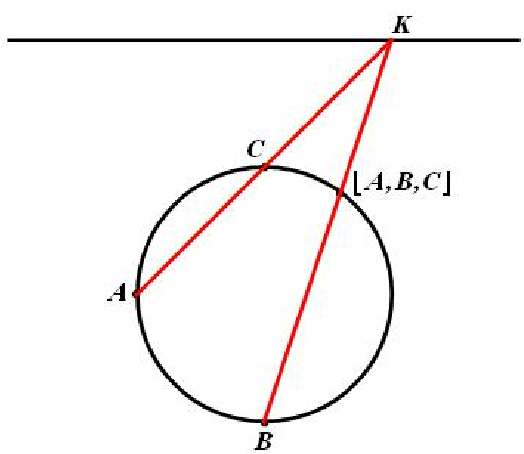

Fig. 5: Libra Operator Induced on a Circle by an Exterior Line.

[2] Unless the line is tangent to the circle, in which case $\kappa([A, B, C]) \equiv B$.

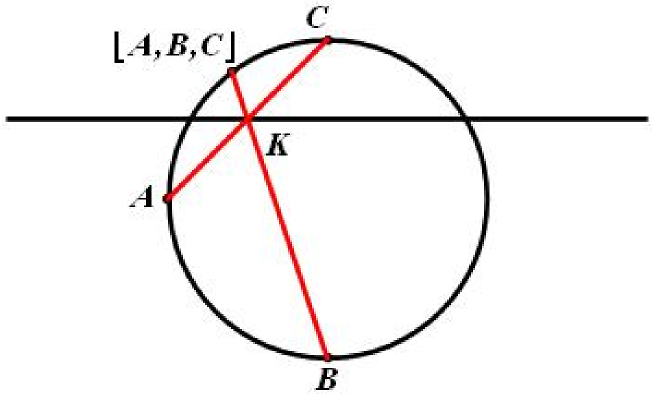

Fig. 6: Libra Operator Induced on a Circle by an Interior Line.

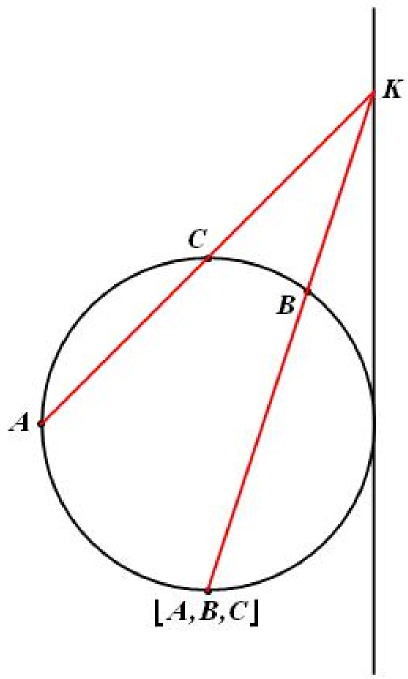

Fig. 7: Libra Operator Induced on a Circle by a Tangent Line.

(1.5) Libras The trinary operators κ described in (1.4) are examples of what we shall call *libra operators*. Libra operators satisfy two axioms and, as the examples κ suggest, these are closely related to group binary operators. One can think of a libra as a group without a specific identity element, just as one can think of an affine space as a vector space without an identity element. The idea of a libra seems to have first appeared in print in [Certaine] and has gone by various other names such as "groud", "heap" and "torsor". The name "libra" has been adopted here because the different libra operators may be thought of as defining different types of equilibria.[3]

The operators κ, however, do not constitute a principal motive for treating libra operators here. One such motive is that the quinary operators discussed in (1.2) and (1.3) are much more easily explained and handled when the concept of a libra operator is available, as will be seen in Section (4). The fundamental properties of libras are introduced in Section (3), and further properties given in Sections (5) and (7).

(1.6) Transformation Libras Many mathematical objects serve an important role as domains for

[3] This is explained in Section (11).

various families of functions or operators. In the case of a meridian, perhaps the most signal such family is that consisting of what we shall term *Möbius transformations*. In its guise as a projective line equipped with homogeneous coordinates $[x, y]$, these transformations go under several names, such as *homographic transformations*, *bi-linear transformations* or *linear fractional transformations*, and are represented as as quotients of linear terms:

$$(\forall\, A, B, C, D \in \mathcal{F} : AD \neq BC) \quad [X, Y] \hookrightarrow \frac{AX + B}{CX + D}.$$

If A, B, C, D are points in $\mathcal{F}$ as above, we shall denote the corresponding transformation as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Thus if a point P in the meridian has homogeneous coordinates $[X, Y]$ we can use matrix notation to compute the homogenous coordinates of the image:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}(P) \quad \text{has coordinates} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = [\, AX + BY \quad CX + DY \,]. \tag{1}$$

Of course the representation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of a Möbius transformation is only unique up to a constant factor of the coefficients A, B, C and D. Consequently the family $\Gamma(\mathcal{M})$ of all such constitutes a three dimensional object.

The matrix equality of (1) suggests the greater generality of viewing Möbius transformations not just as functions from a meridian onto itself, but as a family $\Gamma(M, N)$ of functions from one meridian $\mathcal{M}$ onto another meridian $\mathcal{N}$. This *prima facie* rather naive suggestion proves to be fruitful, in that it leads to a detailed understanding of the topological nature of $\Gamma(\mathcal{M})$ as a model of three dimensional projective space. This serves as the other principal motive for exploiting libras, for $\Gamma(\mathcal{M})$ regarded rather as a family $\Gamma(\mathcal{M}, \mathcal{N})$ of functions from one meridian $\mathcal{M}$ onto another $\mathcal{N}$, becomes a libra rather than a group. The development of the structure of $\Gamma(\mathcal{M}, \mathcal{N})$ is carried out in Section (8) as well as a characterization of precisely which libras are isomorphic to such families.

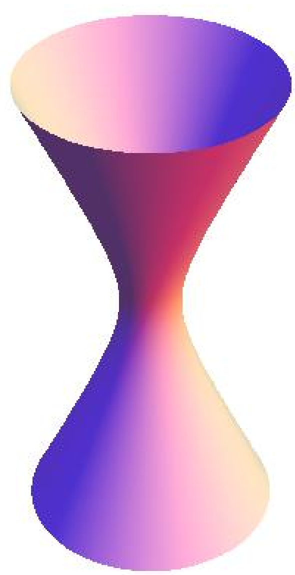

Fig. 8: Section of Quadric Surface over the Real Field

(1.7) Quadric Surfaces Let $\mathbf{S}$ denote a three dimensional projective space over a field of characteristic not equal to 2. A bijective projective mapping ϕ from $\mathbf{S}$ to its dual space consisting of the family $\mathfrak{P}$

of all planes in $\mathbf{S}$, has an adjoint mapping $\phi^{\sim}$ sending $\mathfrak{P}$ to $\mathbf{S}$ and defined by

$$(\forall\, \mathbf{P} \in \mathfrak{P}) \quad \{\phi^{\sim}(\mathbf{P})\} \equiv \bigcap_{p \in P} \phi(p).$$

The mapping ϕ is called a **polarity** provided $\phi^{\sim} \circ \phi$ is the identity map on $\mathbf{S}$. We shall say that a line $\mathbf{K}$ is **quadric relative to the polarity ϕ** provided that, for each $\mathbf{x} \in \mathbf{K}$, $\mathbf{K}$ is a subset of the plane $\phi(\mathbf{x})$. If a polarity has at least three pairwise disjoint quadric lines, we shall call it a **quadric polarity**, and the set of all points $\mathbf{x} \in \mathbf{S}$ such that $\mathbf{x} \in \phi(\mathbf{x})$ is called a **quadric surface**.

It is well-known that each quadric surface $\mathbf{Q}$ is a union of two disjoint families of lines $\mathfrak{C}$ and $\mathfrak{R}$. In fact each of these families is a partition of $\mathbf{Q}$. Furthermore each pair of lines, one from $\mathfrak{C}$ and one from $\mathfrak{R}$, intersect at exactly one point — and each point on the quadric is the intersection of exactly one such pair of lines. These families of lines are called **reguli**, and each line in a regulus is called a **rule**.[4]

It is also well-known that to each triple of pairwise non-intersecting lines in $\mathbf{S}$ corresponds exactly one quadric polarity for which the three lines are rules in one of the reguli. A line in $\mathbf{S}$ is in the other regulus precisely when it intersects each of the three defining lines of the first regulus.

(1.8) Involutions of Quadric Surfaces Suppose that $\mathbf{Q}$ is a quadric surface and that its reguli are $\mathfrak{C}$ and $\mathfrak{R}$. If $\mathbf{P}$ is a plane in $\mathbf{S}$ not tangent to $\mathbf{Q}$, it intersects $\mathbf{Q}$ in a conic. This conic is a meridian (being projectively equivalent to a circle), and so the map sending each $\mathbf{X} \in \mathfrak{C}$ to the intersection point of $\mathbf{X}$ with $\mathbf{P}$ induces a projective structure on $\mathfrak{C}$. Furthermore it is independent of the particular $\mathbf{P}$ used in inducing it. Thus $\mathfrak{C}$ may be regarded as a meridian, and the same is true for $\mathfrak{R}$.

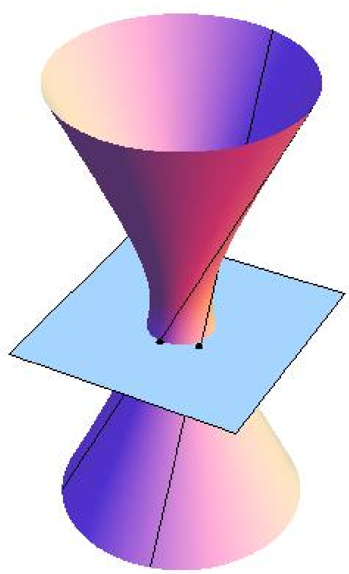

Fig. 9: Associating a Regulus to a Conic Section

Each point $\mathfrak{a}$ of $\mathbf{S}$ which does not lie on $\mathbf{Q}$ induces a natural involution of $\mathbf{Q}$: through each point $\mathbf{x}$ of $\mathbf{Q}$ passes exactly one line also passing through $\mathfrak{a}$, and this line intersects $\mathbf{Q}$ in exactly one other point $\mathfrak{y}$ (unless the line is tangent to the quadric).[5]

[4] *Cf.* [A. Seidenberg] §13.2.
[5] These and related transformations of the quadric are examined in Sections (9) and (10).

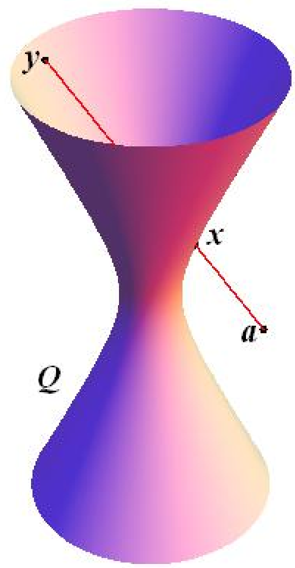

Fig. 10: Transformation of Quadric Surface Q by a Point a not on the Surface

Such a point $\mathfrak{a}$ also induces a natural transformation of $\mathfrak{C}$ onto $\mathfrak{R}$: through each rule $\mathbf{X}$ of $\mathfrak{C}$ passes exactly one plane on which $\mathfrak{a}$ lies. The intersection of this plane with $\mathbf{Q}$ is the union of $\mathbf{X}$ with a rule $\mathbf{Y}$ of the regulus $\mathfrak{R}$. What is more, the family of all such transformations thus described is precisely $\Gamma(\mathfrak{C}, \mathfrak{R})$.

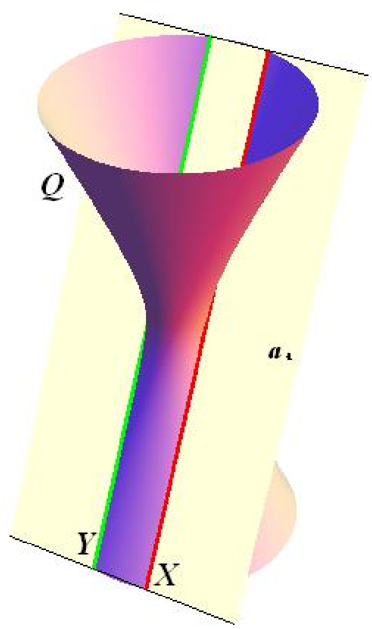

Fig. 11: Transformation of Rules on a Quadric Surface Q by a Point a not on the Surface

A converse to this fact is also true: that if $\mathcal{M}$ are $\mathcal{N}$ are isomorphic meridians, then the libra $\Gamma(\mathcal{M}, \mathcal{N})$, together with the cartesian product $\mathcal{M}\times\mathcal{N}$, can be identified in a natural way with three dimensional projective space, where $\mathcal{M}\times\mathcal{N}$ corresponds to a quadric surface. These facts are set forth in Section (10).

(1.9) Apology The author freely admits that he is not familiar with much of the extensive literature regarding projective geometry — and does not claim credit for any results contained herein which have been obtained earlier and elsewhere. The aim of this paper is rather to illustrate the beauty and variety of what we call here a meridian.

Due to the not inconsiderable amount of notation and terminology introduced, indices of notation and terminology have been included at the end of the paper.

2. The Quadriad Structure of a Meridian

(2.1) Permutation Notation Let Υ be a set $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ with four distinct elements. We shall write Π for the group of permutations of Υ. We denote the transpositions as follows:[6]

$$\begin{aligned}
\boxed{\spadesuit\heartsuit} &\equiv \{[\spadesuit,\heartsuit],[\heartsuit,\spadesuit],[\diamondsuit,\diamondsuit],[\clubsuit,\clubsuit]\}, & \boxed{\heartsuit\diamondsuit} &\equiv \{[\spadesuit,\spadesuit],[\heartsuit,\diamondsuit],[\diamondsuit,\heartsuit],[\clubsuit,\clubsuit]\},\\
\boxed{\spadesuit\diamondsuit} &\equiv \{[\spadesuit,\diamondsuit],[\heartsuit,\heartsuit],[\diamondsuit,\spadesuit],[\clubsuit,\clubsuit]\}, & \boxed{\heartsuit\clubsuit} &\equiv \{[\spadesuit,\spadesuit],[\heartsuit,\clubsuit],[\diamondsuit,\diamondsuit],[\clubsuit,\heartsuit]\},\\
\boxed{\spadesuit\clubsuit} &\equiv \{[\spadesuit,\clubsuit],[\heartsuit,\heartsuit],[\diamondsuit,\diamondsuit],[\clubsuit,\spadesuit]\}, & \boxed{\diamondsuit\clubsuit} &\equiv \{[\spadesuit,\spadesuit],[\heartsuit,\heartsuit],[\diamondsuit,\clubsuit],[\clubsuit,\diamondsuit]\},
\end{aligned}$$

and write Π_2 for the the collection of these six. We denote

$$\begin{aligned}
\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit} &\equiv \boxed{\spadesuit\heartsuit}\circ\boxed{\diamondsuit\clubsuit} = \boxed{\diamondsuit\clubsuit}\circ\boxed{\spadesuit\heartsuit} = \{[\spadesuit,\heartsuit],[\heartsuit,\spadesuit],[\diamondsuit,\clubsuit],[\clubsuit,\diamondsuit]\},\\
\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit} &\equiv \boxed{\spadesuit\diamondsuit}\circ\boxed{\heartsuit\clubsuit} = \boxed{\heartsuit\clubsuit}\circ\boxed{\spadesuit\diamondsuit} = \{[\spadesuit,\diamondsuit],[\heartsuit,\clubsuit],[\diamondsuit,\spadesuit],[\clubsuit,\heartsuit]\},\\
\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit} &\equiv \boxed{\spadesuit\clubsuit}\circ\boxed{\heartsuit\diamondsuit} = \boxed{\heartsuit\diamondsuit}\circ\boxed{\spadesuit\clubsuit} = \{[\spadesuit,\clubsuit],[\heartsuit,\diamondsuit],[\diamondsuit,\heartsuit],[\clubsuit,\spadesuit]\},
\end{aligned}$$

and write Π_{02} for the collection of these three.[7] We denote

$$\begin{aligned}
\boxed{\spadesuit\heartsuit\diamondsuit} &\equiv \boxed{\heartsuit\diamondsuit}\circ\boxed{\spadesuit\diamondsuit} = \{[\spadesuit,\heartsuit],[\heartsuit,\diamondsuit],[\diamondsuit,\spadesuit],[\clubsuit,\clubsuit]\}, & \boxed{\spadesuit\diamondsuit\heartsuit} &\equiv \boxed{\spadesuit\diamondsuit}\circ\boxed{\heartsuit\diamondsuit} = \{[\spadesuit,\diamondsuit],[\heartsuit,\spadesuit],[\diamondsuit,\heartsuit],[\clubsuit,\clubsuit]\},\\
\boxed{\spadesuit\heartsuit\clubsuit} &\equiv \boxed{\heartsuit\clubsuit}\circ\boxed{\spadesuit\clubsuit} = \{[\spadesuit,\heartsuit],[\heartsuit,\clubsuit],[\diamondsuit,\diamondsuit],[\clubsuit,\spadesuit]\}, & \boxed{\spadesuit\clubsuit\heartsuit} &\equiv \boxed{\spadesuit\clubsuit}\circ\boxed{\heartsuit\clubsuit} = \{[\spadesuit,\clubsuit],[\heartsuit,\spadesuit],[\diamondsuit,\diamondsuit],[\clubsuit,\heartsuit]\},\\
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\end{aligned}$$

and write Π_1 for the collection of these eight. We denote

$$\begin{aligned}
\boxed{\spadesuit\heartsuit\diamondsuit\clubsuit} &\equiv \boxed{\spadesuit\clubsuit}\circ\boxed{\spadesuit\diamondsuit}\circ\boxed{\spadesuit\heartsuit} = \{[\spadesuit,\heartsuit],[\heartsuit,\diamondsuit],[\diamondsuit,\clubsuit],[\clubsuit,\spadesuit]\},\\
\boxed{\spadesuit\heartsuit\clubsuit\diamondsuit} &\equiv \boxed{\spadesuit\diamondsuit}\circ\boxed{\spadesuit\clubsuit}\circ\boxed{\spadesuit\heartsuit} = \{[\spadesuit,\heartsuit],[\heartsuit,\clubsuit],[\diamondsuit,\spadesuit],[\clubsuit,\diamondsuit]\},\\
\boxed{\spadesuit\diamondsuit\heartsuit\clubsuit} &\equiv \boxed{\spadesuit\clubsuit}\circ\boxed{\spadesuit\heartsuit}\circ\boxed{\spadesuit\diamondsuit} = \{[\spadesuit,\diamondsuit],[\heartsuit,\clubsuit],[\diamondsuit,\heartsuit],[\clubsuit,\spadesuit]\},\\
\boxed{\spadesuit\diamondsuit\clubsuit\heartsuit} &\equiv \boxed{\spadesuit\heartsuit}\circ\boxed{\spadesuit\clubsuit}\circ\boxed{\spadesuit\diamondsuit} = \{[\spadesuit,\diamondsuit],[\heartsuit,\spadesuit],[\diamondsuit,\clubsuit],[\clubsuit,\heartsuit]\},\\
\boxed{\spadesuit\clubsuit\heartsuit\diamondsuit} &\equiv \boxed{\spadesuit\diamondsuit}\circ\boxed{\spadesuit\heartsuit}\circ\boxed{\spadesuit\clubsuit} = \{[\spadesuit,\clubsuit],[\heartsuit,\diamondsuit],[\diamondsuit,\spadesuit],[\clubsuit,\heartsuit]\},\\
\boxed{\spadesuit\clubsuit\diamondsuit\heartsuit} &\equiv \boxed{\spadesuit\heartsuit}\circ\boxed{\spadesuit\diamondsuit}\circ\boxed{\spadesuit\clubsuit} = \{[\spadesuit,\clubsuit],[\heartsuit,\spadesuit],[\diamondsuit,\heartsuit],[\clubsuit,\diamondsuit]\},
\end{aligned}$$

and write Π_0 for the collection of these six.

The remaining element of Π is the identity permutation. We shall denote it by

$$\boxed{\spadesuit\spadesuit}.$$

Each permutation p of Υ partitions Υ into the family of orbits of that permutation: we shall denote that partition by $\boxed{\mathsf{P}}$. For example we have

$$\boxed{\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}} = \{\{\spadesuit,\heartsuit\},\{\diamondsuit,\clubsuit\}\} \quad \text{and} \quad \boxed{\boxed{\spadesuit\heartsuit\diamondsuit}} = \{\{\spadesuit,\heartsuit,\diamondsuit\},\{\clubsuit\}\}.$$

(2.2) Quadriads For a set $\mathcal{M}$ with at least three points, we write $\mathcal{M}^{\Upsilon}$ for the set of functions from Υ to $\mathcal{M}$. We shall commonly express the values of elements t of $\mathcal{M}^{\Upsilon}$ using subscripts: $\mathsf{t}_{\spadesuit}$, $\mathsf{t}_{\heartsuit}$, $\mathsf{t}_{\diamondsuit}$, and $\mathsf{t}_{\clubsuit}$. We denote[8]

$$\begin{aligned}
\mathcal{M}^{\Upsilon}_{2+} &\equiv \{\mathsf{t}|\Upsilon \to \mathcal{M} : \#\{\mathsf{t}_\flat : \flat \in \Upsilon\} > 2\},\\
\mathcal{M}^{\Upsilon}_{3} &\equiv \{\mathsf{t}|\Upsilon \to \mathcal{M} : \#\{\mathsf{t}_\flat : \flat \in \Upsilon\} = 3\}\\
\text{and} \qquad \mathcal{M}^{\Upsilon}_{4} &\equiv \{\mathsf{t}|\Upsilon \to \mathcal{M} : \#\{\mathsf{t}_\flat : \flat \in \Upsilon\} = 4\}.
\end{aligned}$$

The permutations of Υ act on the elements of $\mathcal{M}^{\Upsilon}_{2+}$ through composition. For instance, for $\mathsf{t} \in \mathcal{M}^{\Upsilon}_{4}$, we have

$$\begin{aligned}
(\mathsf{t}\circ\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit})_{\spadesuit} &= \mathsf{t}_{\heartsuit},\\
(\mathsf{t}\circ\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit})_{\heartsuit} &= \mathsf{t}_{\spadesuit},\\
(\mathsf{t}\circ\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit})_{\diamondsuit} &= \mathsf{t}_{\clubsuit}\\
\text{and} \qquad (\mathsf{t}\circ\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit})_{\clubsuit} &= \mathsf{t}_{\diamondsuit}.
\end{aligned}$$

(2.3) Postulate I Suppose that the set $\mathcal{M}^{\Upsilon}_{2+}$ is equipped with an equivalence relation $\sim$. We shall

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6 For an ordered pair of points x and y we shall use the notation $[x, y]$, and for the value of a function f at an argument x we shall use the notation $f[x]$, that we may reserve normal parentheses ( ) for groupings.

7 These three, together with the identity permutation of Υ, make up the so-called *Klein Four Group*.

8 We denote the cardinality of any set S by $\#S$.

postulate four properties for $\sim$. The first states that equivalence classes are invariant relative to each of the four permutations of the Klein Four Group:

$$(\forall\, \mathfrak{t}\in\mathcal{M}^{\Upsilon}_{2+})(\forall\, \mathrm{x}\in\mathrm{II}_{02})\ \ \mathfrak{t}\circ\mathrm{x}\sim\mathfrak{t}. \tag{1}$$

(2.4) Definition Let $\triangle$ be in Υ and let $\{\flat,\natural,\sharp\}$ consist of the other elements of Υ. If

$$(\forall\, \mathfrak{t}\in\mathcal{M}^{\Upsilon}_{2+})(\forall\, A,B,C\in\mathcal{M}\text{ distinct})(\exists!\, \mathfrak{s}\in\mathcal{M}^{\Upsilon}_{2+})\ \ \mathfrak{s}\sim\mathfrak{t},\ \mathfrak{s}_{\flat}=A,\ \mathfrak{s}_{\sharp}=B,\ \mathfrak{s}_{\natural}=C$$

we shall say that $\triangle$ **complements $\sim$ on $\mathcal{M}$**.

(2.5) Theorem Suppose that $\triangle$ complements $\sim$ on $\mathcal{M}$. Then any other $\triangle'\in\Upsilon$ also complements $\sim$ on $\mathcal{M}$.

Proof. Let $\mathfrak{t}$ be in $\mathcal{M}^{\Upsilon}_{2+}$ and let $A,B,C\in\mathcal{M}$ be distinct. Let $\triangle'\in\Upsilon$ be distinct from $\triangle$ and let $\flat$, $\natural$ and $\sharp$ be such that $\Upsilon=\{\triangle',\flat,\natural,\sharp\}$. It follows from (2.4) that there is a unique element $\mathfrak{s}$ of $\mathcal{M}^{\Upsilon}_{2+}$ such that

$$\mathfrak{s}_{\flat}\equiv A,\ \text{if } \triangle\neq\flat,$$
$$\mathfrak{s}_{\natural}\equiv B,\ \text{if } \triangle\neq\natural,$$
$$\mathfrak{s}_{\sharp}\equiv C,\ \text{if } \triangle\neq\sharp,$$
$$\mathfrak{s}_{\triangle'}\equiv\begin{cases}A, & \text{if } \triangle=\flat\\ B, & \text{if } \triangle=\natural\\ C, & \text{if } \triangle=\sharp\end{cases}$$

and $\mathfrak{s}\sim\mathfrak{t}$. If y is the element of II_{02} that interchanges $\triangle$ and $\triangle'$, then Postulate I ((2.4)) implies that $\mathfrak{s}\sim\mathfrak{s}\circ\mathrm{y}$. It follows that

$$\mathfrak{t}\sim\mathfrak{s}\circ\mathrm{y},\quad (\mathfrak{s}\circ\mathrm{y})_{\flat}=a,\quad (\mathfrak{s}\circ\mathrm{y})_{\natural}=b\ \text{ and }\ (\mathfrak{s}\circ\mathrm{y})_{\sharp}=c.$$

QED

(2.6) Definition Theorem (2.5) means that if any element of Υ complements $\sim$ on $\mathcal{M}$, then all do. In this case, we shall say that $\sim$ **is complemented on** $\mathcal{M}$.

(2.7) Postulate II We postulate that the equivalence relation $\sim$ is complemented on $\mathcal{M}$.

(2.8) Notation We shall denote by

$$\mathfrak{M}$$

the family of $\sim$-equivalence classes. For each triple $(\flat,\sharp,\natural)$ of distinct elements of Υ and each triple (A,B,C) of distinct elements of $\mathcal{M}$, it follows from Postulate II and Theorem (2.5) that there is a unique function $\left[\begin{smallmatrix}A & B & C\\ \flat & \sharp & \natural\end{smallmatrix}\right]$ which sends each $\mathfrak{m}\in\mathfrak{M}$ to the unique $x\in\mathcal{M}$ such that

$$\{[\flat,A],[\sharp,B],[\natural,C],[\triangle,X]\}\in\mathfrak{m}\ \ \text{where } \Upsilon=\{\flat,\sharp,\natural,\triangle\}.$$

We shall write $\mathbf{Mor}(\mathfrak{M},\mathcal{M})$ for the family of all such functions and define

$$\Gamma(\mathcal{M},\sim)\equiv\{\phi\circ\theta^{-1}:\ \phi,\theta\in\mathbf{Mor}(\mathfrak{M},\mathcal{M})\}. \tag{1}$$

(2.9) Definitions and Notation We shall call elements of $\Gamma(\mathcal{M},\sim)$ **projectivities**.

For $\mathfrak{t}\in\mathcal{M}^{\Upsilon}_{3}$ let $\underline{\mathfrak{t}}$ denote the subset $\{\flat,\sharp\}$ of Υ such that $\mathfrak{t}_{\flat}=\mathfrak{t}_{\sharp}$.

We shall say that two elements $\mathfrak{t}$ and $\mathfrak{s}$ of $\mathcal{M}^{\Upsilon}_{2+}$ are **compatible** if the relation

$$\{[\mathfrak{t}_{\spadesuit},\mathfrak{s}_{\spadesuit}],[\mathfrak{t}_{\heartsuit},\mathfrak{s}_{\heartsuit}],[\mathfrak{t}_{\diamondsuit},\mathfrak{s}_{\diamondsuit}],[\mathfrak{t}_{\clubsuit},\mathfrak{s}_{\clubsuit}]\}$$

is a bijection of $\{\mathfrak{t}_{\spadesuit},\mathfrak{t}_{\heartsuit},\mathfrak{t}_{\diamondsuit},\mathfrak{t}_{\clubsuit}\}$ onto $\{\mathfrak{s}_{\spadesuit},\mathfrak{s}_{\heartsuit},\mathfrak{s}_{\diamondsuit},\mathfrak{s}_{\clubsuit}\}$. Evidently $\mathfrak{t}$ and $\mathfrak{s}$ are always compatible if they are both in $\mathcal{M}^{\Upsilon}_{4}$, and they cannot be compatible if one of them is in $\mathcal{M}^{\Upsilon}_{4}$ and one in $\mathcal{M}^{\Upsilon}_{3}$. If they are both in $\mathcal{M}^{\Upsilon}_{3}$, then they are compatible precisely when $\underline{\mathfrak{t}}=\underline{\mathfrak{s}}$.

(2.10) Postulate III Our third postulate is that compatible equivalent members of an element of $\mathfrak{M}$ be related by a single projectivity:

$$(\forall\, \mathfrak{t},\mathfrak{s}\in\mathcal{M}^{\Upsilon}_{2+}\text{ compatible})\ \ \mathfrak{t}\sim\mathfrak{s}\iff(\exists!\,\phi\in\Gamma(\mathcal{M},\sim))\ \ \mathfrak{s}=\phi\circ\mathfrak{t}. \tag{1}$$

(2.11) Notation In the following theorem and elsewhere in the sequel we shall adopt the notation

$$(\forall\, A,B,C,D\in\mathcal{M})\ \ \langle A,B,C,D\rangle\equiv\{[\spadesuit,A],[\heartsuit,B],[\diamondsuit,C],[\clubsuit,D]\}. \tag{1}$$

(2.12) Theorem[9] Let $A, B, C \in \mathcal{M}$ be distinct and $R, S, T \in \mathcal{M}$ be distinct. Then there exists a unique $\phi \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that

$$\phi(A) = R, \ \phi(B) = S, \text{ and } \phi(C) = T. \tag{1}$$

Proof. We first establish the existence of such a ϕ satisfying

$$\phi(A) = R, \ \phi(B) = S, \text{ and } \phi(C) = T. \tag{2}$$

Case $[A, B, C] = [R, S, T]$: Let ϕ be the identity function on $\mathcal{M}$.

Case $[A, B, C] = [R, T, S]$: Postulate I implies that $\langle A, A, B, C\rangle \sim \langle A, A, C, B\rangle$, whence follows from Postulate III the existence of $\alpha \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that $\alpha(A) = A$, $\alpha(B) = C$ and $\alpha(C) = B$. Evidently (2) holds.

Cases $[A, B, C] = [T, S, R]$ and $[A, B, C] = [S, R, T]$: The proofs are similar to that of the immediately previous case $[A, B, C] = [R, T, S]$.

Case $[A, B, C] = [S, T, R]$: As above, we choose $\alpha \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that $\alpha(A) = A$, $\alpha(B) = C$ and $\alpha(C) = B$. Postulate I implies that $\langle B, B, C, A\rangle \sim \langle B, B, A, C\rangle$ whence follows from Postulate III the existence of $\beta \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that $\beta(B) = B$, $\beta(C) = A$ and $\beta(A) = C$. Let $X \in \mathcal{M}$ be distinct from A, B, and C. Postulate III implies that

$$\langle A, B, C, X\rangle \sim \langle \alpha(A), \alpha(B), \alpha(C), \alpha(X)\rangle = \langle A, C, B, \alpha(X)\rangle \sim \langle \beta(A), \beta(C), \beta(B), \beta(\alpha(X))\rangle =$$
$$\langle C, A, B, \beta \circ \alpha(X)\rangle = \langle S, T, R, \beta \circ \alpha(X)\rangle.$$

Postulate III now implies that there exists ϕ such that $\langle \phi(A), \phi(B), \phi(C), \phi(X)\rangle = \langle R, S, T, \beta\circ\alpha(X)\rangle$, whence (2) holds.

Case $[A, B, C] = [T, R, S]$: The proof is similar to that of the case $[A, B, C] = [S, T, R]$.

Now we demonstrate the uniqueness of ϕ. Let θ be another element of $\mathit{\Gamma}(\mathcal{M}, \sim)$ such that $\theta(A) = R$, $\theta(B) = S$, and $\theta(C) = T$. Let X be any element of $\mathcal{M}$ not in $\{A, B, C\}$. We have by Postulate III

$$\langle R, S, T, \phi(X)\rangle = \langle \phi((A), \phi(B), \phi(C), \phi(X)\rangle \sim \langle A, B, C, X\rangle \sim \langle \theta(A), \theta(B), \theta(C), \theta(X)\rangle = \langle R, S, T, \theta(X)\rangle.$$

QED

(2.13) Notation Each element of $\mathcal{M}_3^{\Upsilon}$ is in exactly one of the following sets:[10]

$$\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit} \equiv \{\mathfrak{q} \in \mathcal{M}_3^{\Upsilon} : \underline{\mathfrak{q}} \in \{\{\spadesuit, \heartsuit\}, \{\diamondsuit, \clubsuit\}\}\},$$
$$\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit} \equiv \{\mathfrak{q} \in \mathcal{M}_3^{\Upsilon} : \underline{\mathfrak{q}} \in \{\{\spadesuit, \diamondsuit\}, \{\heartsuit, \clubsuit\}\}\},$$
$$\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit} \equiv \{\mathfrak{q} \in \mathcal{M}_3^{\Upsilon} : \underline{\mathfrak{q}} \in \{\{\spadesuit, \clubsuit\}, \{\heartsuit, \diamondsuit\}\}\}.$$

(2.14) Theorem Each of the sets $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$, $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$ and $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$ is in $\mathfrak{M}$.

Proof. Let $\mathfrak{x}$ and $\mathfrak{y}$ be in $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$. Define

$$\mathfrak{x}' \equiv \begin{cases} \mathfrak{x} & \text{if } \underline{\mathfrak{x}} = \{\spadesuit, \heartsuit\}, \\ \mathfrak{x} \circ \boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit} & \text{if } \underline{\mathfrak{x}} = \{\diamondsuit, \clubsuit\}, \end{cases} \text{ and } \mathfrak{y}' \equiv \begin{cases} \mathfrak{y} & \text{if } \underline{\mathfrak{x}} = \{\spadesuit, \heartsuit\}, \\ \mathfrak{y} \circ \boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit} & \text{if } \underline{\mathfrak{x}} = \{\diamondsuit, \clubsuit\}. \end{cases}$$

Thus $\mathfrak{x}'$ and $\mathfrak{y}'$ are compatible.

By the fundamental theorem there exists a unique $\phi \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that $\phi(\mathfrak{x}'_{\heartsuit}) = \mathfrak{y}'_{\heartsuit}$, $\phi(\mathfrak{x}'_{\diamondsuit}) = \mathfrak{y}'_{\diamondsuit}$, and $\phi(\mathfrak{x}'_{\clubsuit}) = \mathfrak{y}'_{\clubsuit}$. By construction we have $\mathfrak{x}'_{\spadesuit} = \mathfrak{x}'_{\heartsuit}$ and $\mathfrak{y}'_{\spadesuit} = \mathfrak{y}'_{\heartsuit}$. It follows that $\mathfrak{y}' = \phi \circ \mathfrak{x}'$. By Postulate III we have $\mathfrak{y}' \sim \mathfrak{x}'$. Since $\mathfrak{x} \sim \mathfrak{x}'$ and $\mathfrak{y} \sim \mathfrak{y}'$ by Postulate I, it follows that $\mathfrak{x} \sim \mathfrak{y}$. Thus $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$ is a subset of some element of $\mathfrak{M}$.

To show the reverse inclusion, we consider an element $\mathfrak{x}$ of $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$, an element $\mathfrak{z}$ of $\mathcal{M}_{2+}^{\Upsilon}$ equivalent to $\mathfrak{x}$, and deduce that $\mathfrak{z}$ must be in $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$. Choose $\flat, \sharp, \natural \in \Upsilon$ such that $\mathfrak{z}_\flat$, $\mathfrak{z}_\sharp$ and $\mathfrak{z}_\natural$ are distinct. If $\mathfrak{x}_{\spadesuit} = \mathfrak{x}_{\heartsuit}$ and $\{\spadesuit, \heartsuit\} \subset \{\flat, \sharp, \natural\}$, or if $\mathfrak{x}_{\diamondsuit} = x_{\clubsuit}$ and $\{\diamondsuit, \clubsuit\} \subset \{\flat, \sharp, \natural\}$, let $\mathfrak{x}' \equiv \mathfrak{x} \circ \boxed{\spadesuit\clubsuit|\diamondsuit\heartsuit}$ — else let $\mathfrak{x}' \equiv \mathfrak{x}$. Then $\#\{\mathfrak{x}'_\flat, \mathfrak{x}'_\sharp, \mathfrak{x}'_\natural\} = 3$ and $\mathfrak{x}'$ is in $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$. The fundamental theorem implies that there exists $\phi \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that

$$\phi(\mathfrak{x}')_\flat = \mathfrak{z}_\flat, \ \phi(\mathfrak{x}')_\sharp = \mathfrak{z}_\sharp, \text{ and } \phi(\mathfrak{x}')_\natural = \mathfrak{z}_\natural. \tag{1}$$

Let $\triangle$ be such that $\{\flat, \sharp, \natural, \triangle\} = \Upsilon$. Postulate II implies that $\mathfrak{z}_\triangle$ is the unique element of $\mathcal{M}$ such that

$$\{(\flat, \mathfrak{z}_\flat), (\sharp, \mathfrak{z}_\sharp), (\natural, \mathfrak{z}_\natural), (\triangle, \mathfrak{z}_\triangle)\} \sim \mathfrak{x}'. \tag{2}$$

Postulate III implies

$$\mathfrak{x}' \sim \phi \circ \mathfrak{x}' = \{(\flat, \mathfrak{z}_\flat), (\sharp, \mathfrak{z}_\sharp), (\natural, \mathfrak{z}_\natural), (\triangle, \phi(\mathfrak{x}'_\triangle))\}. \tag{3}$$

[9] We shall refer to this as the fundamental theorem.

[10] Recall that for $\mathfrak{t} \in \mathcal{M}_3^{\Upsilon}$, $\underline{\mathfrak{t}}$ is the subset $\{\flat, \sharp\}$ of Υ such that $\mathfrak{t}_\flat = \mathfrak{t}_\sharp$.

From (2) and (3) follows that $\mathfrak{z}_{\triangle}=\phi(\mathfrak{x}'_{\triangle})$. From this and (1) follows that $\mathfrak{z}=\phi\circ\mathfrak{x}'$.

Since, by Postulate I, $\mathfrak{x}'$ is in $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$, either $\mathfrak{x}'_{\spadesuit}=\mathfrak{x}'_{\heartsuit}$ or $\mathfrak{x}'_{\diamondsuit}=\mathfrak{x}'_{\clubsuit}$. Consequently, either $\mathfrak{z}_{\spadesuit}=\mathfrak{z}_{\heartsuit}$ or $\mathfrak{z}_{\diamondsuit}=\mathfrak{z}_{\clubsuit}$. Thus $\mathfrak{z}$ is in $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$. It has now been demonstrated that $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$ is in $\mathfrak{M}$.

That $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$ and $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$ are in $\mathfrak{M}$, can be shown by analogous arguments. QED

(2.15) Theorem The family $\Gamma(\mathcal{M},\sim)$ is a group of transformations of $\mathcal{M}$.

Proof. Let ϕ and θ be in $\Gamma(\mathcal{M},\sim)$ and let $\mathfrak{t}$ be an element $\langle a,b,c,d\rangle$ of $\mathcal{M}_4^{\Upsilon}$. From Postulate III follows

$$\mathfrak{t}\sim\theta\circ\mathfrak{t}\sim\phi\circ\theta\circ\mathfrak{t}.$$

From Postulate III follows that there exists a unique $\gamma\in\Gamma(\mathcal{M},\sim)$ such that

$$\gamma\circ\mathfrak{t}=\phi\circ\theta\circ\mathfrak{t}.$$

Let X be any element of $\mathcal{M}$ and set $\mathfrak{s}\equiv\langle a,b,c,x\rangle$. As above we can find $\delta\in\Gamma(\mathcal{M},\sim)$ such that

$$\delta\circ\mathfrak{s}=\phi\circ\theta\circ\mathfrak{s}. \tag{1}$$

We have $\delta(a)=\delta(\mathfrak{s}_{\spadesuit})=\phi\circ\theta(\mathfrak{s}_{\spadesuit})=\phi\circ\theta(a)=\phi\circ\theta(\mathfrak{t}_{\spadesuit})=\gamma(\mathfrak{t}_{\spadesuit})=\gamma(a)$. Similarly we have $\delta(b)=\gamma(b)$ and $\delta(c)=\gamma(c)$. The fundamental theorem implies that $\gamma=\delta$. It follows that

$$\gamma(x)=\delta(x)=\delta(\mathfrak{s}_{\clubsuit})\overset{\text{by (1)}}{=\!=\!=}\phi\circ\theta(\mathfrak{s}_{\clubsuit})=\phi\circ\theta(x).$$

Thus the element γ of $\Gamma(\mathcal{M},\sim)$ is just the composition $\phi\circ\theta$.

By definition there exist $\alpha,\beta\in\mathbf{Mor}(\mathfrak{M},\mathcal{M})$ such that $\phi=\alpha\circ\beta^{-1}$. Thus $\phi^{-1}=\beta\circ\alpha^{-1}$ and so is in $\mathbf{Mor}(\mathfrak{M},\mathcal{M})$. We have shown that $\Gamma(\mathcal{M},\sim)$ is a group. QED

(2.16) Notation Recall that for distinct $A,B,C\in\mathcal{M}$ the function $\left[\begin{smallmatrix}A&B&C\\ \heartsuit&\diamondsuit&\clubsuit\end{smallmatrix}\right]$ sends each element $\mathfrak{m}$ of $\mathfrak{M}$ to that single element T of $\mathcal{M}$ such that $\langle T,A,B,C\rangle\in\mathfrak{m}$. If A, B and C are distinct elements of $\mathcal{M}$ and U, V and W are distinct elements of $\mathcal{M}$ we apply the fundamental theorem (2.12) to obtain a unique element $\left[\begin{smallmatrix}U&V&W\\ A&B&C\end{smallmatrix}\right]$ of $\Gamma(\mathcal{M})$ such that

$$\begin{bmatrix}U&V&W\\ A&B&C\end{bmatrix}(A)=U,\ \begin{bmatrix}U&V&W\\ A&B&C\end{bmatrix}(B)=V,\ \text{and}\ \begin{bmatrix}U&V&W\\ A&B&C\end{bmatrix}(C)=W. \tag{1}$$

(2.17) Theorem Let $\mathcal{M}$ satisfy Postulates I, II and III, and suppose that $\mathcal{M}$ has at least three distinct points. Then

$$\Gamma(\mathcal{M},\sim)=\{\phi|\mathcal{M}\to\mathcal{M}\text{ a bijection}:(\forall\,\mathfrak{t}\in\mathfrak{M})\quad\mathfrak{t}\sim\phi\circ\mathfrak{t}\}. \tag{1}$$

Proof. Suppose first that ϕ in $\Gamma(\mathcal{M},\sim)$. Let $\mathfrak{t}\in\mathfrak{M}$ and choose $A,B,C,D\in\mathcal{M}$ such that $\langle A,B,C,D\rangle=\mathfrak{t}$. Since $\{A,B,C,D\}$ has at least three elements, the fundamental theorem implies that ϕ is the only element of $\Gamma(\mathcal{M},\sim)$ of which the composition with $\mathfrak{t}$ is $\phi\circ\mathfrak{t}=\langle\phi(A),\phi(B),\phi(C),\phi(D)\rangle$. It follows from Postulate III that $\mathfrak{t}\sim\phi\circ\mathfrak{t}$.

Now we suppose instead that ϕ is in the right-hand set of (1). Let $A,B,$ and C be distinct elements of $\mathcal{M}$. Let $D\equiv\phi(A)$, $E\equiv\phi(B)$ and $F\equiv\phi(C)$. Since $\left[\begin{smallmatrix}D&E&F\\ A&B&C\end{smallmatrix}\right]$ is in $\Gamma(\mathcal{M},\sim)$, we know from the preceding paragraph that, for generic $X\in\mathcal{M}$, $\langle A,B,C,X\rangle\sim\langle D,E,F,\left[\begin{smallmatrix}D&E&F\\ A&B&C\end{smallmatrix}\right](X)\rangle$. But by assumption we have $\langle A,B,C,X\rangle\sim\langle D,E,F,\phi(X)\rangle$, which thus implies

$$\langle D,E,F,\begin{bmatrix}D&E&F\\ A&B&C\end{bmatrix}(X)\rangle\sim\langle D,E,F,\phi(X)\rangle.$$

By Postulate II ((2.7) and (2.6)), it follows that $\phi(X)=\left[\begin{smallmatrix}D&E&F\\ A&B&C\end{smallmatrix}\right](X)$. Hence ϕ is just $\left[\begin{smallmatrix}D&E&F\\ A&B&C\end{smallmatrix}\right]$ and so is in $\Gamma(\mathcal{M},\sim)$. QED

(2.18) Theorem For distinct $A,B,C\in\mathcal{M}$, $\left[\begin{smallmatrix}A&B&C\\ \heartsuit&\diamondsuit&\clubsuit\end{smallmatrix}\right]$ is the unique element of $\mathbf{Mor}(\mathfrak{M},\mathcal{M})$ which takes $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$ to A, $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$ to B, and $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$ to C.

For $\alpha,\beta,\gamma\in\mathbf{Mor}(\mathfrak{M},\mathcal{M})$, the function $\gamma\circ\beta^{-1}\circ\alpha$ is again in $\mathbf{Mor}(\mathfrak{M},\mathcal{M})$.

Proof. By definition, $\langle\left[\begin{smallmatrix}A&B&C\\ \heartsuit&\diamondsuit&\clubsuit\end{smallmatrix}\right](\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}),A,B,C\rangle$ is in $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$. Since $B\neq C$ we have $A=\left[\begin{smallmatrix}A&B&C\\ \heartsuit&\diamondsuit&\clubsuit\end{smallmatrix}\right](\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit})$. That $B=\left[\begin{smallmatrix}A&B&C\\ \heartsuit&\diamondsuit&\clubsuit\end{smallmatrix}\right](\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit})$, and $C=\left[\begin{smallmatrix}A&B&C\\ \heartsuit&\diamondsuit&\clubsuit\end{smallmatrix}\right](\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit})$ follow from analogous reasoning.

Evidently we have

$$\begin{bmatrix}U&V&W\\ R&S&T\end{bmatrix}=\begin{bmatrix}U&V&W\\ \heartsuit&\diamondsuit&\clubsuit\end{bmatrix}\circ\begin{bmatrix}R&S&T\\ \heartsuit&\diamondsuit&\clubsuit\end{bmatrix}^{-1}. \tag{1}$$

If there were another element ϕ of $\mathbf{Mor}(\mathfrak{M},\mathcal{M})$ distinct from $\left[\begin{smallmatrix} A & B & C \\ \heartsuit & \diamondsuit & \clubsuit \end{smallmatrix}\right]$ which sent $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$ to A, $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$ to B and $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$ to C, then $\left[\begin{smallmatrix} A & B & C \\ \heartsuit & \diamondsuit & \clubsuit \end{smallmatrix}\right] \circ \phi^{-1}$ would be an element θ of $\mathit{\Gamma}(\mathcal{M},\sim)$ distinct from the identity transformation. But this would be absurd since θ would leave A, B, and C fixed, and the fundamental theorem would be violated. This shows the uniqueness of $\left[\begin{smallmatrix} A & B & C \\ \heartsuit & \diamondsuit & \clubsuit \end{smallmatrix}\right]$.

Suppose that α sends $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$, $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$, and $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$ respectively to A, B and C respectively, that β sends them to R, S and T respectively, and that γ send them to U, V and W respectively. Then

$$\alpha = \begin{bmatrix} A & B & C \\ \heartsuit & \diamondsuit & \clubsuit \end{bmatrix} \quad \text{and} \quad \gamma \circ \beta^{-1} = \begin{bmatrix} U & V & W \\ R & S & T \end{bmatrix}. \tag{2}$$

Let

$$X \equiv \begin{bmatrix} U & V & W \\ R & S & T \end{bmatrix}(A),\ Y \equiv \begin{bmatrix} U & V & W \\ R & S & T \end{bmatrix}(B),\ \text{and } Z \equiv \begin{bmatrix} U & V & W \\ R & S & T \end{bmatrix}(C).$$

From the fundamental theorem follows that

$$\begin{bmatrix} X & Y & Z \\ A & B & C \end{bmatrix} = \begin{bmatrix} U & V & W \\ R & S & T \end{bmatrix}. \tag{3}$$

Thus

$$\gamma \circ \beta^{-1} \circ \alpha \overset{\text{by (2)}}{=\!=\!=} \begin{bmatrix} U & V & W \\ R & S & T \end{bmatrix} \circ \begin{bmatrix} A & B & C \\ \heartsuit & \diamondsuit & \clubsuit \end{bmatrix} \overset{\text{by (3)}}{=\!=\!=} \begin{bmatrix} X & Y & Z \\ A & B & C \end{bmatrix} \circ \begin{bmatrix} A & B & C \\ \heartsuit & \diamondsuit & \clubsuit \end{bmatrix} = \begin{bmatrix} X & Y & Z \\ \heartsuit & \diamondsuit & \clubsuit \end{bmatrix}.$$

This demonstrates the second part of Theorem (2.18). QED

(2.19) Theorem Let $\mathtt{x}$ be a permutation of Υ. If $\mathfrak{t} \sim \mathfrak{s}$ in $\mathcal{M}_{2+}^{\Upsilon}$, then $\mathfrak{t}\circ\mathtt{x} \sim \mathfrak{s}\circ\mathtt{x}$ as well.

Proof. If $\mathfrak{t}$ and $\mathfrak{s}$ are compatible, by Postulate III, $\mathfrak{s}=\phi\circ\mathfrak{t}$ for some $\phi\in\mathit{\Gamma}(\mathcal{M},\sim)$ and so $\mathfrak{s}\circ\mathtt{x}=\phi\circ\mathfrak{t}\circ\mathtt{x} \sim \mathfrak{t}\circ\mathtt{x}$, again by Postulate III.

If $\mathfrak{t}$ and $\mathfrak{s}$ are incompatible, then Theorem (2.18) implies that both $\mathfrak{t}$ and $\mathfrak{s}$ are in the same element of $\mathcal{A} \equiv \{\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}, \boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}, \boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}\}$. The permutation $\mathtt{x}$ evidently takes each element of $\mathcal{A}$ to another element of $\mathcal{A}$. It follows that $\mathfrak{t}\circ\mathtt{x} \sim \mathfrak{s}\circ\mathtt{x}$. QED

(2.20) Notation For $\mathfrak{t}\in\mathcal{M}_{2+}^{\Upsilon}$ we shall write $\lfloor\mathfrak{t}\rfloor$ for the element of $\mathfrak{M}$ of which $\mathfrak{t}$ is a member.

(2.21) Corollary and Notation Each permutation $\mathtt{x}$ of Υ induces a well-defined bijection of $\mathfrak{M}$ as follows:

$$\overline{\mathtt{x}}|\mathfrak{M} \ni \lfloor\mathfrak{t}\rfloor \hookrightarrow \lfloor\mathfrak{t}\circ\mathtt{x}\rfloor \in \mathfrak{M}. \tag{1}$$

Proof. This follows from applying (2.19) to (2.20). QED

(2.22) Notation We shall write $\mathit{\Gamma}(\mathfrak{M})$ for the set $\{\alpha^{-1}\circ\beta\colon\ \alpha,\beta\in\mathbf{Mor}(\mathfrak{M},\mathcal{M})\}$.

(2.23) Theorem The family $\mathit{\Gamma}(\mathfrak{M})$ is a group under composition, and is isomorphic to $\mathit{\Gamma}(\mathcal{M},\sim)$.

Proof. For any $\theta\in\mathbf{Mor}(\mathfrak{M},\mathcal{M})$ the function

$$\mathit{\Gamma}(\mathcal{M},\sim) \ni \phi \hookrightarrow \theta^{-1}\circ\phi\circ\theta \in \mathit{\Gamma}(\mathfrak{M}) \tag{1}$$

is bijective and preserves the composition operator: thus it is an isomorphism of groups. QED

(2.24) Corollary $(\forall\,\mathfrak{A},\mathfrak{B},\mathfrak{C}\in\mathfrak{M}\text{ distinct})(\forall\,\mathfrak{R},\mathfrak{S},\mathfrak{T}\in\mathfrak{M}\text{ distinct})$

$$(\exists!\,\phi\in\mathit{\Gamma}(\mathfrak{M})) \quad \phi(\mathfrak{A})=\mathfrak{R},\ \phi(\mathfrak{B})=\mathfrak{S},\ \text{and } \phi(\mathfrak{C})=\mathfrak{T}. \tag{1}$$

Proof. This follows from applying (2.23) to the fundamental theorem (2.12). QED

(2.25) Discussion It is a direct consequence of Postulate I that, for each $\mathtt{x}\in\amalg_{02}$, $\overline{\mathtt{x}}$ is the identity transformation.

For $\amalg_2$ and $\amalg_0$ we have

$$\overline{\boxed{\spadesuit\heartsuit}} = \overline{\boxed{\diamondsuit\clubsuit}} = \overline{\boxed{\spadesuit\diamondsuit\heartsuit\clubsuit}} = \overline{\boxed{\spadesuit\clubsuit\heartsuit\diamondsuit}}, \qquad \overline{\boxed{\spadesuit\diamondsuit}} = \overline{\boxed{\heartsuit\clubsuit}} = \overline{\boxed{\spadesuit\heartsuit\diamondsuit\clubsuit}} = \overline{\boxed{\spadesuit\clubsuit\diamondsuit\heartsuit}} \quad \text{and} \quad \overline{\boxed{\spadesuit\clubsuit}} = \overline{\boxed{\heartsuit\diamondsuit}} = \overline{\boxed{\spadesuit\heartsuit\clubsuit\diamondsuit}} = \overline{\boxed{\spadesuit\diamondsuit\clubsuit\heartsuit}}.$$

Furthermore

$\overline{\boxed{\spadesuit\heartsuit}}$ leaves $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$ fixed and interchanges $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$ with $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$,
$\overline{\boxed{\spadesuit\diamondsuit}}$ leaves $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$ fixed and interchanges $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$ with $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$
and $\overline{\boxed{\spadesuit\clubsuit}}$ leaves $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$ fixed and interchanges $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$ with $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$.

For $\amalg_1$ and we have

$$\overline{\boxed{\spadesuit\heartsuit\diamondsuit}} = \overline{\boxed{\spadesuit\clubsuit\heartsuit}} = \overline{\boxed{\heartsuit\clubsuit\diamondsuit}} = \overline{\boxed{\spadesuit\diamondsuit\clubsuit}} \text{ and } \overline{\boxed{\spadesuit\diamondsuit\heartsuit}} = \overline{\boxed{\spadesuit\heartsuit\clubsuit}} = \overline{\boxed{\heartsuit\diamondsuit\clubsuit}} = \overline{\boxed{\spadesuit\clubsuit\diamondsuit}}.$$

Specifically

$$\overline{\boxed{\spadesuit\heartsuit\diamondsuit}} \text{ sends } \boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit} \text{ to } \boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit} \text{ to } \boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit} \text{ and then back to } \boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$$

and

$$\overline{\boxed{\spadesuit\diamondsuit\heartsuit}} \text{ sends } \boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit} \text{ to } \boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit} \text{ to } \boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit} \text{ and then back to } \boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}.$$

(2.26) Definition One consequence of the fundamental theorem is that no projectivity, other than the identity transformation, can have more than two fixed points. The fundamental theorem also implies that every two points are fixed under some transformation, but *per se* it does not insure that there are projectivities with a single fixed point. We shall call a projectivity with a single fixed point a **translation**[11].

(2.27) Example Thus far nothing has been postulated about $\mathcal{M}$ which implies that $\mathcal{M}$ has more than 3 points. In fact, if A, B and C are any three distinct points, then the sets $\mathcal{M}\equiv\{A,B,C\}$ and $\mathfrak{M}\equiv\{\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}, \boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}, \boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}\}$ satisfy Postulates I through III. In this simplest of all cases it follows from the fundamental theorem that $\Gamma(\mathcal{M},\sim)$ (as well as $\mathbf{Mor}(\mathfrak{M},\mathcal{M})$ and $\Gamma(\mathfrak{M})$) have exactly 6 members. Specifically here we have

$$\Gamma(\mathfrak{M}) = \{\overline{\boxed{\spadesuit\spadesuit}}, \overline{\boxed{\spadesuit\heartsuit}}, \overline{\boxed{\spadesuit\diamondsuit}}, \overline{\boxed{\spadesuit\clubsuit}}, \overline{\boxed{\spadesuit\heartsuit\diamondsuit}}, \overline{\boxed{\spadesuit\diamondsuit\heartsuit}}\}.$$

The transformations $\overline{\boxed{\spadesuit\heartsuit}}$ $\overline{\boxed{\spadesuit\diamondsuit}}$, and $\overline{\boxed{\spadesuit\clubsuit}}$ are translations. This is however atypical, and our fourth postulate *infra* will invalidate this example.

(2.28) Notation and Definition For any positive integer n, we write $\underset{\sim}{\Gamma}^n(\mathcal{M})$ for the set of $\phi\in\Gamma(\mathcal{M},\sim)$ such that $\overbrace{\phi\circ\ldots\circ\phi}^{n\text{ copies}}$ equals the identity transformation ι (and does not for any smaller positive n) — $\Gamma^n(\mathfrak{M})$ is defined analogously. An element of $\underset{\sim}{\Gamma}^2(\mathcal{M})$ will sometimes be referred to as an **involution**.

The identity transformation $\overline{\boxed{\spadesuit\spadesuit}}$ is the sole member of $\Gamma^1(\mathfrak{M})$. The fundamental theorem provides numerous elements of $\Gamma^2(\mathfrak{M})$: in particular $\overline{\boxed{\spadesuit\heartsuit}}$, $\overline{\boxed{\spadesuit\diamondsuit}}$, and $\overline{\boxed{\spadesuit\clubsuit}}$. The transformations $\overline{\boxed{\spadesuit\heartsuit\diamondsuit}}$ and $\overline{\boxed{\spadesuit\diamondsuit\heartsuit}}$ are in $\Gamma^3(\mathfrak{M})$, again by the fundamental theorem.

(2.29) Theorem Let A, B, C and D be four distinct points of $\mathcal{M}$. Then there exists $\phi\in\underset{\sim}{\Gamma}^2(\mathcal{M})$ such that $\phi(A)=B$ and $\phi(C)=D$.

Proof. By Postulate I we have $\langle A,B,C,D\rangle\sim\langle B,A,D,C\rangle$ and by Postulate III there exists $\phi\in\Gamma(\mathcal{M},\sim)$ such that $\phi(A)=B$, $\phi(B)=A$, $\phi(C)=D$ and $\phi(D)=C$. Since $\phi\circ\phi$ agrees with the identity element of $\Gamma(\mathcal{M},\sim)$ on A, B and C, it follows from the fundamental theorem that it is the identity: that ϕ is in $\underset{\sim}{\Gamma}^2(\mathcal{M})$. QED

(2.30) Theorem Let ϕ be an element of $\Gamma(\mathcal{M},\sim)$ and let $A,B\in\mathcal{M}$ distinct be such that $\phi(A)=B$ and $\phi(B)=A$. Then ϕ is in $\underset{\sim}{\Gamma}^2(\mathcal{M})$.

Proof. If $\mathcal{M}$ has only three points, then the conclusion follows from Example (2.27). We presume then that $\mathcal{M}$ has at least four points and that $\phi\notin\underset{\sim}{\Gamma}^2(\mathcal{M})$. Then there exists $C,D\in\mathcal{M}$ distinct from A and B such that $\phi(C)=D$ and $\phi(D)\neq C$. By (2.29) there exists $\theta\in\underset{\sim}{\Gamma}^2(\mathcal{M})$ such that $\theta(A)=B$ and $\theta(C)=D$. From the fundamental theorem (2.12) follows that $\theta=\phi$, which is absurd. QED

(2.31) Theorem Let $\mathfrak{A},\mathfrak{B},\mathfrak{C}\in\mathfrak{M}$ be distinct. Then there exists $\theta\in\Gamma^2(\mathfrak{M})$ such that $\theta(\mathfrak{A})=\mathfrak{C}$ and $\theta((\mathfrak{B})=\mathfrak{B}$.

Proof. By (2.24) there exists $\phi\in\Gamma(\mathfrak{M})$ such that

$$\phi((\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}) = \mathfrak{A},\ \phi((\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}) = \mathfrak{B},\ \text{and } \phi((\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}) = \mathfrak{C}.$$

The element $\overline{\boxed{\spadesuit\diamondsuit}}$ of $\Gamma(\mathfrak{M})$ fixes $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$ and interchanges $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$ and $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$. Thus we may let $\theta\equiv\phi\circ\overline{\boxed{\spadesuit\diamondsuit}}\circ\phi^{-1}$. QED

(2.32) Corollary Let $A,B,C\in\mathcal{M}$ be distinct. Then there exists $\theta\in\underset{\sim}{\Gamma}^2(\mathcal{M})$ such that $\theta(A)=C$ and $\theta(B)=B$.

Proof. Apply (2.23) to (2.31). QED

(2.33) Theorem Let ϕ be an element of $\Gamma(\mathcal{M},\sim)$ not in $\underset{\sim}{\Gamma}^2(\mathcal{M})$. Then ϕ is the composition of two involutions. In particular, if $C\in\mathcal{M}$ is such that $\phi(C)\neq C$ and if $\beta\in\Gamma(\mathcal{M},\sim)$ satisfies

$$\beta(\phi^{-1}(C)) = \phi(C),\ \beta(\phi(C)) = \phi^{-1}(C) \text{ and } \beta(C)=C, \tag{1}$$

[11] An element of $\Gamma(\mathfrak{M})$ with a single fixed point will also be called a translation.

then the transformations β and $\alpha \equiv \phi \circ \beta$ are in $\underset{\sim}{\mathit{\Gamma}}^2(\mathcal{M})$ and $\phi = \alpha \circ \beta$.

Proof. That β is in $\underset{\sim}{\mathit{\Gamma}}^2(\mathcal{M})$ follows from (2.30) and (2.33.1). We have

$$\alpha(C) = \phi(\beta(C)) = \phi(C) \quad \text{and} \quad \alpha(\phi(C)) = \phi(\beta(\phi(C))) = \phi(\phi^{-1}(C)) = C$$

which with (2.30) implies that α is in $\underset{\sim}{\mathit{\Gamma}}^2(\mathcal{M})$. That $\phi = \alpha \circ \beta$ follows from the definition of α. QED

(2.34) Theorem Let $\phi, \theta \in \underset{\sim}{\mathit{\Gamma}}^2(\mathcal{M})$ and $P, Q \in \mathcal{M}$ be such that ϕ and θ agree on both P and Q. Suppose that P is distinct from both Q and $\phi(Q)$. Then $\phi = \theta$.

Proof. Let $T \in \mathcal{M}$ be distinct from P and Q.

We first consider the case wherein both P and Q are both left fixed by ϕ. We have $\phi(T) \neq T$ by definition of $\underset{\sim}{\mathit{\Gamma}}^2(\mathcal{M})$. It follows that $\langle P, Q, T, \phi(T)\rangle \in \mathcal{M}_4^{\Upsilon}$. Postulate I implies that $\langle T, \phi(T), P, Q\rangle \sim \langle P, Q, T, \phi(T)\rangle$, and so Postulate III implies that there exists $\mu \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that

$$\mu(T) = P,\ \mu(\phi(T)) = Q,\ \mu(P) = T,\ \text{and}\ \mu(Q) = \phi(T).$$

Similarly, there exists $\nu \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that

$$\nu(T) = P,\ \nu(\theta(T)) = Q,\ \nu(P) = T,\ \text{and}\ \nu(Q) = \theta(T)$$

Both the transformations $\mu^{-1} \circ \phi \circ \mu$ and $\nu^{-1} \circ \theta \circ \nu$ evidently interchange P and Q and leave T fixed. Thus the *fundamental theorem* implies that $\mu^{-1} \circ \phi \circ \mu = \nu^{-1} \circ \theta \circ \nu$. Consequently they have the same fixed points. The fixed points of the first are t and $\phi(T)$; those of the second are T and $\theta(T)$. Hence $\phi(T) = \theta(T)$. Since T was taken arbitrarily, we have $\phi = \theta$.

Now we consider the case wherein $\phi(P) \neq P$. Then ϕ and θ agree not only on P and Q, but also

$$\theta(\phi(P)) = \theta(\theta(P)) = P = \phi(\phi(P)).$$

By the *fundamental theorem* they must be identical.

The demonstration of the remaining case, wherein $\phi(Q) \neq Q$, is analogous. QED

(2.35) Theorem If $\theta \in \underset{\sim}{\mathit{\Gamma}}^2(\mathcal{M})$ is distinct from $\phi \in \underset{\sim}{\mathit{\Gamma}}^2(\mathcal{M})$ but has a fixed point in common with ϕ, then $\phi \circ \theta$ is a translation.[12]

Proof. Choose $P \in \mathcal{M}$ such that $\phi(P) = \theta(P) = P$ and suppose that $Q \in \mathcal{M}$ satisfies $\phi \circ \theta(Q) = Q$. Then $\phi(Q) = \theta(Q)$. If Q were distinct from P, then Theorem (2.24) would imply the absurdity that $\phi = \theta$. QED

(2.36) Discussion and Definition We have yet to introduce the fourth postulate for $\mathcal{M}$ and $\sim$. It will be closely connected with the existence of what we call harmonic pairs and quadric cycles.

An element $\mathfrak{t}$ of $\mathcal{M}_4^{\Upsilon}$ will be said to be a **quadric cycle** provided their exists $\phi \in \mathit{\Gamma}(\mathcal{M}, \sim)$ such that

$$\phi \circ \mathfrak{t} = \mathfrak{t} \circ \boxed{\spadesuit\heartsuit\diamondsuit\clubsuit}.$$

In this case the transformation ϕ must be in $\underset{\sim}{\mathit{\Gamma}}^4(\mathcal{M})$ because $\phi \circ \phi \circ \phi \circ \phi$ evidently leaves $\mathfrak{t}_{\spadesuit}$, $\mathfrak{t}_{\heartsuit}$, $\mathfrak{t}_{\diamondsuit}$, and $\mathfrak{t}_{\clubsuit}$ all fixed.

A ordered quadruple $[A, B, C, D]$ such that $B = \phi(A)$, $C = \phi(B)$, $D = \phi(C)$, and $A = \phi(D)$ is a **ϕ-orbit**. Clearly, necessary and sufficient conditions for $[A, B, C, D]$ to be a ϕ-orbit are for $[B, C, D, A]$, $[C, D, A, B]$, and $[D, A, B, C]$ to be ϕ-orbits. Replacing ϕ by its inverse ϕ^{-1}, we see also that $[A, B, C, D]$ is a ϕ-orbit if and only if $[D, C, B, A]$ is a ϕ^{-1}-orbit. If T is a point in a quadruple ϕ-orbit, we shall say that T and $\phi \circ \phi(t)$ are **symmetric orbit points**. Obviously the set of orbit points of an orbit is a union of two pairs of symmetric orbit points. We shall say that a pair $\{[A, C], [B, D]\}$ is a **harmonic pair** if A and C are symmetric orbit points, and if B and D are symmetric orbit points, both with respect to a common element ϕ of $\mathit{\Gamma}(\mathcal{M}, \sim)$ (which ϕ is necessarily in $\underset{\sim}{\mathit{\Gamma}}^4(\mathcal{M})$).

[12] *Cf.* (2.23).

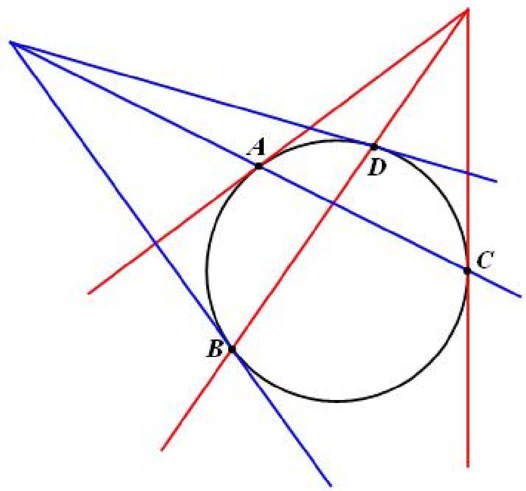


Fig. 12: A Quadric Cycle $[A, B, C, D]$ and Harmonic Pairs $\{[A, C], [B, D]\}$ on the Circle

(2.37) Theorem Let A, B, C and D be distinct elements of $\mathcal{M}$. The the following statements are pairwise equivalent:

(i) $\langle A, B, C, D\rangle$ is a quadric cycle;
(ii) $\{[A, C], [B, D]\}$ is a harmonic pair;
(iii) $(\forall\, \phi \in \Gamma(\mathcal{M}, \sim))\quad \{[\phi(A), \phi(C)], [\phi(B), \phi(D)]\}$ is a harmonic pair;
(iv) $(\exists\, \phi \in \Gamma^2_{\sim}(\mathcal{M}))\;\; \phi(A) = A,\; \phi(B) = D$ and $\phi(C) = C$;
(v) $(\exists\, \alpha, \beta \in \Gamma^2_{\sim}(\mathcal{M}))\;\; \alpha(A) = A,\; \alpha(C) = C,\; \beta(B) = B,\; \beta(D) = D$ and $\alpha \circ \beta = \beta \circ \alpha$.

Furthermore, if $\{[A, C], [B, D]\}$ is a harmonic pair, then
(vi) $(\exists\, \theta \in \Gamma(\mathcal{M}, \sim)$ a translation$)\;\; \theta(A) = A,\; \theta^{-1}(C) = B,$ and $\theta(C) = D$.

Proof. (i)$\Longleftrightarrow$(ii): This follows directly from the definitions.

(ii)$\Longleftrightarrow$(iii): That (iii) implies (ii) follows when α is the identity transformation. Suppose that (ii) holds and that $\phi \in \Gamma(\mathcal{M}, \sim)$. Since (i) and (ii) are equivalent, there exists $\theta \in \Gamma(\mathcal{M}, \sim)$ such that

$$\theta \circ \langle A, B, C, D\rangle = \langle A, B, C, D\rangle \circ \boxed{\spadesuit\heartsuit\diamondsuit\clubsuit}.$$

Thus

$$(\phi \circ \theta \circ \phi^{-1}) \circ \langle \phi(A), \phi(B), \phi(C), \phi(D)\rangle = \phi \circ \theta \circ \langle A, B, C, D\rangle =$$

$$\phi \circ \langle A, B, C, D\rangle \circ \boxed{\spadesuit\heartsuit\diamondsuit\clubsuit} = \langle \phi(A), \phi(B), \phi(C), \phi(D)\rangle \circ \boxed{\spadesuit\heartsuit\diamondsuit\clubsuit},$$

which means that $\langle \phi(A), \phi(B), \phi(C), \phi(D)\rangle$ is a quadric cycle. Since (i) and (ii) are equivalent, it follows that $\{[\phi(A), \phi(C)], [\phi(B), \phi(D)]\}$ is a harmonic pair.

(i)$\Longrightarrow$(iv): Let $\eta \in \Gamma^4_{\sim}(\mathcal{M})$ be such that $\eta \circ \mathfrak{t} = \mathfrak{t} \circ \boxed{\spadesuit\heartsuit\diamondsuit\clubsuit}$. By (2.29) there exists $\theta \in \Gamma^2_{\sim}(\mathcal{M})$ such that $\theta(A) = D$ and $\theta(B) = C$. Then

$$\eta \circ \theta(A) = \theta(D) = A,\; \eta \circ \theta(B) = \eta(C) = D,\; \eta \circ \theta(C) = \eta(B) = C, \text{ and } \eta \circ \theta(D) = \eta(A) = B.$$

Letting $\phi \equiv \eta \circ \theta$, we see from (2.30) that ϕ is an involution.

(iv)$\Longrightarrow$(v): Applying (iv) twice we obtain α in $\Gamma^2_{\sim}(\mathcal{M})$ such that

$$\alpha(A) = A,\; \alpha(C) = C,\; \alpha(B) = D.$$

Applying (2.32) we choose $\beta \in \Gamma^2_{\sim}(\mathcal{M})$

$$\beta(A) = C,\; \beta(B) = B, \text{ and } \beta(C) = A.$$

Then

$$\alpha \circ \beta(A) = \alpha(C) = C \quad \text{ and } \quad \alpha \circ \beta(C) = \alpha(A) = A$$

which by (2.30) implies that

$$\alpha \circ \beta$$

is an involution: that is, $\alpha\circ\beta=\beta\circ\alpha$.

(v)$\Longrightarrow$(i). We let α and β be as in (v). Since α is not the identity, the fundamental theorem guarantees that α cannot fix B. However

$$\beta(\alpha(B)) = \alpha(\beta(B)) = \alpha(B).$$

The fundamental theorem also guarantees that β only fixes B and D — consequently

$$\alpha(B) = D \Longrightarrow \alpha(D) = B. \tag{1}$$

A By (2.29) there exists $\theta\in\Gamma^2_\sim(\mathcal{M})$ such that $\theta(A)=D$ and $\theta(B) = C$. Letting $\phi\equiv\alpha\circ\theta$ we have

$$\phi(A) = \alpha\circ\theta(A) = \alpha(D) \overset{\text{by (1)}}{=\!=\!=} B,\ \phi(B) = \alpha\circ\theta(B) = \alpha(C) \overset{\text{by (v)}}{=\!=\!=} C,$$
$$\phi(C) = \alpha\circ\theta(C) = \alpha(B) \overset{\text{by (1)}}{=\!=\!=} D \quad\text{and}\quad \phi(D) = \alpha\circ\theta(D) = \alpha(A) \overset{\text{by (v)}}{=\!=\!=} A,$$

which implies (i).

We have now shown that (i) through (v) are pairwise equivalent. It but remains to show that (iv) implies (vi). By (2.32) there exists $\alpha\in\Gamma^2_\sim(\mathcal{M})$ such that $\alpha(A)=A$, $\alpha(C)=D$, and $\alpha(D)=C$. Letting $\theta\equiv\alpha\circ\phi$ we have

$$\theta(A) = \alpha\circ\phi(A) \overset{\text{by (iv)}}{=\!=\!=} \alpha(A) = A$$

and

$$\theta(B) = \alpha\circ\phi(B) \overset{\text{by (iv)}}{=\!=\!=} \alpha(D) = C,\ \text{and}\ \theta(C) = \alpha\circ\phi(C) \overset{\text{by (iv)}}{=\!=\!=} \alpha(C) = D.$$

From (2.35) follows that θ is a translation. QED

(2.38) Theorem Let $\sim$ be an equivalence relation satisfying Postulates I, II, and III. Then the following two statements are pairwise equivalent:

(i) $(\forall\,\tau$ a translation$)(\forall\,\alpha\in\Gamma^2_\sim(\mathcal{M}):(\exists\,A,B\in\mathcal{M}:\ \tau(A)=A=\alpha(A)$ and $\tau(B) = \alpha(B)))\quad \tau\circ\alpha\in\Gamma^2_\sim(\mathcal{M})$;

(ii) $(\forall\,A,C\in\mathcal{M}$ distinct$)(\forall\,\tau$ a translation: $\tau(A)=A)\quad \{A,\tau^{-1}(C),C,\tau(C)\}$ is a quadric cycle.

Furthermore, either of (i) and (ii) implies the following two statements:

(iii) $(\forall\,A,B,C\in\mathcal{M}$ distinct$)(\exists!\,D\in\mathcal{M})\quad \{A,B,C,D\}$ is a quadric cycle;

(iv) $(\forall\,\phi\in\Gamma^2_\sim(\mathcal{M}),B\in\mathcal{M}:\phi(B)=B)(\exists!\,D\in\mathcal{M})\quad D\neq B$ and $\phi(D)=D$;

(v) $(\forall\,\tau,\theta$ translations$)(\forall\,A,X\in\mathcal{M}:A=\tau(A)=\theta(A)\neq\tau(X)=\theta(X))\quad \tau=\theta$.

Proof. (i)$\Longrightarrow$(ii): By (2.31) there exists $\alpha\in\Gamma^2_\sim(\mathcal{M})$ such that

$$\alpha(\tau^{-1}(C)) = C \quad\text{and}\quad \alpha(A) = A. \tag{1}$$

By (i), $\tau\circ\alpha$ is an involution. We have

$$\tau\circ\alpha(A) \overset{\text{by (1) and (i)}}{=\!=\!=\!=\!=} A,\quad \tau\circ\alpha(\tau^{-1}(C)) \overset{\text{by (1)}}{=\!=\!=} \tau(C) \quad\text{and}\quad \tau\circ\alpha(C) = \tau(\tau^{-1}(C)) \overset{\text{by (1)}}{=\!=\!=} C \tag{2}$$

Since $\tau\circ\alpha$ is an involution, we also have $\tau\circ\alpha(\tau(C))=\tau^{-1}(C)$. This, with (2), and in view of and (2.37.i) and (2.37.ii), implies that $\{A,\tau^{-1}(C),C,\tau(C)\}$ is a quadric cycle.

(i)$\Longrightarrow$(v): Let $C\equiv\tau(X)$ and choose α as in (1). From (1) and (2) we have

$$\tau\circ\alpha(A) = A,\ \tau\circ\alpha(X) = \tau(C),\ \tau\circ\alpha(C) = C \text{ and } \tau\circ\alpha(\tau(C))=X. \tag{3}$$

As above, we have

$$\theta\circ\alpha(A) = A,\ \theta\circ\alpha(\theta^{-1}(C)) = \theta(C),\ \theta\circ\alpha(C) = \theta(\theta^{-1}(C)) = C \text{ and } \theta\circ\alpha(\theta(C))=\theta^{-1}(C)$$

which can be rewritten as

$$\theta\circ\alpha(A) = A,\ \theta\circ\alpha(X) = \theta(C),\ \theta\circ\alpha(C) = C \text{ and } \theta\circ\alpha(\theta(C))=X. \tag{4}$$

From (4), (3) and the fundamental theorem it follows that $\tau\circ\alpha = \theta\circ\alpha$. Thus $\tau = \theta$.

(ii)$\Longrightarrow$(iii) Choose $X\in\mathcal{M}$ distinct from A, B and C. By (2.32) there exists $\alpha\in\Gamma^2_\sim(\mathcal{M})$ such that $\alpha(A) = A$ and $\alpha(B) = X$. By (2.32) there exists $\beta\in\Gamma^2_\sim(\mathcal{M})$ such that $\beta(A) = A$ and $\beta(X) = C$. By (2.35) the composition $\beta\circ\alpha$ is a translation. By (ii),

$$[A,B,C,\beta\circ\alpha(C)] = [A,(\beta\circ\alpha)^{-1}(C),C,\beta\circ\alpha(C)] \text{ is a quadric cycle.}$$

Let $D\equiv\beta\circ\alpha(C)$. Assume that there were $E\in\mathcal{M}$ distinct from D such that $[A,B,C,E]$ were a quadric cycle. By (2.37.iv) there would exist $\gamma,\delta\in\Gamma^2_\sim(\mathcal{M})$ such that

$$\gamma(A)=A=\delta(A),\quad \gamma(C)=C=\delta(C),\quad \gamma(B)=D,\quad\text{and}\quad \delta(B)=E.$$

By (2.35) $\gamma\circ\delta$ would then be a translation, which has the fixed points A and C: an absurdity.

(iii)$\Longrightarrow$(iv) Let X be distinct from B. If $\phi(X)=X$, let $D\equiv X$. Else we apply (iii) to obtain the unique $D\in\mathcal{M}$ such that $[X,B,\phi(X),D]$ is a quadric cycle. In the latter case (2.37.iv) implies the existence of $\pi\in\Gamma^2_\sim(\mathcal{M})$ such that

$$\pi(B)=B,\quad \pi(X)=\phi(X)\quad \text{and}\quad \pi(D)=D.$$

By (2.34) we have $\phi=\pi$. Thus we have $\phi(D)=D$. If there existed E distinct from D and B such that $\phi(E)=E$, then the fundamental theorem would imply that ϕ were the identity.

(ii)$\Longrightarrow$(i): From the above we know that (iii) holds, and so (iv) holds as well. Let $\theta\equiv\alpha\circ\tau\circ\alpha$. Since A is the only fixed point of τ, it follows that $\alpha(A)$ is the only fixed point of θ: thus θ is a translation as well. From (ii) we know that $[A,\tau^{-1}(B),B,\tau(B)]$ is a quadric cycle, which with (2.37) implies that

$$[A,\tau(B),B,\tau^{-1}(B)] \text{ is a quadric cycle} \tag{5}$$

as well. From (ii) also follows that $[A,\theta^{-1}(B),B,\theta(B)]$ is a quadric cycle. Rewriting this last, we see that

$$[A,\tau(B),B,\theta(B)] \text{ is a quadric cycle.} \tag{6}$$

From (iv), with (5) and (6), follows that $\tau^{-1}(B)=\theta(B)$. We already know that

$$\tau^{-1}(A)=A=\alpha\circ\tau\circ\alpha(A)=\theta(A)\quad\text{and}\quad \tau^{-1}(\tau(B))=B=\alpha\circ\tau(B)=\alpha\circ\tau\circ\alpha(\tau(B))=\theta(\tau(B))$$

and so it follows from the fundamental theorem that $\tau^{-1}=\theta$. Thus, if ι denotes the identity mapping on $\mathcal{M}$, then

$$\iota=\tau\circ\theta=\tau\circ\alpha\circ\tau\circ\alpha\Longrightarrow\tau\circ\alpha\in\Gamma^2_\sim(\mathcal{M}).$$

QED

(2.39) Postulate IV The fourth and last postulate will be that, whenever τ is a translation with fixed point A, then, for any other point B distinct from A, the ordered quadruple $[A,\tau^{-1}(B),B,\tau(B)]$ is a quadric cycle.

(2.40) Definition An equivalence relation $\sim$ on a set $\mathcal{M}$ satisfying all four postulates (2.3), (2.7), (2.10) and (2.39), will be called a **meridian equivalence relation for $\mathcal{M}$.**

The set $\mathfrak{M}$ will be called the **intrinsic meridian model**.

(2.41) Notation It follows from Theorem (2.38.iv) than each of the transformations[13] $\overline{\boxed{\spadesuit\heartsuit}}$, $\overline{\boxed{\spadesuit\diamondsuit}}$ and $\overline{\boxed{\spadesuit\clubsuit}}$ have fixed points other than $\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$, $\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit}$ and $\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}$, respectively. We shall denote these other fixed points by $\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}$, $\boxed{\spadesuit\diamondsuit\ddagger\heartsuit\clubsuit}$ and $\boxed{\spadesuit\clubsuit\ddagger\heartsuit\diamondsuit}$, respectively.

(2.42) Example The elements $\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}$, $\boxed{\spadesuit\diamondsuit\ddagger\heartsuit\clubsuit}$ and $\boxed{\spadesuit\clubsuit\ddagger\heartsuit\diamondsuit}$ may not be distinct. Suppose that $\mathcal{M}$ has exactly four elements A, B, C and D. Let $\sim$ be the equivalence relation on $\mathcal{M}^{\Upsilon}_{2+}$ of which the corresponding partition has the following four elements:

$$\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit},\quad \boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit},\quad \boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit}\quad\text{and}\quad \mathcal{M}^{\Upsilon}_4.$$

Then

$$\mathfrak{M}=\{\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit},\boxed{\spadesuit\diamondsuit|\heartsuit\clubsuit},\boxed{\spadesuit\clubsuit|\heartsuit\diamondsuit},\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}\}$$

and

$$\{\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}\}=\{\boxed{\spadesuit\diamondsuit\ddagger\heartsuit\clubsuit}\}=\{\boxed{\spadesuit\clubsuit\ddagger\heartsuit\diamondsuit}\}=\mathcal{M}^{\Upsilon}_4.$$

(2.43) Theorem Let $\sim$ be a meridian equivalence relation. The following statements hold:

(i) $\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}=\{\mathfrak{t}\in\mathcal{M}^{\Upsilon}_4:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\heartsuit\},\{\mathfrak{t}_\diamondsuit,\mathfrak{t}_\clubsuit\}\}\text{ is a harmonic pair}\}$;

(ii) $\boxed{\spadesuit\diamondsuit\ddagger\heartsuit\clubsuit}=\{\mathfrak{t}\in\mathcal{M}^{\Upsilon}_4:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\diamondsuit\},\{\mathfrak{t}_\heartsuit,\mathfrak{t}_\clubsuit\}\}\text{ is a harmonic pair}\}$;

(iii) $\boxed{\spadesuit\clubsuit\ddagger\heartsuit\diamondsuit}=\{\mathfrak{t}\in\mathcal{M}^{\Upsilon}_4:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\clubsuit\},\{\mathfrak{t}_\heartsuit,\mathfrak{t}_\diamondsuit\}\}\text{ is a harmonic pair}\}$.

Proof. Denote by $\mathfrak{X}$ the set $\{\mathfrak{t}\in\mathcal{M}^{\Upsilon}_4:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\heartsuit\},\{\mathfrak{t}_\diamondsuit,\mathfrak{t}_\clubsuit\}\}$ is a harmonic pair$\}$. Let $\mathfrak{r},\mathfrak{s}\in\mathcal{X}$. By the fundamental theorem there exist $\delta\in\Gamma(\mathcal{M},\sim)$ such that

$$\delta(\mathfrak{r}_\spadesuit)=\mathfrak{s}_\spadesuit,\ \delta(\mathfrak{r}_\heartsuit)=\mathfrak{s}_\heartsuit,\ \text{and}\ \delta(\mathfrak{r}_\diamondsuit)=\mathfrak{s}_\diamondsuit.$$

From Theorem (2.37.iii) follows that

$$\{\{\mathfrak{s}_\spadesuit,\mathfrak{s}_\heartsuit\},\{\mathfrak{s}_\diamondsuit,\delta(\mathfrak{r}_\clubsuit)\}\}\text{ is a harmonic pair.}$$

[13] *Cf.* (2.21.1).

From the uniqueness part of Theorem (2.39.iv) follows that $\mathfrak{r}_{\clubsuit}=\mathfrak{s}_{\clubsuit}$ and so by Postulate III it follows that $\mathfrak{r}\sim\mathfrak{s}$. Thus $\mathcal{X}$ is a subset of a single member $\mathfrak{Y}$ of $\mathfrak{M}$.

Let $\{\{A,C\},\{B,D\}\}$ be a harmonic pair. Then $\langle A,B,C,D\rangle\in\mathfrak{X}$, whence $\lfloor\langle A,B,C,D\rangle\rfloor=\mathfrak{Y}$. We have

$$\overline{\boxed{\spadesuit\heartsuit}}(\lfloor\langle A,B,C,D\rangle\rfloor)=(\lfloor\langle A,B,C,D\rangle\circ\boxed{\spadesuit\heartsuit}\rfloor)=\lfloor\langle C,B,A,D\rangle\rfloor.$$

Since $\{\{C,A\},\{B,D\}\}$ is a harmonic pair, it follows that $\langle C,B,A,D\rangle$ is in $\mathfrak{X}$, and so $\lfloor\langle C,B,A,D\rangle\rfloor=\mathfrak{Y}$. We shave shown that $\overline{\boxed{\spadesuit\heartsuit}}$ leaves $\mathfrak{Y}$ fixed. Thus, by definition, $\mathfrak{Y}=\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}$. Hence $\mathfrak{X}\subset\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}$.

Let $\mathfrak{t}$ be in $\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}$. By Postulate III there exists $\theta\in\Gamma(\mathcal{M},\sim)$ such that $\mathfrak{t}=\theta\circ\langle A,B,C,D\rangle$. From Theorem (2.37.iii) follows that $\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\heartsuit\},\{\mathfrak{t}_\diamondsuit,\mathfrak{t}_\clubsuit\}\}$ is a harmonic pair. Thus $\mathfrak{t}$ is in $\mathfrak{X}$. We have demonstrated that $\mathfrak{X}=\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}$.

The establishment of (ii) and (iii) is analogous to that of (i). QED

(2.44) Corollary Let $\sim$ be a meridian equivalence relation. If any two elements of the set $\{\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit},\boxed{\spadesuit\diamondsuit\ddagger\heartsuit\clubsuit},\boxed{\spadesuit\clubsuit\ddagger\heartsuit\diamondsuit}\}$ are distinct, then all three are.

Proof. Suppose for instance that $\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}\neq\boxed{\spadesuit\diamondsuit\ddagger\heartsuit\clubsuit}$ are distinct: that

$$\{\mathfrak{t}\in\mathcal{M}_4^\Upsilon:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\heartsuit\},\{\mathfrak{t}_\diamondsuit,\mathfrak{t}_\clubsuit\}\}\text{ is a harmonic pair}\}\neq\{\mathfrak{t}\in\mathcal{M}_4^\Upsilon:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\diamondsuit\},\{\mathfrak{t}_\heartsuit,\mathfrak{t}_\clubsuit\}\}\text{ is a harmonic pair}\}.\quad(1)$$

Applying $\overline{\boxed{\spadesuit\heartsuit}}$ to (1) we obtain

$$\{\mathfrak{t}\in\mathcal{M}_4^\Upsilon:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\heartsuit\},\{\mathfrak{t}_\diamondsuit,\mathfrak{t}_\clubsuit\}\}\text{ is a harmonic pair}\}\neq\{\mathfrak{t}\in\mathcal{M}_4^\Upsilon:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\clubsuit\},\{\mathfrak{t}_\heartsuit,\mathfrak{t}_\diamondsuit\}\}\text{ is a harmonic pair}\}\quad(2)$$

and applying $\overline{\boxed{\spadesuit\diamondsuit}}$ to (1) we obtain

$$\{\mathfrak{t}\in\mathcal{M}_4^\Upsilon:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\clubsuit\},\{\mathfrak{t}_\heartsuit,\mathfrak{t}_\diamondsuit\}\}\text{ is a harmonic pair}\}\neq\{\mathfrak{t}\in\mathcal{M}_4^\Upsilon:\{\{\mathfrak{t}_\spadesuit,\mathfrak{t}_\diamondsuit\},\{\mathfrak{t}_\heartsuit,\mathfrak{t}_\clubsuit\}\}\text{ is a harmonic pair}\}.\quad(3)$$

Thus (2) and (3) yield $\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}\neq\boxed{\spadesuit\clubsuit\ddagger\heartsuit\diamondsuit}$ and $\boxed{\spadesuit\clubsuit\ddagger\heartsuit\diamondsuit}\neq\boxed{\spadesuit\diamondsuit\ddagger\heartsuit\clubsuit}$. QED

(2.45) Theorem Let $\sim$ be a meridian equivalence relation. Let P be in $\mathcal{M}$ and let $\alpha,\beta,\gamma\in\Gamma^2_\sim(\mathcal{M})$ agree on P. Then $\alpha\circ\beta\circ\gamma$ either is the identity transformation or is in $\Gamma^2_\sim(\mathcal{M})$.

Proof. For $\alpha\circ\beta\circ\gamma$ to be in $\Gamma^2_\sim(\mathcal{M})$ is equivalent to

$$\alpha\circ\beta\circ\gamma=\gamma\circ\beta\circ\alpha.$$

If $\alpha=\beta$ or $\beta=\gamma$ the above equation is trivial, so we will suppose that $\alpha\neq\beta\neq\gamma$. Let Q be such that $\alpha(Q)\neq\beta(Q)$. By Theorem (2.29) there exists $\delta\in\Gamma^2_\sim(\mathcal{M})$ such that $\delta(P)=P$ and $\delta(\alpha\circ\beta(Q))=\gamma(Q)$. By Theorem (2.35) $\beta\circ\alpha$ and $\gamma\circ\delta$ are translations. Since they agree on P and both take on the value Q at $\alpha\circ\beta(Q)$, it follows from Theorem (2.38.v) that $\beta\circ\alpha=\gamma\circ\delta$. From this follows that $\gamma\circ\beta\circ\alpha=\delta$, which means in particular that $\gamma\circ\beta\circ\alpha$ is in $\Gamma^2_\sim(\mathcal{M})$: it is its own inverse $\alpha\circ\beta\circ\gamma$. QED

(2.46) Theorem Let $\sim$ be a meridian equivalence relation. Let $A,B,C,D\in\mathcal{M}$ be such that $\{A,D\}\cap\{B,C\}=\emptyset$. Then there exists a unique element $\left[\begin{smallmatrix}A&B\\D&C\end{smallmatrix}\right]$ of $\Gamma^2_\sim(\mathcal{M})$ such that

$$\begin{bmatrix}A&B\\D&C\end{bmatrix}(A)=D\text{ and }\begin{bmatrix}A&B\\D&C\end{bmatrix}(B)=C.\quad(1)$$

Proof. If $A\neq D$ and $B\neq C$, this is just (2.29). If $\#\{A,B,C,D\}=3$, then this is just (2.32). If $A=D$ and $B=C$, and if E is any distinct third point, it follows from (2.38.()iii) that there exists $M\in\mathcal{M}$ such that $\{[A,B],[E,M]\}$ is a harmonic pair. From (iii) of (2.38) follows that there exists a unique $\left[\begin{smallmatrix}A&B\\D&C\end{smallmatrix}\right]\in\Gamma^2_\sim(\mathcal{M})$ such that (1) holds. QED

(2.47) Discussion We could as this point enter further into the the description and classification of the elements of $\Gamma(\mathcal{M},\sim)$. However there is an alternate characterization of the meridian equivalence relation which seems a more appropriate setting for that program. We shall set the foundation for this characterization in the next section.

(2.48) Historical For each quadriad $\mathfrak{t}\in\mathcal{M}^\Upsilon_{2+}$ we can form the pair $\mathfrak{Wurf}(\mathfrak{t})\equiv\{[\mathfrak{t}_\heartsuit,\mathfrak{t}_\diamondsuit],[\mathfrak{t}_\spadesuit,\mathfrak{t}_\clubsuit]\}$ of ordered pairs. We note that

$$(\forall\,\mathfrak{s},\mathfrak{t}\in\mathcal{M}^\Upsilon_{2+})\quad\mathfrak{Wurf}(\mathfrak{s})=\mathfrak{Wurf}(\mathfrak{t})\iff\mathfrak{s}=\mathfrak{t}\circ\boxed{\spadesuit\heartsuit|\diamondsuit\clubsuit}$$

Because of Postulate I it follows that, whenever $\mathfrak{Wurf}(\mathfrak{s})=\mathfrak{Wurf}(\mathfrak{t})$, then $\mathfrak{s}\sim\mathfrak{t}$. This means that in obtaining $\mathfrak{M}$ we could alternatively have placed an equivalence relation on the family

$$\mathfrak{W\ddot{u}rfe}\equiv\{\{[A,B],[C,D]\}:A,B,C,D\in\mathcal{M}\text{ and }\#\{A,B,C,D\}>2\}.$$

This is the program followed by Karl Von Staudt in [Von Staudt] pp. 166 *et seq.* and later by Veblen and Young in [Veblen & Young] §55. The latter authors used the terms *throws* for what Von Staudt called *Würfe* and *marks* for the (equivalence class) elements of $\mathfrak{M}$. None of these authors however begins with a system of postulates as was done here.

3. Libras (Part I)

(3.1) Introduction The meridian equivalence relation appears naturally out of a particular algebraic operation. Its introduction will be much simplified if we bring to hand a more fundamental notion which requires introduction here. The basic notion behind it is a set of scales — hence the appellation: "libra". For brevity however, we shall take a short cut past the scales, leaving those for the appendix: Section (11).

(3.2) Definitions Let L be a set and $\lfloor\, , \, , \rfloor | L \times L \times L \to L$ a trinary operator on L for which the following holds:

$$(\forall\, a, b \in L) \quad \lfloor a, a, b \rfloor = b = \lfloor b, a, a \rfloor \tag{1}$$

and

$$(\forall\, a, b, c, d, e \in L) \quad \lfloor \lfloor a, b, c \rfloor, d, e \rfloor = \lfloor a, b, \lfloor c, d, e \rfloor \rfloor. \tag{2}$$

Then $\lfloor\, , \, , \rfloor$ will be said to be a **libra operator** and L, relative to $\lfloor\, , \, , \rfloor$, a **libra**.

A subset B of a libra will be said to be **balanced** provided $\lfloor a, b, c \rfloor$ is in B whenever $a, b, c \in B$.

(3.3) Theorem Let $\lfloor\, , \, , \rfloor | L \times L \times L \to L$ be a libra operator on a set L. Then

$$(\forall\, a, b, c, d, e \in L) \quad \lfloor a, \lfloor d, c, b \rfloor, e \rfloor = \lfloor \lfloor a, b, c \rfloor, d, e \rfloor. \tag{1}$$

Proof. We have

$$\begin{aligned} a &\overset{\text{by } (3.2.1)}{=\!=} \lfloor a, b, b \rfloor \overset{\text{by } (3.2.1)}{=\!=} \lfloor a, b, \lfloor c, c, b \rfloor \rfloor \overset{\text{by } (3.2.1)}{=\!=} \lfloor a, b, \lfloor \lfloor c, d, d \rfloor, c, b \rfloor \rfloor \overset{\text{by } (3.2.2)}{=\!=} \\ &\lfloor \lfloor a, b, \lfloor c, d, d \rfloor \rfloor, c, b \rfloor \overset{\text{by } (3.2.2)}{=\!=} \lfloor \lfloor \lfloor a, b, c \rfloor, d, d \rfloor, c, b \rfloor \overset{\text{by } (3.2.2)}{=\!=} \lfloor \lfloor a, b, c \rfloor, d, \lfloor d, c, b \rfloor \rfloor \end{aligned} \tag{2}$$

whence follows

$$\begin{aligned} &\lfloor a, \lfloor d, c, b \rfloor, e \rfloor \overset{\text{by } (2)}{=\!=} \lfloor \lfloor \lfloor a, b, c \rfloor, d, \lfloor d, c, b \rfloor \rfloor, \lfloor d, c, b \rfloor, e \rfloor \overset{\text{by } (3.2.2)}{=\!=} \\ &\lfloor \lfloor a, b, c \rfloor, d, \lfloor \lfloor d, c, b \rfloor, \lfloor d, c, b \rfloor, e \rfloor \rfloor \overset{\text{by } (3.2.1)}{=\!=} \lfloor \lfloor a, b, c \rfloor, d, e \rfloor. \end{aligned}$$

QED

(3.4) Convention The various compositions of libra operators with libra operators, in view of (3.2.1), (3.2.2) and (3.3.1), may be greatly simplified: we define

$$\lfloor a, b, c, d, e \rfloor \equiv \lfloor \lfloor a, b, c \rfloor, d, e \rfloor = \lfloor a, \lfloor d, c, b \rfloor, e \rfloor. \tag{1}$$

Each such composition may be converted to a form

$$\lfloor a_1, a_2, \lfloor a_3, a_4, \lfloor \cdots \lfloor a_{n-2}, a_{n-1}, a_n \rfloor \cdots \rfloor \rfloor \rfloor \tag{2}$$

for n a positive odd integer. We shall at times adopt the abbreviation

$$\lfloor a_1, a_2, \cdots, a_n \rfloor \tag{3}$$

for (2).

(3.5) Example Let A be an affine space over a field F. Then the **translations** of A form a vector space V over F. The translation of a point $a \in A$ by a vector $v \in V$ is denoted by $v + a$. To any two distinct points $a, b \in A$ corresponds a unique vector (which we denote by $b - a$) such that $(b - a) + a = b$. Then

$$\lfloor a, b, c \rfloor \equiv (a - b) + c \quad (\forall\, a, b, c \in L)$$

defines a libra operator. We have $d = \lfloor a, b, c \rfloor$ precisely when the points a, b, c and d describe the points of a parallelogram.[14]

(3.6) Example Given two sets X and Y of equal cardinality we shall write $\mathfrak{J}(X, Y)$ for the set of all bijections of X onto Y. The set $\mathfrak{J}(X, Y)$ is a libra under the **canonical libra operator**

$$(\forall\, f, g, h \in \mathfrak{J}(X, Y)) \quad [\![f, g, h]\!] \equiv f \circ g^{-1} \circ h. \tag{1}$$

Any balanced subset of $\mathfrak{J}(X, Y)$ will be called a **libra of operators from X to Y**.

The family $\mathbf{Mor}(\mathfrak{M}, \mathcal{M})$ of (2.8.1) is a balanced subset of $\mathfrak{J}(\mathfrak{M}, \mathcal{M})$, and so is a libra of operators from $\mathfrak{M}$ to $\mathcal{M}$.

[14] Taken in clockwise, or counter-clockwise order.

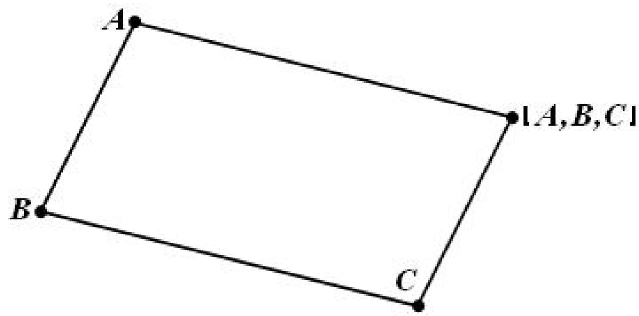


Fig. 13: Affine Libra Operator

(3.7) Theorem Let $\lfloor , , \rfloor$ be a libra operator on a libra L and let e an element of L. Then the binary operation $x \cdot y \equiv \lfloor x, e, y \rfloor$ is a group operation on L, relative to which e is the identity and

$$(\forall\, x \in L) \quad \lfloor e, x, e \rfloor \text{ is the inverse of } x.$$

Proof. For $x, y, z \in L$

$$(x \cdot y) \cdot z = \lfloor \lfloor x, e, y \rfloor, e, z \rfloor \overset{\text{by (3.2.2)}}{=\!=\!=} \lfloor x, e, \lfloor y, e, z \rfloor \rfloor = x \cdot (y \cdot z),$$

$$x \cdot e = \lfloor x, e, e \rfloor \overset{\text{by (3.2.1)}}{=\!=\!=} x \overset{\text{by (3.2.1)}}{=\!=\!=} \lfloor e, e, x \rfloor = e \cdot x,$$

$$x \cdot \lfloor e, x, e \rfloor = \lfloor x, e, \lfloor e, x, e \rfloor \rfloor \overset{\text{by (3.2.2)}}{=\!=\!=} \lfloor \lfloor x, e, e, \rfloor, x, e \rfloor \overset{\text{by (3.2.1)}}{=\!=\!=} \lfloor x, x, e \rfloor \overset{\text{by (3.2.1)}}{=\!=\!=} e$$

and

$$\lfloor e, x, e \rfloor \cdot x = \lfloor \lfloor e, x, e \rfloor, e, x \rfloor \overset{\text{by (3.2.2)}}{=\!=\!=} \lfloor e, x, \lfloor e, e, x \rfloor \rfloor \overset{\text{by (3.2.1)}}{=\!=\!=} \lfloor e, x, x \rfloor \overset{\text{by (3.2.1)}}{=\!=\!=} e.$$

QED

(3.8) Theorem Let G be a group with binary operation $\cdot$. Define the trinary operator $\lfloor , , \rfloor$ by

$$\lfloor a, b, c \rfloor \equiv a \cdot b^{-1} \cdot c$$

for all $a, b, c \in G$. Then $\lfloor , , \rfloor$ is a libra operator.

Proof. For $r, s, t, u, v \in G$,

$$\lfloor r, s, s \rfloor = r \cdot s^{-1} \cdot s = r = s \cdot s^{-1} \cdot r = \lfloor s, s, r \rfloor$$

and

$$\lfloor \lfloor r, s, t \rfloor, u, v \rfloor = (r \cdot s^{-1} \cdot t) \cdot u^{-1} \cdot v = r \cdot s^{-1} \cdot (t \cdot u^{-1} \cdot v) = \lfloor r, s, \lfloor t, u, v \rfloor \rfloor.$$

QED

(3.9) Definition The libra operator defined in (3.8) will the called the **group libra operator**.

(3.10) Definition A function f from one libra L_1 to another libra L_2 which preserves the libra operator is called a **libra homomorphism**. Thus a libra homomorphism f is characterized by

$$(\forall\, a, b, c \in L_1) \quad \lfloor f(a), f(b), f(c) \rfloor = f(\lfloor a, b, c \rfloor). \tag{1}$$

A libra homomorphism which is bijective is a **libra isomorphism**.

(3.11) Theorem Let G and H be two groups, and let f be a group homomorphism from G to H. Then f is also a homomorphism of libras.

Proof. For $a, b, c \in G$ we have

$$\lfloor f(a), f(b), f(c) \rfloor = f(a) \cdot f(b)^{-1} \cdot f(c) = f(a \cdot b^{-1} \cdot c) = f(\lfloor a, b, c \rfloor).$$

QED

(3.12) Definitions and Notation A libra L will be called **abelian** if $\lfloor a, b, c \rfloor = \lfloor c, b, a \rfloor$ for all $a, b, c \in L$. Evidently L is abelian if and only if each of its corresponding groups is abelian.

For $a, b \in L$ we define the functions

$${}_a\pi_b | L \ni x \hookrightarrow \lfloor a, x, b \rfloor \in L, \quad {}_a\rho_b | L \ni x \hookrightarrow \lfloor x, a, b \rfloor \in L, \text{ and } {}_a\lambda_b | L \ni x \hookrightarrow \lfloor a, b, x \rfloor \in L.$$

The functions ${}_a\rho_b$ and ${}_a\lambda_b$, respectively, are called **right translations** and **left translations**, respectively.

When L is abelian, left translations are right translations, and *vice versa*, and so in this case we simply call them **translations**. When L is abelian, the functions ${}_a\pi_b$ are called **(inner) involutions**.

(3.13) Theorem Let $\lfloor , , \rfloor$ be an abelian libra operation on a set L. Let $\Pi(L)$ denote the set of inner involutions on L. Then

(i) each function in $\Pi(L)$ is its own inverse;
(ii) $(\forall\, a, b \in L)(\exists!\, f \in \Pi(L)) \quad f(a) = b$;
(iii) $(\forall\, f, g, h \in \Pi(L)) \quad f \circ g \circ h \in \Pi(L)$.

Proof. For $r, s, t, u, v, w, x \in L$

$$ {}_r\pi_s \circ {}_r\pi_s(x) = \lfloor r, \lfloor r, x, s\rfloor, s\rfloor = \lfloor r, \lfloor s, x, r\rfloor, s\rfloor \overset{\text{by (3.3.1)}}{=\!=} \lfloor r, r, x, s, s\rfloor \overset{\text{by (3.2.1)}}{=\!=} x, $$
$$ {}_r\pi_s(r) = \lfloor r, r, s\rfloor \overset{\text{by (3.2.1)}}{=\!=} s, $$

and, if we let $a \equiv \lfloor r, u, v\rfloor$ and $b \equiv \lfloor w, t, s\rfloor$,

$$ {}_r\pi_s \circ {}_t\pi_u \circ {}_v\pi_w(x) = \lfloor r, \lfloor t, \lfloor v, x, w\rfloor, u\rfloor, s\rfloor \overset{\text{by (3.3.1)}}{=\!=} \lfloor r, \lfloor \lfloor t, w, \lfloor x, v, u\rfloor\rfloor\rfloor, s\rfloor \overset{\text{by (3.3.1)}}{=\!=} $$
$$ \lfloor r, \lfloor x, v, u\rfloor, \lfloor w, t, s\rfloor\rfloor = \lfloor r, u, v, x, w, t, s\rfloor = \lfloor \lfloor r, u, v\rfloor, x, \lfloor w, t, s\rfloor\rfloor = {}_a\pi_b. $$

It remains only to show that ${}_r\pi_s$ is the only element of $\Pi(L)$ which sends r to s. Suppose that ${}_t\pi_u$ is another such. Then $s = \lfloor t, r, u\rfloor$, whence, for each $x \in L$,

$$ {}_r\pi_s(x) = \lfloor r, x, s\rfloor = \lfloor r, x, \lfloor t, r, u\rfloor\rfloor \overset{\text{by (3.3.1)}}{=\!=} \lfloor r, \lfloor r, t, x\rfloor, u\rfloor = $$
$$ \lfloor r, \lfloor x, t, r\rfloor, u\rfloor \overset{\text{by (3.3.1)}}{=\!=} \lfloor \lfloor r, r, t\rfloor, x, u\rfloor \overset{\text{by (3.2.1)}}{=\!=} \lfloor t, x, u\rfloor = {}_t\pi_u(x). $$

QED

(3.14) Corollary Relative to the trinary operator

$$ \Pi(L) \times \Pi(L) \times \Pi(L) \ni ([f, g, h] \hookrightarrow f \circ g \circ h \in \Pi(L), $$

$\Pi(L)$ is a libra itself.

Proof. This follows from (3.13.i) and (3.13.iii). QED

(3.15) Theorem Let Π be a family of bijections of a set S such that

(i) each function in Π is its own inverse;
(ii) $(\forall\, a, b \in S)(\exists!\, f \in \Pi) \quad f(a) = b$;
(iii) $(\forall\, f, g, h \in \Pi) \quad f \circ g \circ h \in \Pi$.

Let $T \equiv \{f \circ g : f, g \in \Pi\}$. Then

(iv) $(\forall\, f, g, h, k \in \Pi : (\exists\, s \in S) \quad f \circ g(s) = h \circ k(s)) \quad f \circ g = h \circ k$;
(v) $(\forall\, a, b \in S)(\exists!\, {}_a\tau_b \in T) \quad {}_a\tau_b(a) = b$;
(vi) T is an abelian group under composition.

Proof. If $f \circ g(s) = h \circ k(s)$, then $h \circ f \circ g(s) = k(s)$ and so (iii) and (ii) imply that $h \circ f \circ g = k$. Thus (iv) holds.

Let f be the function in Π which leaves a fixed and let g be the one which sends a to b. Then $g \circ f(a) = b$. That $g \circ f$ is unique with this property follows from (iv), which proves (v).

For $f, g, h, k \in \Pi$ we have $(f \circ g) \circ (h \circ k) = (f \circ g \circ h) \circ k$ which is in T by (iii). That $f \circ f$ is the identity function ι follows from (i). For $f, g \in \Pi$, we have $(f \circ g) \circ (g \circ f) = \iota$ by (i).[15] Thus T is a group.

For $f, g, h, k \in \Pi$ we have $(f \circ g \circ h)^{-1} = h \circ g \circ f$. By (i) and (iii) this implies that

$$ f \circ g \circ h = h \circ g \circ f. $$

Consequently

$$ (f \circ g) \circ (h \circ k) = (f \circ g \circ h) \circ k = (h \circ g \circ f) \circ k = h \circ (g \circ f \circ k) = h \circ (k \circ f \circ g) = (h \circ k) \circ (f \circ g) $$

which proves (vi). QED

(3.16) Theorem . Let Π be a family of bijections of a set C such that

(i) each function in Π is its own inverse;
(ii) $(\forall\, a, b \in C)(\exists!\, f \in \Pi) \quad f(a) = b$;

---

[15] By ι we mean the identity function.

(iii) $(\forall\, f, g, h \in \Pi) \quad f \circ g \circ h \in \Pi$.

For all $a, b, c \in C$ we denote by ${}_a\phi_c$ the function in Π which sends a to c, and define $\lfloor a, b, c \rfloor \equiv {}_a\phi_c(b)$.

Then $\lfloor\, , \, , \rfloor$ is an abelian libra operator on C.

Proof. Let a, b and c be generic elements of C. We have

$$\lfloor a, a, b \rfloor = {}_a\phi_b(a) = b \quad \text{and} \quad \lfloor a, b, b \rfloor = {}_a\phi_b(b) = a$$

by definition, which is just (3.2.1).

Let a, b, c, d and e be generic elements of C. Let T be as in (3.15). For all $x, y \in C$, let ${}_x\tau_y$ be as in (3.15.v). It follows from (3.15) that

$$(\forall\, x \in C) \quad {}_a\phi_x \circ {}_b\phi_x(b) = a \Longrightarrow (\forall\, x \in C) \quad {}_a\phi_x \circ {}_b\phi_x = {}_b\tau_a \tag{1}$$

and

$$(\forall\, x \in C) \quad {}_x\phi_e \circ {}_x\phi_d(d) = e \Longrightarrow (\forall\, x \in C) \quad {}_x\phi_e \circ {}_x\phi_d = {}_d\tau_e. \tag{2}$$

Letting $u \equiv {}_b\tau_a(c)$ and $v \equiv {}_d\tau_e(c)$, we have

$$\lfloor \lfloor a, b, c \rfloor, d, e \rfloor = \lfloor {}_a\phi_c \circ {}_b\phi_c(c), d, e \rfloor \overset{\text{by (1)}}{=\!=\!=} \lfloor {}_b\tau_a(c), d, e \rfloor = \lfloor u, d, e \rfloor =$$

$${}_u\phi_e \circ {}_u\phi_d(u) \overset{\text{by (2)}}{=\!=\!=} {}_d\tau_e(u) = {}_d\tau_e \circ {}_b\tau_a(c) \overset{\text{by (3.15.vi)}}{=\!=\!=\!=} {}_b\tau_a \circ {}_d\tau_e(c) = {}_b\tau_a(v) \overset{\text{by (1)}}{=\!=\!=} {}_a\phi_v \circ {}_b\phi_v(v) =$$

$$\lfloor a, b, v \rfloor = \lfloor a, b, {}_d\tau_e(c) \rfloor \overset{\text{by (2)}}{=\!=\!=} \lfloor a, b, {}_c\phi_e \circ {}_c\phi_d(c) \rfloor = \lfloor a, b, \lfloor c, d, e \rfloor \rfloor$$

which establishes (3.2.2). QED

(3.17) Example Let P be the real projective plane, let C be some conic in P and let N be some line in P. Each point $q \in N$ not on C corresponds to an involution $\hat{\mathrm{x}}$ on C defined as in the figure below:

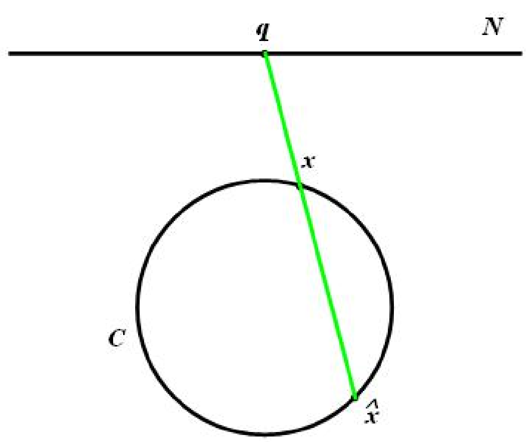


Fig. 14: Involution of a Circle by a Point on a Line.

In the next figure we take three points a, b and c in N and find the element $\lfloor a, b, c \rfloor$ of N such that $\lfloor a, b, c \rfloor\hat{} = \hat{a} \circ \hat{b} \circ \hat{c}$, by picking points x and y on C at random, and checking that the results are the same for each.

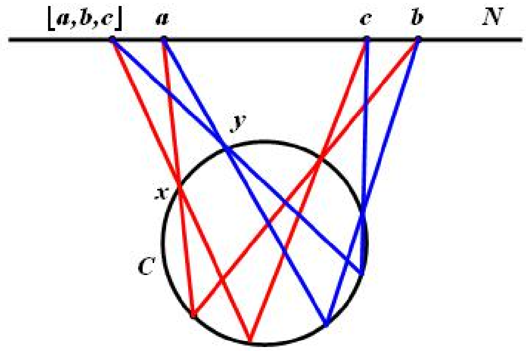


Fig. 15: Libra Operator Induced on a Line by a Circle.

4. Meridians

(4.1) Definition Let $\mathcal{M}$ be a set with at least four elements, and define

$$\mathcal{M}^{(5)} \equiv \{[A,B,C,D,E] \in \mathcal{M}\times\mathcal{M}\times\mathcal{M}\times\mathcal{M}\times\mathcal{M} : \{A,E\} \neq \{B,D\}\}.$$

Suppose that a quinary operation

$$\mathcal{M}^{(5)} \ni [A,B,C,D,E] \hookrightarrow \begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} \in \mathcal{M}$$

satisfies, for all $(\forall\,[A,B,C,D,E],[R,S,T,U,V] \in \mathcal{M}^{(5)})$

$$\begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} = \begin{bmatrix} E & & B \\ & C & \\ D & & A \end{bmatrix} = \begin{bmatrix} B & & A \\ & C & \\ E & & D \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} = F \quad \text{if } F \in \{A,E\} \cap \{B,D\}, \tag{2}$$

$$\begin{bmatrix} A & & B \\ & A & \\ D & & E \end{bmatrix} = E \quad \text{if } \{A,E\} \cap \{B,D\} = \emptyset, \tag{3}$$

$$\begin{bmatrix} \begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} & & B \\ & R & \\ D & & S \end{bmatrix} = \begin{bmatrix} A & & B \\ & C & \\ D & & \begin{bmatrix} E & & B \\ & R & \\ D & & S \end{bmatrix} \end{bmatrix}, \tag{4}$$

and

$$\begin{bmatrix} \begin{bmatrix} A & & B \\ & R & \\ D & & E \end{bmatrix} & & \begin{bmatrix} A & & B \\ & S & \\ D & & E \end{bmatrix} \\ & \begin{bmatrix} A & & B \\ & T & \\ D & & E \end{bmatrix} & \\ \begin{bmatrix} A & & B \\ & U & \\ D & & E \end{bmatrix} & & \begin{bmatrix} A & & B \\ & V & \\ D & & E \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A & & B \\ & \begin{bmatrix} R & & S \\ & T & \\ U & & V \end{bmatrix} & \\ D & & E \end{bmatrix}. \tag{5}$$

Then we shall say that the quinary operator is a **meridian operator** and that $\mathcal{M}$, relative to [:·:], is a **meridian**.[16]

Condition (1) is an abelian or commutative condition, which immediately implies several others: to wit

$$\begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} = \begin{bmatrix} A & & D \\ & C & \\ B & & E \end{bmatrix} = \begin{bmatrix} B & & E \\ & C & \\ A & & D \end{bmatrix} = \begin{bmatrix} E & & D \\ & C & \\ B & & A \end{bmatrix} = \begin{bmatrix} D & & A \\ & C & \\ E & & B \end{bmatrix} = \begin{bmatrix} D & & E \\ & C & \\ A & & B \end{bmatrix}. \tag{6}$$

[16] **meridian**. . . 4.[Ellipt. for meridian circle or line.] a. Astr. (More explicitly celestial m.) The great circle (of the celestial sphere) which passes through the celestial poles and the zenith of any place on the earth's surface ([Oxford Eng. Dict.]).

Condition (3) is reminiscent of (3.2.1) and condition (4) is reminiscent of (3.2.2). Applying the commutative conditions (1) and (6) to (2), (3) and (4), we obtain as well

$$\begin{bmatrix} E & & B \\ & A & \\ D & & A \end{bmatrix} = \begin{bmatrix} A & & E \\ & D & \\ D & & B \end{bmatrix} = \begin{bmatrix} A & & B \\ & B & \\ E & & C \end{bmatrix} = E = \begin{bmatrix} E & & B \\ & C & \\ E & & D \end{bmatrix} = \begin{bmatrix} E & & E \\ & C & \\ B & & D \end{bmatrix} = \begin{bmatrix} A & & E \\ & C & \\ D & & E \end{bmatrix} \tag{7}$$

and

$$\begin{bmatrix} A & & \begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} \\ & R & \\ S & & E \end{bmatrix} = \begin{bmatrix} A & & B \\ & C & \\ \begin{bmatrix} A & & D \\ & R & \\ S & & E \end{bmatrix} & & E \end{bmatrix}. \tag{8}$$

If $\mathcal{M}$ and $\mathcal{N}$ are two meridians and $f|\mathcal{M} \to \mathcal{N}$ satisfies

$$(\forall\, A,B,C,D,E \in \mathcal{M}^{(5)}) \quad f(\begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix}) = \begin{bmatrix} f(A) & & f(B) \\ & f(C) & \\ f(D) & & f(E) \end{bmatrix},$$

then f is a **homomorphism of meridians**. Condition (4) just states that each mapping

$$\mathcal{M} \ni X \hookrightarrow \begin{bmatrix} A & & B \\ & X & \\ D & & E \end{bmatrix} \in \mathcal{M}$$

is a homomorphism of meridians. The next theorem shows that each mapping $\mathcal{M} \ni X \hookrightarrow \begin{bmatrix} X & & B \\ & C & \\ D & & E \end{bmatrix} \in \mathcal{M}$ is as well.

To assist in the the proof of that theorem and elsewhere we define, for B and D fixed in $\mathcal{M}$, the auxiliary trinary operator ${}^{B}\llcorner ,, \lrcorner^{D}$ on $\{x \in \mathcal{M} :\ B{\neq}X{\neq}D\}$ by

$${}^{B}\llcorner A,C,E \lrcorner^{D} = \begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix}. \tag{9}$$

It follows from (2) and (3) that it is a libra operator — from (1) follows that it is abelian.

(4.2) Theorem Let $[\because]$ be a meridian operator on a set M. For $\lceil R,S,T,U,V \rceil \in \mathcal{M}^{(5)}$ and $B,C,D,E \in \mathcal{M}$

$$\begin{bmatrix} \begin{bmatrix} R & & B \\ & C & \\ D & & E \end{bmatrix} & & \begin{bmatrix} S & & B \\ & C & \\ D & & E \end{bmatrix} \\ & \begin{bmatrix} T & & B \\ & C & \\ D & & E \end{bmatrix} & \\ \begin{bmatrix} U & & B \\ & C & \\ D & & E \end{bmatrix} & & \begin{bmatrix} V & & B \\ & C & \\ D & & E \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} R & & S \\ & T & \\ U & & V \end{bmatrix} & & B \\ & C & \\ D & & E \end{bmatrix}.$$

Proof. Upon replacing E by C, (4.1.5) implies

$$\begin{bmatrix} \begin{bmatrix} A & & B \\ & R & \\ D & & C \end{bmatrix} & & \begin{bmatrix} A & & B \\ & S & \\ D & & C \end{bmatrix} \\ & \begin{bmatrix} A & & B \\ & T & \\ D & & C \end{bmatrix} & \\ \begin{bmatrix} A & & B \\ & U & \\ D & & C \end{bmatrix} & & \begin{bmatrix} A & & B \\ & V & \\ D & & C \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A & & B \\ & \begin{bmatrix} R & & S \\ & T & \\ U & & V \end{bmatrix} & \\ D & & C \end{bmatrix}.$$

Applying (4.1.5) once more we have

$$\begin{bmatrix} \begin{bmatrix} A & & B \\ & \begin{bmatrix} A & & B \\ & R & \\ D & & C \end{bmatrix} & \\ D & & E \end{bmatrix} & & \begin{bmatrix} A & & B \\ & \begin{bmatrix} A & & B \\ & S & \\ D & & C \end{bmatrix} & \\ D & & E \end{bmatrix} \\ & \begin{bmatrix} A & & B \\ & \begin{bmatrix} A & & B \\ & T & \\ D & & C \end{bmatrix} & \\ D & & E \end{bmatrix} & \\ \begin{bmatrix} A & & B \\ & \begin{bmatrix} A & & B \\ & U & \\ D & & C \end{bmatrix} & \\ D & & E \end{bmatrix} & & \begin{bmatrix} A & & B \\ & \begin{bmatrix} A & & B \\ & V & \\ D & & C \end{bmatrix} & \\ D & & E \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A & & B \\ & \begin{bmatrix} A & & B \\ & \begin{bmatrix} R & & S \\ & T & \\ U & & V \end{bmatrix} & \\ D & & C \end{bmatrix} & \\ D & & E \end{bmatrix}.$$

Abbreviating the libra notation $\overset{B}{\lfloor},,\overset{D}{\rfloor}$ of (4.1.9) to $\lfloor,,\rfloor$, the above becomes

$$\begin{bmatrix} \lfloor A, \lfloor A, R, C\rfloor, E\rfloor & & \lfloor A, \lfloor A, S, C\rfloor, E\rfloor \\ & \lfloor \mathrm{A}, \lfloor \mathrm{A}, \mathrm{T}, \mathrm{C}\rfloor, \mathrm{E}\rfloor & \\ \lfloor A, \lfloor A, U, C\rfloor, E\rfloor & & \lfloor A, \lfloor A, V, C\rfloor, E\rfloor \end{bmatrix} = \lfloor \mathrm{A}, \lfloor \mathrm{A}, \begin{bmatrix} R & & S \\ & T & \\ U & & V \end{bmatrix}, \mathrm{C}\rfloor, \mathrm{E}\rfloor$$

which by (3.3.1), (3.2.1) and the fact that $\lfloor,,\rfloor$ is abelian, reduces to

$$\begin{bmatrix} \lfloor R, C, E\rfloor & & \lfloor S, C, E\rfloor \\ & \lfloor \mathrm{T}, \mathrm{C}, \mathrm{E}\rfloor & \\ \lfloor U, C, E\rfloor & & \lfloor V, C, E\rfloor \end{bmatrix} = \lfloor \begin{bmatrix} R & & S \\ & T & \\ U & & V \end{bmatrix}, \mathrm{C}, \mathrm{E}\rfloor$$

which is just the equality we set out to establish, but in libra operator notation. QED

(4.3) Theorem Let $[\because]$ be a meridian operator on a set M. For $(R, B, C, D, S) \in \mathcal{M}^{(5)}$ and $A, E \in \mathcal{M}$ such that $\{A, E\} \neq \{B, D\}$, we have

$$\begin{bmatrix} \begin{bmatrix} A & & B \\ & R & \\ D & & C \end{bmatrix} & & B \\ & S & \\ D & & E \end{bmatrix} = \begin{bmatrix} A & & B \\ & \begin{bmatrix} R & & B \\ & C & \\ D & & S \end{bmatrix} & \\ D & & E \end{bmatrix}.$$

Proof. From Theorem (3.3) we have

$$\begin{bmatrix} \begin{bmatrix} A & & B \\ & R & \\ D & & C \end{bmatrix} & & B \\ & S & \\ D & & E \end{bmatrix} = {}^{B}\llcorner\ {}^{B}\llcorner A,R,C\lrcorner^{D},S,E\lrcorner^{D} = {}^{B}\llcorner A,\ {}^{B}\llcorner S,C,R\lrcorner^{D},E\lrcorner^{D} = {}^{B}\llcorner A,\ {}^{B}\llcorner R,C,S\lrcorner^{D},E\lrcorner^{D} = \begin{bmatrix} A & & B \\ & \begin{bmatrix} R & & B \\ & C & \\ D & & S \end{bmatrix} & \\ D & & E \end{bmatrix}.$$

QED

(4.4) Definition and Notation Let $\mathcal{X}$ be any set and $\mathit{\Pi}$ a family of self-inverse bijections of $\mathcal{X}$ such that

$$(\forall\, A,B,D,E \in \mathcal{X}\colon \{A,E\}\cap\{B,D\}=\emptyset)(\exists!\ \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}} \in \mathit{\Pi})\quad \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}(A)=E \quad\text{and}\quad \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}(B)=D; \tag{1}$$

$$(\forall\, A\in\mathcal{X})(\forall\, \alpha,\beta,\gamma\in\mathit{\Pi}\colon \alpha(A)=\beta(A)=\gamma(A))\quad \alpha\circ\beta\circ\gamma\in\mathit{\Pi}; \tag{2}$$

$$(\forall\, \alpha,\beta\in\mathit{\Pi})\quad \alpha\circ\beta\circ\alpha\in\mathit{\Pi}. \tag{3}$$

In this case we shall say that $\mathit{\Pi}$ is a **meridian family of involutions of the set $\mathcal{X}$**.

(4.5) Theorem Let $\mathcal{M}$ be a set and let $\mathit{\Pi}$ be a meridian family of involutions of $\mathcal{M}$. For $(A,B,C,D,E)\in\mathcal{M}^{(5)}$ define

$$\begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} \equiv \begin{cases} \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}(C), & \text{if } \{A,E\}\cap\{B,D\}=\emptyset; \\ A, & \text{if } A=B \text{ or } A=D; \\ E, & \text{if } E=B \text{ or } E=D. \end{cases}$$

Then $[\because]$ is a meridian operator on $\mathcal{M}$.

Proof. $\xrightarrow{(4.1.1)}$ For $(A,B,C,D,E)\in\mathcal{M}^{(5)}$ we have

$$\begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} = \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}(C) = \boxed{\substack{E\leftrightarrow A\\ B\leftrightarrow D}}(C) = \begin{bmatrix} E & & B \\ & C & \\ D & & A \end{bmatrix}$$

and

$$\begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} = \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}(C) = \boxed{\substack{B\leftrightarrow D\\ A\leftrightarrow E}}(C) = \begin{bmatrix} B & & A \\ & C & \\ E & & D \end{bmatrix}.$$

$\xrightarrow{(4.1.2)}$ Follows directly from the definition of $[\because]$.

$\xrightarrow{(4.1.3)}$ Let B and D be in $\mathcal{M}$, and let $\mathit{\Pi}_{B,D}\equiv\{\boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}\ :\ A,E\in\mathcal{M} \text{ and } \{A,E\}\cap\{B,D\}=\emptyset\}$. Let $\mathcal{M}_{B,D}\equiv\{X\in\mathcal{M}\colon\ B\neq X\neq D\}$. It follows from (4.4) that, if we replace $\mathit{\Pi}$ in (3.16) by $\mathit{\Pi}_{B,D}$ and C in (3.16) by $\mathcal{M}_{B,D}$, then (i), (ii) and (iii) of (3.16) are satisfied. By (3.16), the operator

$$\mathcal{M}_{B,D}\times\mathcal{M}_{B,D}\times\mathcal{M}_{B,D} \ni (A,C,E) \hookrightarrow \lfloor A,C,E\rfloor \equiv \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}(C) \in \mathcal{M}_{B,D} \tag{1}$$

is a libra operator. Consequently,

$$(\forall\, A,E\in\mathcal{M}_{B,D})\quad \begin{bmatrix} A & & B \\ & A & \\ D & & E \end{bmatrix} = \lfloor A,A,E\rfloor \xlongequal{\text{by (3.2.1)}} E$$

which is (4.1.3).

$\xrightarrow{(4.1.4)}$ For $A,C,E,R,S\in\mathcal{M}_{B,D}$ we have

$$\begin{bmatrix} \begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix} & & B \\ & R & \\ D & & S \end{bmatrix} \xlongequal{\text{by (1)}} \lfloor\lfloor A,C,E\rfloor,R,S\rfloor \xlongequal{\text{by (3.2.2)}} \lfloor A,C,\lfloor E,R,S\rfloor\rfloor \xlongequal{\text{by (1)}} \begin{bmatrix} A & & B \\ & C & \\ D & & \begin{bmatrix} E & & B \\ & R & \\ D & & S \end{bmatrix} \end{bmatrix}.$$

$\xRightarrow{(4.1.5)}$ From (4.4.3) follows that, for all $\alpha, \beta \in \mathit{\Pi}$, $\alpha\circ\beta\circ\alpha$ is in $\mathit{\Pi}$. Letting $\alpha \equiv \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}$ and $\beta \equiv \boxed{\substack{R\leftrightarrow V\\ S\leftrightarrow U}}$, we obtain

$$\alpha\circ\beta\circ\alpha\left(\begin{bmatrix} A & & B\\ & R & \\ D & & E\end{bmatrix}\right) = \alpha\circ\beta\circ\alpha\circ\alpha(R) = \alpha\circ\beta(R) = \alpha(V) = \begin{bmatrix} A & & B\\ & V & \\ D & & E\end{bmatrix}$$

and

$$\alpha\circ\beta\circ\alpha\left(\begin{bmatrix} A & & B\\ & S & \\ D & & E\end{bmatrix}\right) = \alpha\circ\beta\circ\alpha\circ\alpha(S) = \alpha\circ\beta(S) = \alpha(U) = \begin{bmatrix} A & & B\\ & U & \\ D & & E\end{bmatrix}$$

whence follows that

$$\alpha\circ\beta\circ\alpha = \boxed{\begin{matrix} \begin{bmatrix} A & & B\\ & R & \\ D & & E\end{bmatrix} \leftrightarrow \begin{bmatrix} A & & B\\ & V & \\ D & & E\end{bmatrix} \\ \\ \begin{bmatrix} A & & B\\ & S & \\ D & & E\end{bmatrix} \leftrightarrow \begin{bmatrix} A & & B\\ & U & \\ D & & E\end{bmatrix}\end{matrix}}. \tag{2}$$

Consequently

$$\begin{bmatrix} \begin{bmatrix} A & & B\\ & R & \\ D & & E\end{bmatrix} & & \begin{bmatrix} A & & B\\ & S & \\ D & & E\end{bmatrix} \\ & \begin{bmatrix} A & & B\\ & T & \\ D & & E\end{bmatrix} & \\ \begin{bmatrix} A & & B\\ & U & \\ D & & E\end{bmatrix} & & \begin{bmatrix} A & & B\\ & V & \\ D & & E\end{bmatrix}\end{bmatrix} = \boxed{\begin{matrix} \begin{bmatrix} A & & B\\ & R & \\ D & & E\end{bmatrix} \leftrightarrow \begin{bmatrix} A & & B\\ & V & \\ D & & E\end{bmatrix} \\ \\ \begin{bmatrix} A & & B\\ & S & \\ D & & E\end{bmatrix} \leftrightarrow \begin{bmatrix} A & & B\\ & U & \\ D & & E\end{bmatrix}\end{matrix}}(\alpha(T)) \overset{\text{by (2)}}{=\!=\!=} \alpha\circ\beta\circ\alpha\circ\alpha(T) = \alpha\circ\beta(T) = \begin{bmatrix} A & & B\\ & \begin{bmatrix} R & & S\\ & T & \\ U & & V\end{bmatrix} & \\ D & & E\end{bmatrix}$$

which is (4.1.5). QED

(4.6) Definition and Notation Let $[\because]$ be a meridian operator for a meridian $\mathcal{M}$. We define

$$(\forall\, A, B, D, E \in \mathcal{M} : \{A, E\} \cap \{B, D\} = \emptyset) \quad \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}} | \mathcal{M} \ni X \hookrightarrow \begin{bmatrix} A & & B\\ & X & \\ D & & E\end{bmatrix} \in \mathcal{M}$$

and for $B, C, D, E \in \mathcal{M}$ such that $E \notin \{B, D\}$

$$\boxed{\substack{B\updownarrow E\\ D\nearrow C}} | \mathcal{M} \ni X \hookrightarrow \begin{bmatrix} X & & B\\ & C & \\ D & & E\end{bmatrix} \in \mathcal{M}.$$

Functions of the form $\boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}$ are called **meridian involutions** and functions of the form $\boxed{\substack{A\updownarrow E\\ B\nearrow D}}$ are called **meridian lations**. We shall write $\mathit{\Pi}(\mathcal{M})$ for the family of all meridian involutions and $\mathit{\Lambda}(\mathcal{M})$ for the family of all meridian lations. We write $\mathit{\Gamma}(\mathcal{M})$ for the smallest balanced subset of $\mathfrak{J}(\mathcal{M}, \mathcal{M})$ containing $\mathit{\Lambda}(\mathcal{M}) \cup \mathit{\Pi}(\mathcal{M})$ as a subset.

(4.7) Theorem Let $\mathcal{M}$ be a meridian and $[\because]$ its meridian operator. Then, for $M, B, C, D, E \in \mathcal{M}$ such that $\{B, C\} \cap \{D, M\} = \emptyset$ and $\{B, E\} \cap \{D, M\} = \emptyset$,

$$\boxed{\substack{M\updownarrow E\\ D\nearrow C}} = \boxed{\substack{B\leftrightarrow E\\ D\leftrightarrow M}} \circ \boxed{\substack{B\leftrightarrow C\\ D\leftrightarrow M}}.$$

Proof. For $X \in \mathcal{M}$ we have

$$\boxed{\substack{B\leftrightarrow E\\ D\leftrightarrow M}} \circ \boxed{\substack{B\leftrightarrow C\\ D\leftrightarrow M}}(X) = {}^{D}\llcorner B,\ {}^{D}\llcorner B, X, C\lrcorner^{M}, E\lrcorner^{M} = {}^{D}\llcorner X, C, E\lrcorner^{M} = \boxed{\substack{M\updownarrow E\\ D\nearrow C}}(X).$$

QED

(4.8) Corollary The smallest balanced subset of $\mathfrak{J}(\mathcal{M},\mathcal{M})$ containing $\Pi(\mathcal{M})$ as a subset is $\Gamma(\mathcal{M})$.

(4.9) Theorem Let $\mathcal{M}$ be a meridian and $[\because]$ its meridian operator. Then $\Pi(\mathcal{M})$ is a meridian family of involutions on $\mathcal{M}$.

Proof. Let $\phi \in \Pi(\mathcal{M})$. Then $\phi = \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}$ for some $A,B,D,E \in \mathcal{M}$. For $X \in \mathcal{M}$ we have

$$\phi\circ\phi(X) = \begin{bmatrix} A & & B \\ & \begin{bmatrix} A & & B \\ & X & \\ D & & E \end{bmatrix} & \\ D & & E \end{bmatrix} \overset{\text{by (4.3)}}{=\!=\!=\!=} \begin{bmatrix} \begin{bmatrix} A & & B \\ & A & \\ D & & X \end{bmatrix} & & B \\ & & E \\ & D & & E \end{bmatrix} \overset{\text{by (4.1.3)}}{=\!=\!=\!=} \begin{bmatrix} X & & B \\ & E & \\ D & & E \end{bmatrix} = X$$

which establishes that ϕ is self-inverse.

Let $A,B,D,E \in \mathcal{M}$ satisfy $\{A,E\} \cap \{B,D\} = \emptyset$. From (4.1.3) and (4.1.7) we know that $\boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}(A) = E$ and $\boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}(B) = D$. Suppose that $\phi(A) = E$ and $\phi(B) = D$ for some other $\phi \in \Pi(\mathcal{M})$. Choose $R,S,U,V \in \mathcal{M}$ such that $\phi = \boxed{\substack{R\leftrightarrow V\\ S\leftrightarrow U}}$. We need to show that $\phi = \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}$ so, without loss of generality, we may suppose that $S \notin \{A,B,D,E\}$. As we are dealing with an involution, since $\phi(U) = S$, it follows that $U \notin \{A,B,D,E\}$ as well. We have

$$\overset{S}{\llcorner} A,A,E \overset{U}{\lrcorner} = E \quad \text{and} \quad \overset{S}{\llcorner} R,A,V \overset{U}{\lrcorner} = \boxed{\substack{R\leftrightarrow V\\ S\leftrightarrow U}}(A) = \phi(A) = E.$$

By (3.13.ii) it follows that $\overset{S}{\llcorner} A,X,E \overset{U}{\lrcorner} = \overset{S}{\llcorner} R,X,V \overset{U}{\lrcorner}$ for all $X \in \mathcal{M}$. This just means that $\phi = \boxed{\substack{R\leftrightarrow V\\ S\leftrightarrow U}} = \boxed{\substack{A\leftrightarrow E\\ S\leftrightarrow U}}$. So it will suffice in proving (4.4.1) to show that $\boxed{\substack{A\leftrightarrow E\\ S\leftrightarrow U}} = \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}$. We have

$$\overset{A}{\llcorner} B,B,D \overset{E}{\lrcorner} = D \quad \text{and} \quad \overset{A}{\llcorner} S,B,U \overset{E}{\lrcorner} = \boxed{\substack{A\leftrightarrow E\\ S\leftrightarrow U}}(B) = \phi(B) = D$$

and so by (3.13.ii) it follows that $\overset{A}{\llcorner} B,X,D \overset{E}{\lrcorner} = \overset{A}{\llcorner} S,X,U \overset{E}{\lrcorner}$ for all $X \in \mathcal{M}$. This means that $\boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}} = \boxed{\substack{A\leftrightarrow E\\ S\leftrightarrow U}}$. It follows that $\phi = \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}$, and so (4.4.1) is verified.

Let A, α, β, and γ be as in the hypothesis to (4.4.2). Let $B \in \mathcal{M}$ be distinct from A. We define $D \equiv \alpha(A)$, $E \equiv \alpha(B)$, $M \equiv \beta(B)$, and $N \equiv \gamma(B)$. By (4.4.1) we know that

$$\alpha = \boxed{\substack{B\leftrightarrow E\\ A\leftrightarrow D}}, \quad \beta = \boxed{\substack{B\leftrightarrow M\\ A\leftrightarrow D}}, \quad \text{and } \gamma = \boxed{\substack{B\leftrightarrow N\\ A\leftrightarrow D}}.$$

For $X \in \mathcal{M}$ we have

$$\beta\circ\gamma(X) = \begin{bmatrix} B & & A \\ & \begin{bmatrix} B & & A \\ & X & \\ D & & N \end{bmatrix} & \\ D & & M \end{bmatrix} = \overset{A}{\llcorner} B, \overset{A}{\llcorner} B,X,N \overset{D}{\lrcorner},M \overset{D}{\lrcorner} = \overset{A}{\llcorner} \overset{A}{\llcorner} B,B,X \overset{D}{\lrcorner},N,M \overset{D}{\lrcorner} = \overset{A}{\llcorner} X,N,M \overset{D}{\lrcorner}$$

whence follows that

$$\alpha\circ\beta\circ\gamma(X) = \boxed{\substack{B\leftrightarrow E\\ A\leftrightarrow D}}(\overset{A}{\llcorner} X,N,M \overset{D}{\lrcorner}) = \overset{A}{\llcorner} B, \overset{A}{\llcorner} X,N,M \overset{D}{\lrcorner},E \overset{D}{\lrcorner} = \overset{A}{\llcorner} B,X, \overset{A}{\llcorner} N,M,E \overset{D}{\lrcorner} \overset{D}{\lrcorner}.$$

If $C \equiv \overset{A}{\llcorner} N,M,E \overset{D}{\lrcorner}$, this just means that $\alpha\circ\beta\circ\gamma = \boxed{\substack{B\leftrightarrow C\\ A\leftrightarrow D}}$, whence follows the conclusion of (4.4.2).

Let α and β be in $\Pi(\mathcal{M})$. Then there exist $A,B,D,E,R,S,U,V \in \mathcal{M}$ such that $\alpha = \boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}$ and $\beta = \boxed{\substack{R\leftrightarrow V\\ S\leftrightarrow U}}$.

By (4.1.5), for $X\in\mathcal{M}$,

$$\boxed{\substack{A\leftrightarrow E\\ B\leftrightarrow D}}\circ\boxed{\substack{R\leftrightarrow V\\ S\leftrightarrow U}}(X)=\begin{bmatrix} A & & B\\ & \begin{bmatrix} R & & S\\ & X & \\ U & & V\end{bmatrix} & \\ D & & E\end{bmatrix}=\begin{bmatrix} \begin{bmatrix} A & & B\\ & R & \\ D & & E\end{bmatrix} & & \begin{bmatrix} A & & B\\ & S & \\ D & & E\end{bmatrix}\\ & \begin{bmatrix} A & & B\\ & X & \\ D & & E\end{bmatrix} & \\ \begin{bmatrix} A & & B\\ & U & \\ D & & E\end{bmatrix} & & \begin{bmatrix} A & & B\\ & V & \\ D & & E\end{bmatrix}\end{bmatrix}.$$

Setting $M\equiv\begin{bmatrix} A & & B\\ & R & \\ D & & E\end{bmatrix}$, $N\equiv\begin{bmatrix} A & & B\\ & S & \\ D & & E\end{bmatrix}$, $P\equiv\begin{bmatrix} A & & B\\ & U & \\ D & & E\end{bmatrix}$, $Q\equiv\begin{bmatrix} A & & B\\ & V & \\ D & & E\end{bmatrix}$ and $Y\equiv\alpha(X)$ we obtain

$$\alpha\circ\beta(X)=\begin{bmatrix} M & & N\\ & Y & \\ P & & Q\end{bmatrix}=\boxed{\substack{M\leftrightarrow Q\\ N\leftrightarrow P}}\circ\alpha(X).$$

It follows that $\alpha\circ\beta\circ\alpha=\boxed{\substack{M\leftrightarrow Q\\ N\leftrightarrow P}}$, which establishes (4.4.3). QED

(4.10) Theorem Let $\sim$ be a meridian equivalence relation for a set $\mathcal{M}$. For (A,B,C,D,E) in $\mathcal{M}^{(5)}$ such that $\{A,E\}\neq\{B,D\}$ let

$$\begin{bmatrix} A & & B\\ & C & \\ D & & E\end{bmatrix}\equiv\begin{cases} \left[\begin{smallmatrix} A & B\\ D & C\end{smallmatrix}\right](C) & \text{if } \{A,E\}\cap\{B,D\}=\emptyset,\ \left[\begin{smallmatrix} A & B\\ D & C\end{smallmatrix}\right]\in\Gamma^2_\sim(\mathcal{M}),\ \left[\begin{smallmatrix} A & B\\ D & C\end{smallmatrix}\right](A)=E, \text{ and } \left[\begin{smallmatrix} A & B\\ D & C\end{smallmatrix}\right](B)=D;\\ A & \text{if } A\in\{B,D\};\\ E & \text{if } E\in\{B,D\}\end{cases}$$

(where $\left[\begin{smallmatrix} A & B\\ D & C\end{smallmatrix}\right]$ is as in Theorem (2.46)). Then $[\because]$ is a meridian operator, relative to which $\Pi(\mathcal{M})=\Gamma^2_\sim(\mathcal{M})$ and $\Gamma(\mathcal{M})=\Gamma(\mathcal{M},\sim)$.

Proof. Let $\Pi\equiv\Gamma^2_\sim(\mathcal{M})$ in Theorem (4.5). Then condition (4.4.1) holds by the definition of $\Gamma^2_\sim(\mathcal{M})$, (4.4.2) holds by Theorem (2.45), and (4.4.3) holds since $\Gamma(\mathcal{M},\sim)$ is a group[17]. It follows from Theorem (4.5) that $[\because]$ is a meridian operator on $\mathcal{M}$.

That $\Pi(\mathcal{M})=\Gamma^2_\sim(\mathcal{M})$ follows from Theorem (4.9).

From (2.33) follows that $\Gamma(\mathcal{M},\sim)\subset\Gamma(\mathcal{M})$. Thus (2.15) implies that $\Gamma(\mathcal{M})=\Gamma(\mathcal{M},\sim)$. QED

(4.11) Theorem Suppose that we have a meridian operator $[\because]$ on a set $\mathcal{M}$ with at least four elements. Let 0, 1, and ∞ be three distinct elements of $\mathcal{M}$. For $X,Y,R,S\in\mathcal{M}$, none of which equals ∞, define

$$X+Y\equiv\begin{bmatrix} X & & \infty\\ & 0 & \\ \infty & & Y\end{bmatrix}, \text{ and } R\cdot S\equiv\begin{bmatrix} S & & 0\\ & 1 & \\ \infty & & R\end{bmatrix}. \tag{1}$$

Then $\mathcal{F}\equiv\{X\in\mathcal{M}:\ x\neq\infty\}$, relative to these two binary operators, is a field with additive identity 0 and multiplicative identity 1.

For $(R,S,U,V)\in\mathcal{F}$ with $R\cdot V\neq S\cdot U$, let

$$(\forall\, X\in\mathcal{M})\quad \begin{pmatrix} R & S\\ U & V\end{pmatrix}(X)\equiv\begin{cases}\frac{R\cdot X+S}{U\cdot X+V} & \text{if } X\in\mathcal{F};\\ \frac{R}{U} & \text{if } X=\infty\end{cases} \tag{2}$$

where the value in either case is ∞ when the denominator is 0. Define

$$\Gamma_{(0,1,\infty)}(\mathcal{M})\equiv\{\begin{pmatrix} R & S\\ U & V\end{pmatrix}:\ (R,S,U,V)\in\mathcal{F} \text{ with } R\cdot V\neq S\cdot U\}. \tag{3}$$

[17] *Cf.* (2.15).

Then $$\Gamma_{(0,1,\infty)}(\mathcal{M}) = \Gamma(\mathcal{M}). \tag{4}$$

Proof. Since $\overset{\infty}{\llcorner},,,\overset{\infty}{\lrcorner}$ is an abelian libra operator, it follows from Theorem (3.7) that $+$ is an abelian group operator with identity 0. Since $\overset{o}{\llcorner},,,\overset{\infty}{\lrcorner}$ is an abelian libra operator, it follows from Theorem (3.7) that $\cdot$ is an abelian group operator with identity 1. It remains to show the distributive law. We shall adopt the common practice of suppressing the "dot" in products.

For $A, B, C \in \mathcal{M}$ such that $B \neq \infty \neq C$ and $\infty \neq A \neq 0$

$$A(B+C) = \left[\begin{array}{ccc}\left[\begin{array}{ccc}B&&\infty\\&0&\\\infty&&C\end{array}\right]&&0\\&1&\\\infty&&A\end{array}\right] \overset{\text{by (4.2)}}{=\!=\!=\!=} \left[\begin{array}{ccc}\left[\begin{array}{ccc}B&&0\\&1&\\\infty&&A\end{array}\right]&&\left[\begin{array}{ccc}\infty&&0\\&1&\\\infty&&A\end{array}\right]\\&\left[\begin{array}{ccc}0&&0\\&1&\\\infty&&A\end{array}\right]&\\\left[\begin{array}{ccc}\infty&&0\\&1&\\\infty&&A\end{array}\right]&&\left[\begin{array}{ccc}C&&0\\&1&\\\infty&&A\end{array}\right]\end{array}\right] \overset{\text{by (4.1.2)}}{=\!=\!=\!=} \left[\begin{array}{ccc}AB&&\infty\\&0&\\\infty&&AC\end{array}\right] = AB + AC.$$

Thus $\mathcal{F}$ is a field.

Our next task is to show that $$\Gamma_{(0,1,\infty)}(\mathcal{M}) \subset \Gamma(\mathcal{M}). \tag{5}$$

A function of the form $\mathcal{F} \ni X \hookrightarrow AX+B \in F$, for $A, B \in F$ is called an **affine function** – this function is the restriction to $\mathcal{F}$ of $\begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}$. Consequently we shall call functions of the type $\begin{pmatrix} R & S \\ 0 & V \end{pmatrix}$ **affine elements of $\Gamma_{(0,1,\infty)}$**. We shall call the function $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the **inversion of $\Gamma_{(0,1,\infty)}$**. Suppose $\begin{pmatrix} R & S \\ U & V \end{pmatrix}$ is not affine. Then we may choose $D \in F$ such that $DU = R$, after which we choose $A \in F$ such that $S = DV + A$. We now evidently have

$$\begin{pmatrix} R & S \\ U & V \end{pmatrix} = \begin{pmatrix} A & D \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} U & V \\ 0 & 1 \end{pmatrix}.$$

Thus, if we can show that affine elements of $\Gamma_{(0,1,\infty)}$ and the inversion are in $\Gamma(\mathcal{M})$, we will have demonstrated (5). The equality

$$(\forall\, X \in \mathcal{M}) \quad \begin{pmatrix} R & S \\ 0 & 1 \end{pmatrix}(X) = \left[\begin{array}{ccc}\left[\begin{array}{ccc}X&&0\\&1&\\\infty&&R\end{array}\right]&&\infty\\&0&\\\infty&&S\end{array}\right] = \boxed{\substack{\infty\updownarrow S\\ \infty\nearrow 0}} \circ \boxed{\substack{0\updownarrow R\\ \infty\nearrow 1}}(X)$$

shows that affine functions are in $\Gamma(\mathcal{M})$. To show that the inversion is in $\Gamma(\mathcal{M})$, it will suffice to show that it equals $\boxed{\substack{0\leftrightarrow 1\\ \infty\leftrightarrow 1}}$. By definition, $\frac{1}{X}$ is the unique element whose product with X is 1. But

$$X \cdot \boxed{\substack{0\leftrightarrow 1\\ \infty\leftrightarrow 1}}(X) = X \cdot \left[\begin{array}{ccc}1&&0\\&X&\\\infty&&1\end{array}\right] = \left[\begin{array}{ccc}X&&0\\&1&\\\infty&&\left[\begin{array}{ccc}1&&0\\&X&\\\infty&&1\end{array}\right]\end{array}\right] = \overset{\infty}{\llcorner} X,1,\ \overset{\infty}{\llcorner} 1,X,1 \overset{0}{\lrcorner}\overset{0}{\lrcorner} = \overset{\infty}{\llcorner} X,1,1,X,1 \overset{0}{\lrcorner} = 1.$$

Thus $\frac{1}{X} = \boxed{\substack{0\leftrightarrow 1\\ \infty\leftrightarrow 1}}(X)$, and so (5) holds.

To complete the proof of (4), it will suffice to show the opposite inclusion to that in (5). In view of (4.8), that will follow once we have shown that each element of $\Pi(\mathcal{M})$ is in $\Gamma_{(0,1,\infty)}(\mathcal{M})$. We consider then $\boxed{\substack{A\leftrightarrow D\\ B\leftrightarrow C}}$ for generic $A, B, C, D \in \mathcal{M}$ such that $\{A, B\} \cap \{C, D\} = \emptyset$. We consider several cases serially:

Case: $A = D = \infty$ or $B = C = \infty$. Without loss of generality we presume $A = D = \infty$. We have $\begin{pmatrix} -1 & B+C \\ 0 & 1 \end{pmatrix}(\infty) = \infty$ and $\begin{pmatrix} -1 & B+C \\ 0 & 1 \end{pmatrix}(B) = -B+B+C = C \Longrightarrow \begin{pmatrix} -1 & C+B \\ 0 & 1 \end{pmatrix} = \boxed{\substack{A\leftrightarrow D \\ B\leftrightarrow C}}$.

Case: $\#\{X \in \{A,B,C,D\} : X = \infty\} = 1$. Without loss of generality we presume that $A = \infty$. We have $\begin{pmatrix} D & CB\text{-}CD\text{-}DB \\ 1 & \text{-}D \end{pmatrix}(\infty) = D$ and $\begin{pmatrix} D & CB\text{-}CD\text{-}DB \\ 1 & \text{-}D \end{pmatrix}(B) = \dfrac{DB+CB\text{-}CD\text{-}DB}{B\text{-}D} = C \Longrightarrow$

$$\begin{pmatrix} D & CB\text{-}CD\text{-}DB \\ 1 & \text{-}D \end{pmatrix} = \boxed{\substack{A\leftrightarrow D \\ B\leftrightarrow C}}.$$

Case: $\infty \notin \{A,B,C,D\}$. We have

$$\begin{pmatrix} AD\text{-}BC & (A\text{-}B\text{-}C+D)AD+(BC\text{-}AD)(A+D) \\ A\text{-}B\text{-}C+D & BC\text{-}AD \end{pmatrix}(A) = D \quad \text{and}$$

$$\begin{pmatrix} AD\text{-}BC & (A\text{-}B\text{-}C+D)AD+(BC\text{-}AD)(A+D) \\ A\text{-}B\text{-}C+D & BC\text{-}AD \end{pmatrix}(B) = C \quad \Longrightarrow$$

$$\begin{pmatrix} AD\text{-}BC & (A\text{-}B\text{-}C+D)AD+(BC\text{-}AD)(A+D) \\ A\text{-}B\text{-}C+D & BC\text{-}AD \end{pmatrix} = \boxed{\substack{A\leftrightarrow D \\ B\leftrightarrow C}}.$$

It follows that (4) holds. QED

(4.12) Theorem Let $\mathcal{M}$ be a meridian (relative to a meridian operator $[\because]$ with at least four elements. Let $0, 1, \infty \in \mathcal{M}$ be distinct, and we shall adopt the notation of Theorem (4.11). The **cross-ratio relative to** $(0,1,\infty)$ is defined by $(\forall\, A,B,C,D \in \mathcal{M} : \#\{A,B,C,D\} > 2)$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{(0,1,\infty)} \equiv \begin{cases} \frac{(C-A)\cdot(D-B)}{(C-B)\cdot(D-A)}, & \text{if } \infty \notin \{A,B,C,D\}, C \neq B \text{ and } D \neq A; \\ \infty, & \text{if } C = B \text{ or } D = A; \\ 0, & \text{if } C = A = \infty \text{ or } D = B = \infty; \\ 1, & \text{if } C = D = \infty \text{ or } B = A = \infty; \\ \frac{D-B}{C-B}, & \text{if } A = \infty \text{ and } \infty \notin \{B,C,D\} \text{ and } C \neq B; \\ \frac{C-A}{D-A}, & \text{if } B = \infty \text{ and } \infty \notin \{A,C,D\} \text{ and } D \neq A; \\ \frac{D-B}{D-A}, & \text{if } C = \infty \text{ and } \infty \notin \{A,B,D\} \text{ and } D \neq A; \\ \frac{C-A}{C-B}, & \text{if } D = \infty \text{ and } \infty \notin \{A,B,C\} \text{ and } C \neq B. \end{cases} \tag{1}$$

Define $\sim$ on $\mathcal{M}^{\Upsilon}_{2^+}$ by

$$(\forall\, \mathfrak{t}, \mathfrak{s} \in \mathcal{M}^{\Upsilon}_{2^+}) \quad \mathfrak{t} \sim \mathfrak{s} \Longleftrightarrow \begin{bmatrix} \mathfrak{t}_\heartsuit & \mathfrak{t}_\spadesuit \\ \mathfrak{t}_\clubsuit & \mathfrak{t}_\diamondsuit \end{bmatrix}_{(0,1,\infty)} = \begin{bmatrix} \mathfrak{s}_\heartsuit & \mathfrak{s}_\spadesuit \\ \mathfrak{s}_\clubsuit & \mathfrak{s}_\diamondsuit \end{bmatrix}_{(0,1,\infty)}. \tag{2}$$

Then
(i) $\sim$ is a meridian equivalence relation for $\mathcal{M}$;
(ii) $\Gamma(\mathcal{M}) = \Gamma(\mathcal{M}, \sim)$;
(iii) $\Pi(\mathcal{M}) = \Gamma^2_\sim(\mathcal{M})$;
(iv) $(\forall\, A,B,C \in \mathcal{M}$ distinct$)(\forall\, R,S,T \in \mathcal{M}$ distinct$)(\exists!\, \alpha \in \Gamma(\mathcal{M}))\quad \alpha(A) = R,\ \alpha(B) = S$ and $\alpha(C) = T$;
(v) $\sim$ is independent of the choice of $0, 1, \infty \in \mathcal{M}$ distinct.

Proof. That Postulate I (2.3) holds is immediate from the definition. It also follows from the definition of $\sim$ that each element $\mathfrak{X} \in \mathfrak{M}$ is of the form

$$R_{\mathfrak{M}} \equiv \{\mathfrak{t} \in \mathcal{M}^{\Upsilon}_{2^+} : \begin{bmatrix} \mathfrak{t}_\heartsuit & \mathfrak{t}_\spadesuit \\ \mathfrak{t}_\clubsuit & \mathfrak{t}_\diamondsuit \end{bmatrix}_{(0,1,\infty)} = R\} \tag{3}$$

for some $R \in \mathcal{M}$. Solving the equation in (1) for $\mathfrak{t}_\diamondsuit$, we obtain

$$\mathfrak{t}_\diamondsuit = \begin{pmatrix} \mathfrak{t}_\heartsuit \cdot (\mathfrak{t}_\spadesuit - \mathfrak{t}_\clubsuit) & \mathfrak{t}_\spadesuit \cdot (\mathfrak{t}_\clubsuit - \mathfrak{t}_\heartsuit) \\ \mathfrak{t}_\spadesuit - \mathfrak{t}_\clubsuit & \mathfrak{t}_\clubsuit - \mathfrak{t}_\heartsuit \end{pmatrix}(R). \tag{4}$$

It follows that

$$\mathbf{Mor}(\mathfrak{M}, \mathcal{M}) = \{\mathfrak{M} \ni R_{\mathfrak{M}} \hookrightarrow \alpha(R) \in \mathcal{M} : \alpha \in \Gamma(\mathcal{M})\}. \tag{5}$$

From (5)) and (2.8.1) follows that

$$\varGamma(\mathcal{M},\sim) = \{\alpha\circ\beta\colon \alpha,\beta\in\varGamma(\mathcal{M})\} = \varGamma(\mathcal{M})$$

which establishes (ii), as well as Postulate II (2.7) and Postulate III (2.10). That (iii) holds is now evident from (ii) and the definitions of $\varPi(\mathcal{M})$ and $\varGamma^2_\sim(\mathcal{M})$.

That (iv) holds follows from the fundamental theorem (2.12). This fundamental theorem (iv) along with Postulate III implies (v).

Let $\tau\in\varGamma(\mathcal{M})$ be a translation in the sense of (2.26). Let α be any element of $\varGamma^2_\sim(\mathcal{M})$ which agrees with τ at its fixed point, and at some other point. In view of (v) we can choose 0, 1 and ∞ such that

$$\alpha(\infty) = \tau(\infty) = \infty,\quad \alpha(0) = \tau(0) = 1. \tag{6}$$

Choose $R,S,T,U\in\mathcal{M}$ such that $\tau = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$. That $\tau(\infty)=\infty$ implies that $T=0$ so that we may, and shall presume that $U=1$. Thus

$$\tau(0) = R\cdot 0 + S = 1 \Longrightarrow S=1.$$

The equation $RX+1 = X$ has the solution $\frac{-1}{R-1}$ if $R\neq 1$. But τ, being a translation, fixes only ∞. Thus R must be 1:

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The transformation α being an involution, must be of the form $\begin{pmatrix} A & B \\ C & -A \end{pmatrix}$ for $A,B,C\in\mathcal{M}$. We have

$$\infty = \alpha(\infty) \Longrightarrow B = 0$$

and

$$1 = \alpha(0) = \begin{pmatrix} A & B \\ 0 & -A \end{pmatrix}(0) = \frac{B}{-A}.$$

Thus we may take $A=-1$ and $B=1$. We have

$$\tau\circ\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\circ\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

which last is evidently an involution. This, with (2.38) establishes the validity of Postulate IV (2.39). QED

(4.13) Theorem Let $(\mathcal{F},+,\cdot)$ be a field with additive identity 0 and multiplicative identity 1. Let ∞ any point not in $\mathcal{F}$ and let $\mathcal{M}\equiv\{\infty\}\cup\mathcal{F}$. For $(R,S,U,V)\in\mathcal{F}$ with $RV\neq SU$, let

$$(\forall\, X\in\mathcal{M})\quad \begin{pmatrix} R & S \\ U & V \end{pmatrix}(X) \equiv \begin{cases} \frac{R\cdot X+S}{U\cdot X+V} & \text{if } X\in\mathcal{F}; \\ \frac{R}{U} & \text{if } x=\infty \end{cases} \tag{1}$$

where the value in either case is ∞ when the denominator is 0. Let

$$\varPi(\mathcal{M}) \equiv \{\begin{pmatrix} R & S \\ U & V \end{pmatrix} : R+V = 0 \text{ and } RV\neq SU\}.$$

Then $\varPi(\mathcal{M})$ is a meridian family of involutions on $\mathcal{M}$.

Proof. For $R,S,U\in\mathcal{M}$ such that $SU + R^2\neq 0$

$$\begin{pmatrix} R & S \\ U & -R \end{pmatrix}\circ\begin{pmatrix} R & S \\ U & -R \end{pmatrix} = \begin{pmatrix} R^2+SU & RS-RS \\ UR-RU & US+R^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which shows the mapping is its own inverse.

Let A, B, D and E be as in the hypothesis to (4.4.1). For $R,S,U,X,Y\in\mathcal{M}$ such that $SU+R^2\neq 0$, the equation $\begin{pmatrix} R & S \\ U & -R \end{pmatrix}(X)=Y$ resolves into

$$\begin{cases} U=0 & \text{if } X=\infty=Y; \\ R=UY & \text{if } X=\infty \text{ and } Y\in\mathcal{F}; \\ UXY=R(X+Y)+S & \text{if } X,Y\in\mathcal{F}. \end{cases}$$

Without loss of generality we need consider just the following cases: $A=E=\infty$ with $B,D\in\mathcal{F}$; $A=\infty=B$ with

$D, E\in\mathcal{F}$; and $A, B, D, E\in\mathcal{F}$. The solution is $\boxed{\substack{R\leftrightarrow R\\ S\leftrightarrow U}}$ where

$$\begin{cases} R=1,\ U=0,\ S=-(B+D) & \text{if } A=E=\infty \text{ with } B, D\in\mathcal{F};\\ R=E,\ U=1,\ S=BD-E(B+D) & \text{if } A=\infty \text{ with } B, D, E\in\mathcal{F};\\ R=AE-BD,\ U=A+E-B-D,\ S=AU-R(A+E) & \text{if } A, B, D, E\in\mathcal{F}. \end{cases}$$

Furthermore these solutions are unique up to a constant factor, which would not change the value of $\boxed{\substack{R\leftrightarrow R\\ S\leftrightarrow D}}$. This establishes (4.4.1).

Let A, α, β, and γ be as in the hypothesis to (4.4.2), and let $B\equiv\alpha(A)$. From (4.4.1) there exists $\delta\in\Pi(\mathcal{M})$ such that $\delta(A)=1$, $\delta(1)=A$, $\delta(-1)=B$, and $\delta(B)=-1$. Then $\delta\circ\alpha\circ\delta$, $\delta\circ\beta\circ\delta$, and $\delta\circ\gamma\circ\delta$ all interchange 1 with -1. Direct calculation shows that there exist $M, N, O, S\in\mathcal{F}$ such that $\alpha=\begin{pmatrix} M & S\\ -S & -M\end{pmatrix}$, $B=\begin{pmatrix} N & S\\ -S & -N\end{pmatrix}$, and $B=\begin{pmatrix} O & S\\ -S & -O\end{pmatrix}$. Direct calculation also shows

$$\alpha\circ\beta\circ\gamma=\begin{pmatrix} MNO(N-M-O)S^2 & S(MN+ON-MO)-S^3\\ -S(MN+ON-MO)+S^3 & -MNO(N-M-O)\end{pmatrix}$$

which is in $\Pi(L)$. This establishes (4.4.2).

Let α and β be as in the (4.4.3). Choose $A, B, D, R, S, U\in\mathcal{F}$ such that

$$\begin{pmatrix} -A & B\\ D & A\end{pmatrix}=\alpha \quad\text{and}\quad \begin{pmatrix} -R & S\\ U & R\end{pmatrix}=\beta.$$

We have

$$\alpha\circ\beta\circ\alpha=\begin{pmatrix} -A & B\\ D & A\end{pmatrix}\circ\begin{pmatrix} -R & S\\ U & R\end{pmatrix}\circ\begin{pmatrix} -A & B\\ D & A\end{pmatrix}=\begin{pmatrix} -A & B\\ D & A\end{pmatrix}\circ\begin{pmatrix} RA+SD & -RB+SA\\ -UA+RD & UB+RA\end{pmatrix}=$$

$$\begin{pmatrix} -A^2R-ADS-ABU+BDR & ABR-A^2S+B^2U+ABR\\ ADR+D^2S-UA^2+ARD & A^2R+ADS+ABU-BDR\end{pmatrix}.$$

This last is in $\Pi(L)$, and so (4.4.3) holds. QED

(4.14) Remarks The circle of theorems (4.5), (4.9), (4.10), (4.11), (4.12) and (4.13) show that a meridian as derived from a quadric equivalence relation, a meridian operator, a meridian family of involutions, or from a field, is in each case essentially the same object. So far as we know, the equivalence of the latter two characterizations is due to J. Tits ([Tits]).

(4.15) Notation Let $\mathcal{M}$ be a meridian. We shall denote by $\Gamma T(\mathcal{M})$ the set of (meridian) translations in $\Gamma(\mathcal{M})$:

$$\Gamma T(\mathcal{M})\equiv\{\alpha\in\Gamma(\mathcal{M}) : \alpha \text{ has exactly one fixed point}\}.$$

(4.16) Theorem Let $\mathcal{M}$ be a meridian relative to a meridian operator. Then

$$\Gamma T(\mathcal{M})=\{\mathcal{M}\ni X\hookrightarrow\begin{bmatrix} X & & B\\ & C & \\ B & & E\end{bmatrix}\in\mathcal{M} : B, C, E\in\mathcal{M} \text{ distinct}\}.$$

Proof. We first take three distinct elements from $\mathcal{M}$, denote them by ∞, 0, and 1, and define operators $+$ and $\cdot$ as in (4.11.1). We have

$$(\forall X\in\mathcal{F})\quad \begin{bmatrix} X & & \infty\\ & 0 & \\ \infty & & 1\end{bmatrix}=X+1 \quad\text{and}\quad \begin{bmatrix} \infty & & \infty\\ & 0 & \\ \infty & & 1\end{bmatrix}=\infty.$$

Evidently this function $\mathcal{M}\ni X\hookrightarrow\begin{bmatrix} X & & \infty\\ & 0 & \\ \infty & & 1\end{bmatrix}\in\mathcal{M}$ is a translation.

Now let $\tau\in\Gamma(\mathcal{M})$ be a translation. We shall denote its fixed point by ∞. Let 0 be any other point and denote $\tau(0)$ by 1. Adopting $+$ and $\cdot$ to 0, 1, and ∞, we define $\theta|\mathcal{M}\ni X\hookrightarrow X+1\in\mathcal{M}$ for $X\in\mathcal{F}$ and $\theta(\infty)\equiv\infty$. Evidently θ is a translation and agrees with τ at both ∞ and 0. It follows from Theorem (2.38.v) that $\tau=\theta$. QED

(4.17) Definition and Notation . Let $\mathcal{M}$ be a meridian relative to a meridian operator $[\because]$. We define the set of **meridian dilations** by

$$\Gamma\Delta(\mathcal{M}) \equiv \{\mathcal{M} \ni X \hookrightarrow \begin{bmatrix} X & & B \\ & C & \\ D & & E \end{bmatrix} \in \mathcal{M} : B, C, D, E \in \mathcal{M} \text{ distinct}\}.$$

(4.18) Theorem Let $\mathcal{M}$ be a meridian relative to a meridian operator $[\because]$. Then a necessary and sufficient condition for $\delta \in \Gamma(\mathcal{M})$ to be a meridian dilation is for it to have two distinct fixed points.

Proof. If δ is a dilation, then it is of the form $\mathcal{M} \ni X \hookrightarrow \begin{bmatrix} X & & B \\ & C & \\ D & & E \end{bmatrix} \in \mathcal{M}$. From (4.1.2) follows that δ fixes both B and D.

Suppose now that δ fixes two distinct points ∞ and 0 in $\mathcal{M}$. Let 1 any other point of $\mathcal{M}$ and define operators $+$ and $\cdot$ as in Theorem (4.11.1). Let $E \equiv \delta(1)$. We have

$$\begin{bmatrix} \infty & & \infty \\ & 1 & \\ 0 & & E \end{bmatrix} = \infty = \delta(\infty), \quad \begin{bmatrix} 0 & & \infty \\ & 1 & \\ 0 & & E \end{bmatrix} = 0 = \delta(0) \quad \text{and} \quad \begin{bmatrix} 1 & & \infty \\ & 1 & \\ 0 & & E \end{bmatrix} = E = \delta(1).$$

The function $\mathcal{M} \ni X \hookrightarrow \begin{bmatrix} X & & \infty \\ & 1 & \\ 0 & & E \end{bmatrix} \in \mathcal{M}$ equals δ at three distinct points and, by the fundamental theorem, must be δ. QED

(4.19) Definition and Notation Let $\mathcal{M}$ be a meridian. An element of $\Gamma(\mathcal{M})$ which is neither a translation nor a dilation will be called a **meridian rotation**. We denote the set of all rotations as follows:

$$\Gamma R(\mathcal{M}) \equiv \{\rho \in \Gamma(\mathcal{M}) : \rho \text{ has no fixed points.}\}$$

(4.20) Theorem Let $\mathcal{M}$ be a meridian and let θ be an element of $\Gamma(\mathcal{M})$. Then

(i) θ is an involution$\iff$ $(\exists\, A, B \in \mathcal{M}$ distinct$)$ $\quad \theta(A) = B$ and $\theta(B) = A$;

(ii) θ is a translation$\iff$ $(\exists\, \pi, \sigma \in \Pi(\mathcal{M}) \cap \Gamma\Delta(\mathcal{M})$ with a single common fixed point$)$ $\quad \theta = \pi \circ \sigma$;

(iii) θ is a dilation$\iff$ $\theta \notin \Gamma T(\mathcal{M})$ and either θ is an involution or
$(\exists\, \pi \in \Pi(\mathcal{M}) \cap, \sigma \in \Pi(\mathcal{M})$ agreeing on two points$)$ $\quad \theta = \pi \circ \sigma$;

(iv) θ is a rotation$\iff$ either θ is an involution with no fixed point or
$(\exists\, \pi \in \Pi(\mathcal{M}) \cap \Gamma\Delta(\mathcal{M}), \sigma \in \Pi(\mathcal{M})$ agreeing at no point$)$ $\quad \theta = \pi \circ \sigma$;

(v) $\Gamma(\mathcal{M}) = \Pi(\mathcal{M}) \cup \{\pi \circ \sigma : \pi \in \Pi(\mathcal{M}) \cap \Gamma\Delta(\mathcal{M}), \sigma \in \Pi(\mathcal{M})\}$.

Proof. $\overset{\text{(i)}}{\Longrightarrow}$: Trivial.

$\overset{\text{(i)}}{\Longleftarrow}$: This follows from (2.30).

$\overset{\text{(ii)}}{\Longrightarrow}$: If θ is a translation with fixed point ∞, 0 is another point, and $1 \equiv \theta(0)$, then $\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ where $+$ and $\cdot$ are as in (4.11.1). Since $\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are in $\Pi(\mathcal{M}) \cap \Gamma\Delta(\mathcal{M})$ and both fix ∞, and since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have established the $\Longrightarrow$ part of (ii).

$\overset{\text{(ii)}}{\Longleftarrow}$: Trivial.

$\overset{\text{(iii)}}{\Longrightarrow}$: Since θ is a dilation, it has two fixed points by (4.18), and so cannot be a translation. We shall presume that θ is not an involution. We shall denote the two fixed points of θ by 0 and ∞, write 1 for a third point, and define $+$ and $\cdot$ as in (4.11.1). If $R \equiv \theta(1)$ then $\theta = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$ and so

$$\theta = \theta = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ R & 0 \end{pmatrix}.$$

Both factors are in $\Pi(\mathcal{M})$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which fixes both 1 and -1, is in $\Gamma\Delta(\mathcal{M})$.

$\overset{\text{(iii)}}{\Longleftarrow}$: Let $\theta = \pi \circ \sigma$ for $\pi, \sigma \in \Pi(\mathcal{M})$ and suppose that θ is not a translation and that π and σ agree on exactly two points A and B. If $\pi(A) = A$ and $\pi(B) = B$ then, by (2.34), π would equal σ and so π would be the identity mapping, which is absurd. If $\pi(A) = A$ then θ would be a translation (with fixed point A) and so $B \equiv \pi(A) = \sigma(A)$ is not A. Evidently $\theta(A) = A$ and $\theta(B) = B$. Hence θ is a dilation.

$\overset{\text{(iv)}}{\Longrightarrow}$: We presume that θ is not an involution. Then there exist $\infty, 1, 0 \in \mathcal{M}$ distinct such that $\theta(\infty) = 1$ and $\theta(1) = 0$. Let $M \equiv \theta(0)$. Direct calculation yields

$$\theta = \begin{pmatrix} -M & M \\ -M & 1 \end{pmatrix} = \begin{pmatrix} M & -M \\ 1 & -M \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Both factors are in $\Pi(\mathcal{M})$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which fixes both 1 and -1, is in $\Gamma\Delta(\mathcal{M})$.

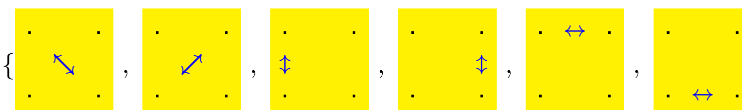
$\overset{\text{(iv)}}{\Longleftarrow}$: Let $\theta = \pi \circ \sigma$ for $\pi, \sigma \in \Pi(\mathcal{M})$, $\pi \in \Gamma\Delta(\mathcal{M})$ and suppose that π and σ agree on no point. Then $\pi \circ \sigma$ can have no fixed point, whence θ is a rotation. This establishes the $\Longleftarrow$ part of (iv).

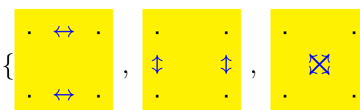
Part (v) now follows from (ii), (iii), and (iv). QED

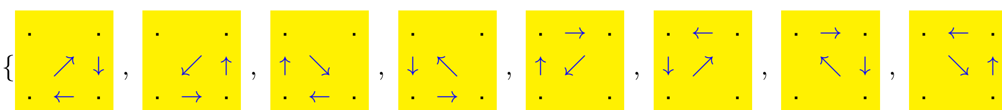
(4.21) Example We return to the example (2.42), which was a meridian $\mathcal{M}$ consisting of four elements A, B, C and D which we shall picture as four points arranged as the vertices of a square in a plane:

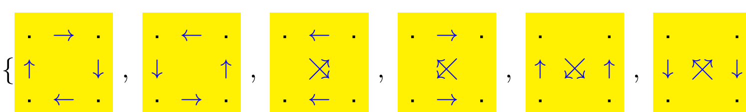
$$\begin{array}{cc} \text{A} & \text{B} \\ & \\ \text{C} & \text{D} \end{array}.$$

It follows from the fundamental theorem that we may regard $\Gamma(\mathcal{M})$ as the group of permutations of these four points. There are of course 24 of them, and we shall denote them with arrows to show the orbits of the permutations.[18] We have

$\Pi(\mathcal{M}) \cap \Delta(\mathcal{M}) = \{$  $\}$

$\Pi(\mathcal{M}) \cap R(\mathcal{M}) = \{$  $\}$

$T(\mathcal{M}) = \{$  $\}$

$R(\mathcal{M}) \cap \Gamma^4(\mathcal{M}) = \{$  $\}$

(4.22) Example Let $\mathcal{M}$ be the real projective line. Then $\begin{bmatrix} A & & C \\ & E & \\ D & & B \end{bmatrix}$ is just $\mu(A, B, C, D, E)$ of Figure (1) in Section (1).

(4.23) Example Let $\mathcal{M}$ be the circle. Then $\begin{bmatrix} A & & C \\ & E & \\ D & & B \end{bmatrix}$ is just $\mu(A, B, C, D, E)$ of Figure (4) in Section (1).

(4.24) Definition Let $\mathcal{M}$ be a meridian relative to a meridian operator $[\because]$. Any bijection from $\mathcal{M}$ to $\mathcal{M}$ which preserves $[\because]$ is called an **automorphism** of $\mathcal{M}$.

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[18] If a point is fixed, we shall picture the point without arrows.

(4.25) Theorem Let $\mathcal{M}$ be a meridian relative to a meridian operator $[:\cdot:]$. Let 0, 1 and ∞ be three distinct points of $\mathcal{M}$ and define $\mathcal{F}$ and operators $+$ and $\cdot$ as in Theorem (4.11). Let ϕ be a field automorphism of $\mathcal{F}$, and let α be the bijection of $\mathcal{M}$ which fixes ∞ and equals ϕ on $\mathcal{F}$. Then α is a meridian automorphism

Proof. For $A, B, C, D, E \in \mathcal{F}$ such that $\{A, E\} \neq \{B, D\}$ we have

$$\begin{pmatrix} AE-BD & ABD+BDE-ABE-ADE \\ A+E-B-D & BD-AE \end{pmatrix}(A) =$$

$$\frac{A^2E-ABD+ABD+BDE-ABE-ADE}{A^2+AE-AB-AD-AE+BD} = \frac{E(A^2+BD-AB-AD)}{A^2+BD-AB-AD} = E$$

and

$$\begin{pmatrix} AE-BD & ABD+BDE-ABE-ADE \\ A+E-B-D & BD-AE \end{pmatrix}(B) = \frac{ABE-B^2D+ABD+BDE-ABE-ADE}{AB+BE-B^2-BD-AE+BD} =$$

$$\frac{D(-B^2+AB+BE-AE)}{-B^2+AB+BE-AE} = D.$$

From (4.11) we know that $\begin{pmatrix} AE-BD & ABD+BDE-ABE-ADE \\ A-E+B-D & BD-AE \end{pmatrix}$ is in or $\Gamma(\mathcal{M})$, whence follows that it is in or $\Pi(\mathcal{M})$ as well. Thus we have

$$\begin{pmatrix} AE-BD & ABD+BDE-ABE-ADE \\ A-E+B-D & BD-AE \end{pmatrix} = \boxed{\begin{smallmatrix} A\leftrightarrow E \\ B\leftrightarrow D \end{smallmatrix}} \qquad (1).$$

Since ϕ is a field automorphism, we evidently have

$$(\forall X \in \mathcal{F}) \quad \alpha \circ \begin{pmatrix} AE-BD & ABD+BDE-ABE-ADE \\ A-E+B-D & BD-AE \end{pmatrix} \circ \alpha^{-1}(X) =$$

$$\alpha\left(\tfrac{(AE-BD)\cdot\alpha^{-1}(X)+ABD+BDE-ABE-ADE}{(A-E+B-D)\cdot\alpha^{-1}(X)+BD-AE}\right) =$$

$$\frac{(\alpha(A)\alpha(E)-\alpha(B)\alpha(D))X+\alpha(A)\alpha(B)\alpha(D)+\alpha(B)\alpha(D)\alpha(E)-\alpha(A)\alpha(B)\alpha(E)-\alpha(A)\alpha(D)\alpha(E)}{(\alpha(A)-\alpha(E)+\alpha(B)-\alpha(D))X+\alpha(B)\alpha(D)-\alpha(A)\alpha(E)} =$$

$$\begin{pmatrix} \alpha(A)\alpha(E)-\alpha(B)\alpha(D) & \alpha(A)\alpha(B)\alpha(D)+\alpha(B)\alpha(D)\alpha(E)-\alpha(A)\alpha(B)\alpha(E)-\alpha(A)\alpha(D)\alpha(E) \\ \alpha(A)-\alpha(E)+\alpha(B)-\alpha(D) & \alpha(B)\alpha(D)-\alpha(A)\alpha(E) \end{pmatrix}(X).$$

Computing as above we see that

$$\alpha \circ \begin{pmatrix} AE-BD & ABD+BDE-ABE-ADE \\ A-E+B-D & BD-AE \end{pmatrix} \circ \alpha^{-1} = \boxed{\begin{smallmatrix} \alpha(A)\leftrightarrow\alpha(E) \\ \alpha(B)\leftrightarrow\alpha(D) \end{smallmatrix}}$$

which implies

$$\alpha \circ \boxed{\begin{smallmatrix} A\leftrightarrow E \\ B\leftrightarrow D \end{smallmatrix}} = \boxed{\begin{smallmatrix} \alpha(A)\leftrightarrow\alpha(E) \\ \alpha(B)\leftrightarrow\alpha(D) \end{smallmatrix}} \circ \alpha.$$

Evaluating both sides of the above at C and rewriting in meridian operator notation, we have

$$\alpha(\begin{bmatrix} A & & B \\ & C & \\ D & & E \end{bmatrix}) = \begin{bmatrix} R & & S \\ & T & \\ U & & V \end{bmatrix}, \quad \text{where} \begin{cases} R=\alpha(A), \\ S=\alpha(B), \\ T=\alpha(C), \\ U=\alpha(D), \\ V=\alpha(E). \end{cases}$$

Thus α is a meridian automorphism. QED

(4.26) Theorem Let $\mathcal{M}$ be a meridian relative to a meridian operator $[:\cdot:]$ and let α be a meridian automorphism. Let 0, 1 and ∞ be three distinct points of $\mathcal{M}$ and define $\mathcal{F}$ and operators $+$ and $\cdot$ as in Theorem (4.11). Let $0' \equiv \alpha(0)$, $1' \equiv \alpha(1)$ and $\infty' \equiv \alpha(\infty)$, and let $\mathcal{F}'$ be the corresponding field with operators $+'$ and $\cdot'$. Then the restriction of α to $\mathcal{F}$ is an isomorphism of fields from $\mathcal{F}$ onto $\mathcal{F}'$.

Proof. For $A, B \in \mathcal{F}$ we shall write R for $\alpha(A)$, S for $\alpha(B)$, T for $\alpha(0)$, U for $\alpha(\infty)$ and V for $\alpha(1)$. We have

$$\alpha(A) + \alpha(B) = \begin{bmatrix} R & U \\ & T \\ U & S \end{bmatrix} = \alpha \circ \begin{bmatrix} A & \infty \\ & 0 \\ \infty & B \end{bmatrix} = \alpha(A + B)$$

and

$$\alpha(A) \cdot \alpha(B) = \begin{bmatrix} S & T \\ & V \\ U & R \end{bmatrix} = \alpha \circ \begin{bmatrix} B & 0 \\ & 1 \\ \infty & A \end{bmatrix} = \alpha(A \cdot B).$$

QED

5. Libras: Part II

(5.1) Definitions and Notation Let $\lfloor\, ,\, ,\, \rfloor$ be a libra operation on a set L. Recall that a subset B of L is balanced if $\lfloor x,y,z\rfloor \in B$ whenever $x,y,z\in B$. If B is a balanced subset of L, then, for all $r,s\in L$

$$\lfloor r,s,B\rfloor \equiv \{\lfloor r,s,b\rfloor : b\in B\}, \quad \lfloor r,B,s\rfloor \equiv \{\lfloor r,b,s\rfloor : b\in B\} \quad \text{and } \lfloor B,r,s\rfloor \equiv \{\lfloor b,r,s\rfloor : b\in B\} \tag{1}$$

are also balanced. We call the sets $\lfloor r,s,B\rfloor$ **left translates of** B and the sets $\lfloor B,r,s\rfloor$ **right translates of** B. We write

$$\|B\| \tag{2}$$

for the family of left translates of B and

$$\overline{\overline{\underline{\underline{B}}}} \tag{3}$$

for the family of right translates of B, respectively. Sets of the form $\lfloor r,B,s\rfloor$ are called **translates of** B, and the family of all such will be denoted by

$$\boxed{\boxed{B}}. \tag{4}$$

The elements of

$$\left\|\overline{\overline{\underline{\underline{B}}}}\right\| \equiv \|B\| \cup \overline{\overline{\underline{\underline{B}}}} \tag{5}$$

will be referred to as **linear translates of** B, and the other elements of $\boxed{\boxed{B}}$ as **skew translates of** B.

(5.2) Theorem Let B be a balanced subset of L. Then

(i) $(\forall\, b\in B) \quad B = \lfloor b,B,b\rfloor$;

(ii) $\|B\| \cup \overline{\overline{\underline{\underline{B}}}} \subset \boxed{\boxed{B}}$;

(iii) a translate of a translate of B is again a translate of B.

Proof. That $\lfloor b,B,b\rfloor \subset B$ for $b\in B$ is trivial. Let $x,b\in B$. Since B is balanced, we have $\lfloor b,x,b\rfloor\in B$. Then

$$x \xlongequal{\text{by (3.2.1)}} \lfloor b,b,x\rfloor \xlongequal{\text{by (3.2.1)}} \lfloor\lfloor b,b,x\rfloor,b,b\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor b,\lfloor b,x,b\rfloor,b\rfloor \in \lfloor b,B,b\rfloor,$$

which implies that $B\subset \lfloor b,B,b\rfloor$. Hence (i) holds.

For $r,s\in L$ and $b\in B$,

$$\lfloor r,s,B\rfloor \xlongequal{\text{by (i)}} \lfloor r,s,\lfloor b,B,b\rfloor\rfloor \xlongequal{\text{by (3.2.2)}} \lfloor\lfloor r,s,b\rfloor,B,b\rfloor$$

and

$$\lfloor B,r,s\rfloor \xlongequal{\text{by (i)}} \lfloor\lfloor b,B,b\rfloor,r,s\rfloor \xlongequal{\text{by (3.2.2)}} \lfloor b,B,\lfloor b,r,s\rfloor\rfloor$$

which implies (ii).

For $r,s,t,u\in L$ we have, for any $b\in B$

$$\lfloor r,\lfloor t,B,u\rfloor,s\rfloor \xlongequal{\text{by (i)}} \lfloor r,\lfloor t,\lfloor b,B,b\rfloor,u\rfloor,s\rfloor \xlongequal{\text{by (3.2.2)}} \lfloor r,\lfloor\lfloor t,b,B\rfloor,b,u\rfloor,s\rfloor \xlongequal{\text{by (3.3.1)}}$$
$$\lfloor\lfloor r,u,b\rfloor,\lfloor t,b,B\rfloor,s\rfloor\lfloor\lfloor r,u,b\rfloor,B,\lfloor b,t,s\rfloor\rfloor,$$

whence (iii). QED

(5.3) Definitions By a **homogeneous aggregate of translates**, or more simply, an **aggregate**, we shall mean a family $\mathcal{T}$ of balanced sets, each one of which is a translate of each other one, and each translate of a member of $\mathcal{T}$ again a member of $\mathcal{T}$. It follows from Theorem (3.2) that the translates of any balanced set comprise an aggregate, and that an aggregate is the family of translates of any one of its members:

$$(\forall\,\mathcal{T} \text{ an aggregate})(\forall\, T\in\mathcal{T}) \quad \boxed{\boxed{T}} = \mathcal{T}. \tag{1}$$

The family of all singletons is evidently a homogeneous aggregate of translates. We shall call it the **point aggregate of** L.

(5.4) Theorem Let $\mathcal{T}$ be a aggregate and let $B,C\in\mathcal{T}$. Then

(i) $C\in\|B\| \iff B\in\|C\|$;

(ii) $C\in \overline{\overline{\underline{\underline{B}}}} \iff B\in \overline{\overline{\underline{\underline{C}}}}$.

Proof. If $C \in \|B\|$ then $C = \lfloor r, s, B \rfloor$ for $r, s \in L$. Then

$$\lfloor s, r, C \rfloor = \lfloor s, r, \lfloor r, s, B \rfloor \rfloor \xlongequal{\text{by (3.2.2)}} \lfloor \lfloor s, r, r \rfloor, s, B \rfloor \xlongequal{\text{by (3.2.1)}} \lfloor s, s, B \rfloor \xlongequal{\text{by (3.2.1)}} B$$

which show that $B \in \|C\|$. The reverse implication follows by interchanging the roles of B and C in the above. Hence (i) holds.

That (ii) holds follows from an analogous argument. QED

(5.5) Definition We say that a balanced set B is **normal**[19] if each right translate of B is also a left translate of B.

(5.6) Lemma We have the following for any libra L:

(i) $(\forall\, a, b, c \in L)(\exists!\, x \in L) \quad a = \lfloor x, b, c \rfloor$;
(ii) $(\forall\, a, b, c \in L)(\exists!\, x \in L) \quad a = \lfloor b, x, c \rfloor$;
(iii) $(\forall\, a, b, c \in L)(\exists!\, x \in L) \quad a = \lfloor b, c, x \rfloor$;
(iv) $(\forall\, B \subset L$ balanced$)(\forall\, b, y \in L : b \in B) \quad \lfloor b, B, y \rfloor = \lfloor B, b, y \rfloor$;
(v) $(\forall\, B \subset L$ balanced$)(\forall\, b, y \in L : b \in B) \quad \lfloor y, b, B \rfloor = \lfloor y, B, b \rfloor$.

Proof. $\stackrel{\text{(i)}}{\Longrightarrow}$: If $x \in L$ is such that $a = \lfloor x, b, c \rfloor$, then

$$\lfloor a, c, b \rfloor = \lfloor \lfloor x, b, c \rfloor, c, b \rfloor \xlongequal{\text{by (3.2.2)}} \lfloor x, b, \lfloor c, c, b \rfloor \rfloor \xlongequal{\text{by (3.2.1)}} x$$

so x is unique. That $x \equiv \lfloor a, c, b \rfloor$ satisfies $a = \lfloor x, b, c \rfloor$ is a direct computation.

$\stackrel{\text{(ii)}}{\Longrightarrow}$: If $x \in L$ is such that $a = \lfloor b, x, c \rfloor$, then

$$\lfloor c, a, b \rfloor = \lfloor c, \lfloor b, x, c \rfloor, b \rfloor \xlongequal{\text{by (3.3.1)}} \lfloor \lfloor c, c, x \rfloor, b, b \rfloor \xlongequal{\text{by (3.2.1)}} = x$$

so x is unique. That $x \equiv \lfloor c, a, b \rfloor$ satisfies $a = \lfloor b, x, c \rfloor$ is a direct computation.

$\stackrel{\text{(iii)}}{\Longrightarrow}$: Follows by an argument analogous to that showing (i).

$\stackrel{\text{(iv)}}{\Longrightarrow}$: For $c \in B$ we have

$$\lfloor b, c, y \rfloor \xlongequal{\text{by (3.2.1)}} \lfloor b, \lfloor b, b, c \rfloor, y \rfloor \xlongequal{\text{by (3.3.1)}} \lfloor \lfloor b, c, b \rfloor, b, y \rfloor \in \lfloor B, b, y \rfloor$$

and

$$\lfloor c, b, y \rfloor \xlongequal{\text{by (3.2.1)}} \lfloor \lfloor b, b, c \rfloor, b, y \rfloor \xlongequal{\text{by (3.3.1)}} \lfloor b, \lfloor b, c, b \rfloor, y \rfloor \in \lfloor b, B, y \rfloor$$

which shows (iv).

$\stackrel{\text{(v)}}{\Longrightarrow}$: The proof is analogous to that of (iv). QED

(5.7) Theorem Let B be a balanced subset of a libra L and let $\mathcal{T}$ be the smallest aggregate containing B. Then the followings statements are pairwise equivalent.

(i) B is normal;
(ii) each left translate of B is a right translate of B;
(iii) $\mathcal{T}$ is a partition of L;
(iv) $\|B\| = \mathcal{T}$;
(v) $\overline{\overline{\underline{\underline{B}}}} = \mathcal{T}$;
(vi) $(\forall\, A \in \mathcal{T}) \quad A$ is normal.

Proof. Suppose that (i) holds. Let $x, y \in L$ and $b \in B$. By (i) there exist $r, s \in L$ such that

$$\lfloor \lfloor b, y, x \rfloor, b, B \rfloor = \lfloor B, r, s \rfloor. \tag{1}$$

By (2..i) we have

$$\begin{aligned}\lfloor b, \lfloor B, x, y \rfloor, b, \rfloor &\xlongequal{\text{by (3.3.1)}} \lfloor \lfloor b, y, x \rfloor, B, b \rfloor \xlongequal{\text{by (5.2.i)}} \lfloor \lfloor b, y, x \rfloor, \lfloor b, B, b \rfloor, b \rfloor \xlongequal{\text{by (3.3.1)}} \\ &\lfloor \lfloor b, y, x \rfloor, b, \lfloor B, b, b \rfloor \rfloor \xlongequal{\text{by (3.2.1)}} \lfloor \lfloor b, y, x \rfloor, b, B \rfloor \xlongequal{\text{by (1)}} \lfloor B, r, s \rfloor\end{aligned} \tag{2}$$

and so

$$\lfloor B, x, y \rfloor \xlongequal{\text{by (3.2.1)}} \lfloor \lfloor b, b, \lfloor B, x, y \rfloor \rfloor, b, b \rfloor \xlongequal{\text{by (3.3.1)}} \lfloor b, \lfloor b, \lfloor B, x, y \rfloor, b \rfloor, b \rfloor \xlongequal{\text{by (2)}}$$

[19] This term is carried over from group theory and chosen here for historical rather than descriptive reasons. A balanced subset is "normal" if it is a level set of a libra homomorphism.

$$\lfloor b, \lfloor B, r, s\rfloor, b\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor\lfloor b, s, r\rfloor, B, b\rfloor \xlongequal{\text{by (5.2.i)}} \lfloor\lfloor b, s, r\rfloor, \lfloor b, B, b\rfloor, b\rfloor \xlongequal{\text{by (3.3.1)}}$$
$$\lfloor\lfloor b, s, r\rfloor, b, \lfloor b, b, B\rfloor\rfloor \xlongequal{\text{by (3.2.1)}} \lfloor\lfloor b, s, r, \rfloor, b, B\rfloor.$$

It follows that (ii) holds.

Suppose that (ii) holds. Assume that (iii) does not hold. Then there exist $A, C \in \mathcal{T}$ such that $A \neq C$ and $A \cap C \neq \emptyset$. In view of (3.3.1) we can choose $u, v \in L$ such that $B = \lfloor u, A, v\rfloor$. Let $D \equiv \lfloor u, C, v\rfloor$. Then $B \neq D$ and $B \cap D \neq \emptyset$. Choose b from $B \cap D$ and choose $d \in D$ such that $d \notin B$. Since $D = \lfloor d, b, B\rfloor$ is a left translate of B, (ii) implies that there exist $r, t \in L$ such that $D = \lfloor B, r, t\rfloor$. Thus

$$d \in D = \lfloor B, r, t\rfloor \xlongequal{\text{by (3.2.1)}} \lfloor B, \lfloor r, b, b\rfloor, t\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor B, b, \lfloor b, r, t\rfloor\rfloor \text{ and } b \in D = \lfloor B, b, \lfloor b, r, t\rfloor\rfloor.$$

By (5.6.i) we can choose $m, n \in B$ such that $d = \lfloor m, b, \lfloor b, r, t\rfloor\rfloor$ and $b = \lfloor n, b, \lfloor b, r, t\rfloor\rfloor$. We have

$$\lfloor b, r, t\rfloor \xlongequal{\text{by (3.2.1)}} \lfloor\lfloor b, n, n\rfloor, \lfloor r, b, b\rfloor, t\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor\lfloor b, n, n\rfloor, b, \lfloor b, r, t\rfloor\rfloor \xlongequal{\text{by (3.2.2)}} \lfloor b, n, \lfloor n, b, \lfloor b, r, t\rfloor\rfloor\rfloor = \lfloor b, n, b\rfloor$$

which yields

$$d = \lfloor m, b, \lfloor b, r, t\rfloor\rfloor = \lfloor m, b, \lfloor b, n, b\rfloor\rfloor \Longrightarrow d \in B.$$

Since this is absurd, it follows that (iii) holds.

Suppose that (iii) holds. Since $\|B\|$ is a sub-family of $\mathcal{T}$ and $\|B\|$ is also a partition, they must be the same partition of L. Hence (iv) holds.

Now suppose that (iv) holds. Let A be in $\mathcal{T}$. Let b be in B. By (iv) there exist $x, y \in L$ such that $\lfloor b, A, b\rfloor = \lfloor x, y, B\rfloor$. We have

$$A \xlongequal{\text{by (3.2.1)}} \lfloor\lfloor b, b, A\rfloor, b, b\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor b, \lfloor b, A, b\rfloor, b\rfloor = \lfloor b, \lfloor x, y, B\rfloor, b\rfloor \xlongequal{\text{by (3.3.1)}}$$
$$\lfloor\lfloor b, B, y\rfloor, x, b\rfloor \xlongequal{\text{by (5.6.iv)}} \lfloor\lfloor B, b, y\rfloor, x, b\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor B, b, \lfloor y, x, b\rfloor\rfloor \xlongequal{\text{by (3.2.1)}}$$
$$\lfloor\lfloor b, b, B\rfloor, b, \lfloor y, x, b\rfloor\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor b, \lfloor b, B, b\rfloor, \lfloor y, x, b\rfloor\rfloor \xlongequal{\text{by (5.2.i) and by (5.6.iv)}} \lfloor B, b, \lfloor y, x, b\rfloor\rfloor.$$

This implies (v).

Now suppose that (v) holds. Let A be in $\mathcal{T}$. Then by (v) A is a right translate of B, whence follows that B is a right translate of A. Suppose we have shown that each right translate of A is left translate of B. Then A itself will be a left translate of B, whence follows that B will be a left translate of A – and so each right translate of A, being a left translate of B, will be also a left translate of B. Thus, to show that A is normal, it will suffice to show that, for each $x, y \in L$, $\lfloor A, x, y\rfloor$ is a left translate of B. To this end we let b be in B and apply (v) to find $r, s \in L$ such that $\lfloor\lfloor b, y, x\rfloor, A, b\rfloor = \lfloor B, r, s\rfloor$. We have by (2.i)

$$\lfloor A, x, y\rfloor \xlongequal{\text{by (3.2.1)}} \lfloor\lfloor b, b, \lfloor A, x, y\rfloor\rfloor, b, b\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor b, \lfloor b, \lfloor A, x, y\rfloor, b\rfloor, b\rfloor \xlongequal{\text{by (3.3.1)}}$$
$$\lfloor b, \lfloor\lfloor b, y, x\rfloor, A, b\rfloor, b\rfloor = \lfloor b, \lfloor B, r, s\rfloor, b\rfloor \xlongequal{\text{by (3.3.1)}} \lfloor\lfloor b, s, r\rfloor, B, b\rfloor \xlongequal{\text{by (5.6.iv)}} \lfloor\lfloor b, s, r\rfloor, b, B\rfloor$$

which implies (vi).

That (vi) implies (i) is trivial. QED

(5.8) Definition We say that a homogeneous aggregate of balanced sets is **normal** provided all of its elements are normal balanced sets. By Theorem (5.7) an aggregate is normal if and only if any one of its elements is normal.

(5.9) Definitions and Notation A libra homomorphism of L into a libra of operators[20] from a set X to another set Y is called a **representation of L on $X \times Y$**.[21]. Here X and Y are referred to as the **representation spaces**. If ϕ is the representation (homomorphism) and x an element of the representation space X, it will be customary herein to write the value of ϕ at a by ϕ_a.

If a representation is injective, we say that it is **faithful**. If for all $x \in X$ and $y \in Y$ there exists $a \in L$ such that $\phi_a(x) = y$, we shall say that the representation is **homogeneous**.

For a representation ϕ of a libra L on $X \times Y$ and $[x, y] \in X \times Y$, we shall use the notation

$$[x \overset{\phi}{=} y] \equiv \{a \in L : \phi_a(x) = y\}. \tag{1}$$

[20] *Cf.* (3.6).

[21] If $X = Y$, the representation is said to be **on X**

and

$$\mathcal{T}_{\phi} \equiv \{[x \overset{\phi}{=} y] : x \in X, y \in Y\}. \tag{2}$$

(5.10) Theorem Let ϕ be a homogeneous representation of a libra L on $X \times Y$. Then $\mathcal{T}_{\phi}$ is an aggregate of balanced sets.

Furthermore, the following statements are pairwise equivalent:
(i) $\mathcal{T}$ is normal;
(ii) $(\forall\, a, b \in L) \quad \phi_a = \phi_b \Longleftrightarrow (\forall\, x \in X)\ \phi_a(x) = \phi_b(x)$;
(iii) $(\forall\, a, b \in L) \quad \phi_a = \phi_b \Longleftrightarrow (\exists\, x \in X)\ \phi_a(x) = \phi_b(x)$.

Proof. Let T be in $\mathcal{T}_{\phi}$. Then there exist $x \in X$ and $y \in Y$ for which $T = [x \overset{\phi}{=} y]$. Let $a, b \in L$. Let $u \in X$ satisfy $\phi_b(u) = y$. Then

$$(\forall\, c \in T) \quad \phi_{\lfloor a,c,b \rfloor}(u) \xlongequal{\text{by (3.6.1)}} \phi_a \circ {\phi_c}^{-1}(\phi_b(u)) = \phi_a(x) \Longrightarrow \lfloor a, T, b \rfloor = [u \overset{\phi}{=} \phi_a(x)]. \tag{1}$$

Now let S be any other element of $\mathcal{T}$, so that there exist $w \in X, z \in Y$ such that $S = [w \overset{\phi}{=} z]$. Choose $d \in L$ such that $\phi_d(w) = y$ and choose $e \in L$ such that $\phi_e(x) = z$. Then, by replacing T in equation (1) with S, replacing a with d, and replacing b with e, we obtain

$$\lfloor d, S, e \rfloor = [x \overset{\phi}{=} y] = T.$$

This proves that $\mathcal{T}$ is an aggregate.

(i)$\Longrightarrow$(ii): Let (i) hold. Let $a, b \in L$ and $x \in X$ and suppose that $\phi_a(x) = \phi_b(x)$. Let t be in X. Since $\mathcal{T}$ is normal, there exist $r, s \in L$ such that $\lfloor r, s, [t \overset{\phi}{=} \phi_a(t)] \rfloor = [x \overset{\phi}{=} \phi_a(x)]$. Since a is in $[x \overset{\phi}{=} \phi_a(x)]$, there exists $c \in [t \overset{\phi}{=} \phi_a(t)]$ such that $a = \lfloor r, s, c \rfloor$. We have

$$\phi_a(t) = \phi_{\lfloor r,s,c \rfloor}(t) = \phi_r \circ {\phi_s}^{-1}(\phi_c(t)) = \phi_r \circ {\phi_s}^{-1}(\phi_a(t)). \tag{2}$$

Since b is in $[x \overset{\phi}{=} \phi_a(x)]$, there exists $d \in [t \overset{\phi}{=} \phi_a(t)]$ such that $b = \lfloor r, s, d \rfloor$. We have

$$\phi_b(t) = \phi_{\lfloor r,s,d \rfloor}(t) = \phi_r \circ {\phi_s}^{-1}(\phi_d(t)) = \phi_r \circ {\phi_s}^{-1}(\phi_a(t)) \xlongequal{\text{by (2)}} \phi_a(t).$$

Thus (ii) holds.

(ii)$\Longrightarrow$(iii): Trivial.

(iii)$\Longrightarrow$(ii): Suppose that (iii) holds, that m is in L and that t is any element of X. Since ϕ is homogeneous, there exists $w \in L$ such that $\phi_w(x) = \phi_m(t)$. If a and b in L satisfy $\phi_a(t) = \phi_b(t)$, then

$$\phi_{\lfloor a,m,w \rfloor}(x) = \phi_a \circ {\phi_m}^{-1}(\phi_w(x)) = \phi_a \circ {\phi_m}^{-1}(\phi_m(t)) = \phi_a(t) =$$
$$\phi_b(t) = \phi_b \circ {\phi_m}^{-1}(\phi_m(t)) = \phi_b \circ {\phi_m}^{-1}(\phi_w(x)) = \phi_{\lfloor b,m,w \rfloor}(x)$$

which by (iii) implies $\phi_{\lfloor a,m,w \rfloor} = \phi_{\lfloor b,m,w \rfloor}$. Thus

$$\phi_a = \llbracket \llbracket \phi_a, \phi_m, \phi_w \rrbracket, \phi_w, \phi_m \rrbracket = \llbracket \phi_{\lfloor a,m,w \rfloor}, \phi_w, \phi_m \rrbracket =$$
$$\llbracket \phi_{\lfloor b,m,w \rfloor}, \phi_w, \phi_m \rrbracket = \llbracket \llbracket \phi_b, \phi_m, \phi_w \rrbracket, \phi_w, \phi_m \rrbracket = \phi_b,$$

which proves (ii).

(ii)$\Longrightarrow$(i): Let B be a generic element of $\mathcal{T}_{\phi}$. Then there exist $x \in X$ and $y \in Y$ such that $B = [x \overset{\phi}{=} y]$. Let R be any right translate of B. Then there exist $q, r \in L$ such that $R = \lfloor B, r, q \rfloor$. Since all the elements of B agree at x, it follows from (ii) that they agree on $\phi_r \circ {\phi_q}^{-1}(x)$ as well — let $u \in Y$ be this common value and let $v \equiv {\phi_r}^{-1}(u)$. Since ϕ is a homogeneous representation, there exists $s \in L$ such that $\phi(s)_v = y$. For all $a \in B$ we have

$$\phi_{\lfloor \lfloor s,r,a \rfloor, r,q \rfloor}(x) = \phi(s) \circ {\phi_r}^{-1} \circ \phi_a \circ {\phi_r}^{-1} \circ \phi_q(x) = \phi(s) \circ {\phi_r}^{-1}(u) = y.$$

It now follows from (ii) that $\lfloor \lfloor s, r, B \rfloor, r, q \rfloor = [x \overset{\phi}{=} y]$, whence follows that $\lfloor \lfloor s, r, B \rfloor, r, q \rfloor = B$. We have
$\lfloor r, s, B \rfloor = \lfloor r, s, \lfloor \lfloor s, r, B \rfloor, r, q \rfloor \rfloor = \lfloor \lfloor r, s, \lfloor s, r, B \rfloor \rfloor, r, q \rfloor = \lfloor \lfloor \lfloor r, s, s \rfloor, r, B \rfloor, r, q \rfloor = \lfloor \lfloor r, r, B \rfloor, r, q \rfloor = \lfloor B, r, q \rfloor$.

It follows that B is normal. QED

(5.11) Definition We shall say that a homogeneous representation is **normal** if any of the conditions of Theorem (5.10) hold.

(5.12) Theorem Let ϕ be a normal homogeneous representation of a libra L on $X \times Y$. Let $r, s, t \in Y$. Suppose $x, m \in X$ and $a, b, c, u, v, w \in L$ satisfy

$$r = \phi_a(x) = \phi_u(m), \ s = \phi_b(x) = \phi_v(m), \text{ and } t = \phi_c(x) = \phi_w(m). \tag{1}$$

Then

$$\phi_{\lfloor a,b,c \rfloor}(x) = \phi_{\lfloor u,v,w \rfloor}(m). \tag{2}$$

Proof. Let d be any element of L and choose $e \in L$ such that $\phi_e(x) = \phi_d(m)$. Evidently

$$\phi_{\lfloor a,e,d \rfloor}(m) = r = \phi_u(m), \ \phi_{\lfloor b,e,d \rfloor}(m) = s = \phi_v(m), \text{ and } \phi_{\lfloor c,e,d \rfloor}(m) = t = \phi_w(m).$$

Since ϕ is normal, it follows from (5.10.ii) that $\phi_{\lfloor a,e,d \rfloor} = \phi_u$, $\phi_{\lfloor b,e,d \rfloor} = \phi_v$, and $\phi_{\lfloor c,e,d \rfloor} = \phi_w$. Thus

$$\phi_{\lfloor u,v,w \rfloor}(m) = \phi_{\lfloor \lfloor a,e,d \rfloor, \lfloor b,e,d \rfloor, \lfloor c,e,d \rfloor \rfloor}(m) =$$

$$\phi_{\lfloor a,e,d,d,e,b,c,e,d \rfloor}(m) = \phi_{\lfloor a,b,c,e,d \rfloor}(m) = \phi_{\lfloor a,b,c \rfloor}(x)$$

which proves equation (2). QED

(5.13) Notation Let ϕ be a normal homogeneous representation of a libra L on $X \times Y$. For $r, s, t \in Y$ we define

$$\lfloor r, s, t \rfloor_{\phi} \equiv \phi_{\lfloor a,b,c \rfloor}(x) \quad (\forall\, x \in X, \ a, b, c \in L{:}\, r = \phi_a(x), \ s = \phi_b(x), \text{ and } t = \phi_c(x)).$$

In view of Theorem (5.12), $\lfloor , , \rfloor_{\phi}$ is a well-defined libra operation on Y.

(5.14) Theorem Let ϕ be a faithful normal homogeneous representation of L on $X \times Y$. Then, for each $[x, y] \in X \times Y$, the set $[x \overset{\phi}{=} y]$ is a singleton. In particular

$$(\forall\, x \in X) \quad L \ni a \hookrightarrow \phi_a(x) \in Y \text{ is a bijection.} \tag{1}$$

Proof. Let x be in X and y in Y. Assume that there exist distinct elements a and b of $[x \overset{\phi}{=} y]$. Since ϕ is faithful there exists $w \in X$ such that $\phi_a(w) \neq \phi_b(w)$. Since a is in $[x \overset{\phi}{=} y] \cap [w \overset{\phi}{=} \phi_a(w)]$, it follows from (5.7.iii) that $[x \overset{\phi}{=} y] = [w \overset{\phi}{=} \phi_a(w)]$. Hence b is in $[w \overset{\phi}{=} \phi_a(w)]$, which is absurd. QED

(5.15) Definitions, Notation, and Discussion . In the sequel we shall be much concerned with representations which are not normal, and will treat these specifically in the following section. For this we shall need some definitions.

Let $\lfloor , , \rfloor$ be a libra operator for a libra L. The **obverse** of $\lfloor , , \rfloor$ is the trinary operator defined by

$$(\forall\, a, b, c \in L) \quad \lceil a, b, c \rceil \equiv \lfloor c, b, a \rfloor. \tag{1}$$

If ρ is a representation of L on $(X \times Y)$, then the **obverse of** ρ is the representation of L on $Y \times X$ defined by

$$(\forall\, x \in L) \quad \tilde{\rho}_x \equiv {\rho_x}^{-1}. \tag{2}$$

The obverse representation is a representation of L *relative to the obverse operator* $\lceil , , \rceil$ – not relative to the libra operator $\lfloor , , \rfloor$[22]

The **symmetrization of** L is the set $L \times L$ equipped with the **symmetrization operator** $\models , , \dashv$:

$$(\forall\, [a, z], [b, y], [c, x] \in L \times L) \quad \models [a, z], [b, y], [c, x] \dashv \equiv [\lfloor a, b, c \rfloor, \lceil z, y, x \rceil] = [\lfloor a, b, c \rfloor, \lfloor x, y, z \rfloor]. \tag{3}$$

The **symmetrization of the representation** ρ is the representation $\overset{\leftrightarrow}{\rho}$ of the libra $L \times L$ on $(X \times Y) \times (X \times Y)$ defined by

$$(\forall\, x \in X, y \in Y)(\forall\, [a, b] \in L \times L) \quad \overset{\leftrightarrow}{\rho}_{[a,b]}([x, y]) \equiv [\tilde{\rho}_b(y), \rho_a(x)]. \tag{4}$$

[22] Unless they are the same of course, which is the case when L is abelian.

Two representations ϕ on $(X \times Y)$ and η on $(M \times N)$ are said to be **equivalent** if there exist bijections μ from X to M and ν from Y to N such that

$$(\forall\, a \in L) \quad \eta_a = \nu \circ \phi_a \circ \mu^{-1} \quad : vid. \qquad \begin{array}{ccc} X & \xrightarrow{\phi_a} & Y \\ {\scriptstyle\mu}\big\downarrow & a \in L & \big\downarrow{\scriptstyle\nu} \\ M & \xrightarrow{\eta_a} & N \end{array} .$$

6. Cartesian Aggregates

(6.1) Definitions and Notation Let $\mathcal{T}$ be an aggregate[23] of balanced subsets of a libra L. By a **row of $\mathcal{T}$** we shall mean a sub-family $\mathcal{R}$ of $\mathcal{T}$ such that each member of $\mathcal{R}$ is a right translate of each other member of $\mathcal{R}$, and such that each right translate of a member of $\mathcal{R}$ is again a member of $\mathcal{R}$. By a **column of $\mathcal{T}$** we shall mean a sub-family $\mathcal{C}$ of $\mathcal{T}$ such that each member of $\mathcal{C}$ is a left translate of each other member of $\mathcal{C}$, and such that each left translate of a member of $\mathcal{C}$ is again a member of $\mathcal{C}$. Each row is a partition of L, and each column is a partition of L.

We shall write $\boxed{\equiv}$ for the family of all rows in $\mathcal{T}$, and write $\boxed{|||}$ for the family of all columns in $\mathcal{T}$.

(6.2) Theorem Let $a, b \in L$, $X \in \mathcal{X} \in \boxed{|||}$ and $Y \in \mathcal{Y} \in \boxed{\equiv}$. Then

(i) $\{\lfloor a, W, b\rfloor : W \in \mathcal{X}\} = \overline{\overline{\underline{\underline{\lfloor a, X, b\rfloor}}}}$;[24]

(ii) $\{\lfloor a, W, b\rfloor : W \in \mathcal{Y}\} = \|\lfloor a, Y, b\rfloor\|$.

Proof. For $r, s \in L$ holds

$$\lfloor a, \lfloor r, s, X\rfloor, b\rfloor = \lfloor a, X, \lfloor s, r, b\rfloor\rfloor = \lfloor a, X, b, b, s, r, b\rfloor = \lfloor \lfloor a, X, b\rfloor, b, \lfloor s, r, b\rfloor\rfloor$$

whence follows that $\{\lfloor a, W, b\rfloor : W \in \mathcal{X}\} \subset \overline{\overline{\underline{\underline{\lfloor a, X, b\rfloor}}}}$. For $t, u \in L$ holds

$$\lfloor \lfloor a, X, b\rfloor, t, u\rfloor = \lfloor a, X, b, t, u, b, b\rfloor = \lfloor a, \lfloor \lfloor b, u, t\rfloor, b, X\rfloor, b\rfloor$$

whence follows that $\overline{\overline{\underline{\underline{\lfloor a, X, b\rfloor}}}} \subset \{\lfloor a, W, b\rfloor : W \in \mathcal{X}\}$. It follows that (i) holds.

The proof of (ii) is analogous to that of (i). QED

(6.3) Notation Elements a and b of L produce bijections as follows:

$$a \text{\textcircled{\tiny $\mathcal{T}$}} b | \mathcal{T} \ni B \hookrightarrow \lfloor a, B, b\rfloor \in \mathcal{T} \tag{1}$$

$$a \hat{\text{\textcircled{\tiny $\mathcal{T}$}}} b | \boxed{|||} \ni \mathcal{X} \hookrightarrow \{\lfloor a, X, b\rfloor : X \in \mathcal{X}\} \in \boxed{\equiv}, \tag{2}$$

and

$$a \check{\text{\textcircled{\tiny $\mathcal{T}$}}} b | \boxed{\equiv} \ni \mathcal{Y} \hookrightarrow \{\lfloor a, X, b\rfloor : X \in \mathcal{Y}\} \in \boxed{|||}. \tag{3}$$

(6.4) Theorem For each $b \in L$ and all $r, s, t \in L$

$$\lfloor r, s, t\rfloor \text{\textcircled{\tiny $\mathcal{T}$}} b = \llbracket r \text{\textcircled{\tiny $\mathcal{T}$}} b, s \text{\textcircled{\tiny $\mathcal{T}$}} b, t \text{\textcircled{\tiny $\mathcal{T}$}} b \rrbracket.$$

Proof. . For $S \in \mathcal{T}$

$$(r \text{\textcircled{\tiny $\mathcal{T}$}} b) \circ (s \text{\textcircled{\tiny $\mathcal{T}$}} b)^{-1} \circ (t \text{\textcircled{\tiny $\mathcal{T}$}} b)(S) = (r \text{\textcircled{\tiny $\mathcal{T}$}} b) \circ (s \text{\textcircled{\tiny $\mathcal{T}$}} b)^{-1}(\lfloor t, S, b\rfloor) = r \text{\textcircled{\tiny $\mathcal{T}$}} b(\lfloor b, \lfloor t, S, b\rfloor, s\rfloor) =$$
$$\lfloor r, \lfloor b, \lfloor t, S, b\rfloor, s\rfloor, b\rfloor = \lfloor r, \lfloor b, b, S, t, s\rfloor, b\rfloor = \lfloor r, s, t, S, b, b, b\rfloor = \lfloor \lfloor r, s, t\rfloor, S, b\rfloor = (\lfloor r, s, t\rfloor \text{\textcircled{\tiny $\mathcal{T}$}} b)(S).$$

QED

(6.5) Notation We define for a in L

$$a^{\text{\textcircled{\tiny $\mathcal{T}$}}} \equiv a \text{\textcircled{\tiny $\mathcal{T}$}} a \quad \text{and} \quad a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}} \equiv a \hat{\text{\textcircled{\tiny $\mathcal{T}$}}} a. \tag{1}$$

(6.6) Theorem For $a, b, c \in L$

$$\lfloor a, b, c\rfloor^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}} = \llbracket a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}}, b^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}}, c^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}} \rrbracket.$$

Proof. For $X \in \mathcal{T}$

$$a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}} \circ b^{\check{\text{\textcircled{\tiny $\mathcal{T}$}}}} \circ c^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}}(\|X\|) = a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}} \circ b^{\check{\text{\textcircled{\tiny $\mathcal{T}$}}}} \circ c^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}}(\|\lfloor\lfloor c, b, a\rfloor, \lfloor a, b, c\rfloor, X\rfloor\|) = a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}} \circ b^{\check{\text{\textcircled{\tiny $\mathcal{T}$}}}} \circ c^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}}(\|\lfloor c, b, a, c, b, a, X\rfloor\|) =$$

$$a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}} \circ b^{\check{\text{\textcircled{\tiny $\mathcal{T}$}}}}(\overline{\overline{\underline{\underline{\lfloor c, X, a, b, c, a, b, c, c\rfloor}}}}) = a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}} \circ b^{\check{\text{\textcircled{\tiny $\mathcal{T}$}}}}(\overline{\overline{\underline{\underline{\lfloor c, X, a, b, c, a, b\rfloor}}}}) = a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}}(\|\lfloor b, b, a, c, b, a, X, c, b\rfloor\|) =$$

$$a^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}}(\|\lfloor a, c, b, a, X, c, b\rfloor\| = \overline{\overline{\underline{\underline{\lfloor a, b, c, X, a, b, c, a, a\rfloor}}}} = \overline{\overline{\underline{\underline{\lfloor \lfloor a, b, c\rfloor, X, \lfloor a, b, c\rfloor\rfloor}}}} = \lfloor a, b, c\rfloor^{\hat{\text{\textcircled{\tiny $\mathcal{T}$}}}}(\|X\|).)$$

[23] *Cf.* (5.3).

[24] *Cf.* (5.1).

QED

(6.7) Discussion and Notation The analogue of Theorem 6.6 for $\textcircled{\scriptsize T}$ does not hold, and in fact, a necessary and sufficient condition for $\lfloor a,b,c\rfloor^{\textcircled{\scriptsize T}}$ to equal $a^{\textcircled{\scriptsize T}} \circ {b^{\textcircled{\scriptsize T}}}^{-1} \circ c^{\textcircled{\scriptsize T}}$ is for $\lfloor a,b,c\rfloor$ to equal $\lfloor c,b,a\rfloor$. It follows that, in general, $\{a^{\textcircled{\scriptsize T}} \colon a \in L\}$ may not be a balanced subset of the libra $\mathfrak{F}(\mathcal{T},\mathcal{T})$. It is a simple exercise to show that

$$(\forall\, a,b,m,n,r,s \in L) \quad (a \textcircled{\scriptsize T} b) \circ (m \textcircled{\scriptsize T} n)^{-1} \circ (r \textcircled{\scriptsize T} s) = \lfloor a,m,r\rfloor \textcircled{\scriptsize T} \lfloor b,n,s\rfloor. \tag{1}$$

It follows that $\{x \textcircled{\scriptsize T} y \colon x,y \in \mathrm{L}\}$ is a balanced subset. Since it contains $\{a^{\textcircled{\scriptsize T}} \colon a \in L\}$, one may ask if it itself is the smallest balanced set containing it. We shall return to this question in (9) *infra*.

For $a,b,m,n \in L$ we shall adopt the notation $\llcorner a, m \textcircled{\scriptsize T} n, b \lrcorner$ for $(a \textcircled{\scriptsize T} b) \circ (n \textcircled{\scriptsize T} m)$:

$$\llcorner a, m \textcircled{\scriptsize T} n, b \lrcorner | \mathcal{T} \ni A \hookrightarrow \{\lfloor a,m,x,n,b\rfloor \colon x \in A\} \in \mathcal{T}. \tag{2}$$

For $a,b,c,d,r,s,t,u \in L$ and $A \in \mathcal{T}$, the computation

$$\llcorner a, b \textcircled{\scriptsize T} c, d \lrcorner \circ \llcorner r, s \textcircled{\scriptsize T} t, u \lrcorner (A) = \llcorner a, b \textcircled{\scriptsize T} c, d \lrcorner (\lfloor r,s,A,t,u\rfloor) = \lfloor a,b,r,s,A,t,u,c,d\rfloor \tag{3}$$

shows that

$$\llcorner a, b \textcircled{\scriptsize T} c, d \lrcorner \circ \llcorner r, s \textcircled{\scriptsize T} t, u \lrcorner = \llcorner \lfloor a,b,r\rfloor, t \textcircled{\scriptsize T} s, \lfloor u,c,d\rfloor \lrcorner = \llcorner a, \lfloor s,r,b\rfloor \textcircled{\scriptsize T} \lfloor c,u,t\rfloor, d \lrcorner.$$

We shall write

$$\mathfrak{Libra}(\mathcal{T}) \equiv \{a \textcircled{\scriptsize T} b \colon a,b \in L\} \quad \text{and} \quad \mathfrak{Group}(\mathcal{T}) \equiv \{\llcorner a, b \textcircled{\scriptsize T} c, d \lrcorner \colon a,b,c,d \in L\}. \tag{4}$$

Theorem (6.6) implies that $\mathfrak{Libra}(\mathcal{T})$ is a libra. Furthermore, $\mathfrak{Group}(\mathcal{T}))$ is a group since equation (2) implies that $\llcorner a, a \textcircled{\scriptsize T} a, a \lrcorner$ is an identity for each $a \in L$ and equation (3) shows that, for all $a,b,c,d \in L$,

$$\llcorner a, b \textcircled{\scriptsize T} c, d \lrcorner^{-1} = \llcorner b, a \textcircled{\scriptsize T} d, c \lrcorner. \tag{5}$$

(6.8) Definitions Let ϕ be a faithful representation of a libra L on $X \times Y$. If

$$(\forall\, x, r \in X \text{ distinct})(\forall\, y, s \in Y \text{ distinct})(\exists\, a \in L) \quad \phi_a(x) = y \text{ and } \phi_a(r) \neq s, \tag{1}$$

we shall say that ϕ is **cartesian**. In particular, a cartesian representation is homogeneous.

Theorem (6.4) says that, for each $b \in L$ the function $L \ni a \hookrightarrow a \textcircled{\scriptsize T} b \in \mathfrak{I}(\mathcal{T},\mathcal{T})$ is a representation of L on $\mathcal{T}$. Theorem (6.6) says the the function sending each $a \in L$ to $a^{\widehat{\textcircled{\scriptsize T}}}$ is a representation of L on $(\text{▥}, \text{▤})$. This latter will be called the **left $\mathcal{T}$-inner representation of L**.

(6.9) Theorem Let ρ be a cartesian representation of a libra L on $X \times Y$. Define[24]

$$\mu | X \ni x \hookrightarrow \{[x \overset{\rho}{=} y] \colon y \in Y\} \text{ and } \nu | Y \ni y \hookrightarrow \{[x \overset{\rho}{=} y] \colon x \in X\}.$$

Then ρ is equivalent to the $\mathcal{T}$-inner representation $\widehat{\textcircled{\scriptsize T}}$:

$$(\forall\, a \in L) \quad a^{\widehat{\textcircled{\scriptsize T}}} = \nu \circ \rho_a \circ \mu^{-1} \quad : vid. \qquad \begin{tikzcd} X \arrow[r, "\rho_a"] \arrow[d, "\mu"'] \arrow[dr, phantom, "a \in L"] & Y \arrow[d, "\nu"] \\ \text{▥}(\mathcal{T}_\rho) \arrow[r, "a^{\widehat{\textcircled{\scriptsize T}}}"] & \text{▤}(\mathcal{T}_\rho) \end{tikzcd} . \tag{1}$$

Proof. We must show that $\mu(x)$ is an element of $\text{▥}(\mathcal{T}_\rho)$ for each $x \in X$. Let y be in Y and $a,b \in L$. Then, letting $k \equiv \rho_a \circ {\rho_b}^{-1}(y)$,

$$\lfloor a,b,[x \overset{\rho}{=} y]\rfloor = \{\lfloor a,b,t\rfloor \colon \rho_t(x) = y\} = \{\rho_a) \circ {\rho_b}^{-1} \circ \rho_t \colon \rho_t(x) = y\} = [x \overset{\rho}{=} k].$$

Let s be another element of Y. Let u be in $[x \overset{\rho}{=} y]$ and, exploiting the fact that ρ is cartesian, find $v \in L$ such that $\rho_v(x) = s$. Since $\rho_u(x) = y$, we have, for all $t \in L$ such that $\rho_t(x) = y$

$$\rho_v \circ {\rho_u}^{-1} \circ \rho_t(x) = \rho_v(x) = s$$

whence follows that $\lfloor v,u,[x \overset{\rho}{=} y]\rfloor = [x \overset{\rho}{=} s]$. Consequently $\mu(x)$ is in $\text{▥}(\mathcal{T}_\rho)$.

That each $\nu(y)$ is an element of $\text{▤}(\mathcal{T}_\rho)$ is shown by an analogous argument. QED

(6.10) Theorem Let ρ be as in Theorem (6.9). Then

[24] *Cf.* (5.8).

(i) $(\forall\, a, b \in L \text{ distinct})(\exists\, T \in \mathcal{T}_\rho) \quad a \in T \text{ and } b \notin T$;
(ii) $(\forall\, T \in \mathcal{T}_\rho)(\forall\, a, b \in L) \quad \lfloor a, T, b \rfloor = T \iff a, b \in T$.

Proof. We first prove (i). Since ρ is faithful, there exists $x \in X$ such $\rho_a(x) \neq \rho_b(x)$. Setting $y \equiv \rho_a(x)$, we have $a \in [x \overset{\rho}{=} y]$ but $b \notin [x \overset{\rho}{=} y]$.

We now prove (ii). If $a, b \in T$, then it is trivial that $\lfloor a, T, b \rfloor = T$. We show that the reverse implication holds. Suppose that $\lfloor r, T, s \rfloor = T$ for $T \in \mathcal{T}_\rho$ and some $r, s \in L$. Then $\lfloor r, t, s \rfloor = u$ for $t, u \in T$, and so $r = \lfloor u, s, t \rfloor$, whence follows that r must be in T if s is in T. Similarly, s must be in T if r is in T. Thus we may presume that neither r nor s is in T. Choose $[x, y] \in X \times Y$ such that $T = [x \overset{\rho}{=} y]$. Let $m \equiv \rho_r(x)$ and $n \equiv {\rho_s}^{-1}(y)$. Because ρ is cartesian, there exists $t \in L$ such that $\rho_t(x) = y$ and $\rho_t(n) \neq m$. Then t is in T and so in $\lfloor r, T, s \rfloor$ as well. Thus $t = \lfloor r, w, s \rfloor$ for $w \in T$, and so $w = \lfloor s, t, r \rfloor$. Consequently

$$y = \rho_w(x) = \rho_s \circ {\rho_t}^{-1} \circ \rho_r(x) = \rho_s \circ {\rho_t}^{-1}(m) \neq \rho_s(n) = y$$

which is absurd. This establishes (ii). QED

(6.11) Definition We shall say that an aggregate $\mathcal{T}$ is **cartesian** if both the conditions of Theorem (6.10) are satisfied:

$$(\forall\, a, b \in L \text{ distinct})(\exists\, T \in \mathcal{T}) \quad a \in T \text{ and } b \notin T \tag{1}$$

$$(\forall\, T \in \mathcal{T})(\forall\, a, b \in L) \quad \lfloor a, T, b \rfloor = T \iff a, b \in T \tag{2}$$

(6.12) Theorem Let $\mathcal{T}$ be a cartesian aggregate on a libra L. Then

$$(\forall\, a, b, c, d \in L) \quad [a, b] = [c, d] \iff a \circledcirc_{\mathcal{T}} b = c \circledcirc_{\mathcal{T}} d. \tag{1}$$

Proof. We have

$$a \circledcirc_{\mathcal{T}} b = c \circledcirc_{\mathcal{T}} d \iff (\forall\, X \in \mathcal{T}) \quad \lfloor a, X, b \rfloor = \lfloor c, X, d \rfloor \iff$$

$$(\forall\, X \in \mathcal{T}) \quad X = \lfloor b, b, X, a, a \rfloor = \lfloor b, \lfloor a, X, b \rfloor, a \rfloor = \lfloor b, \lfloor c, X, d \rfloor, a \rfloor = \lfloor b, d, X, c, a \rfloor \xLeftrightarrow{\text{by (5.2.i)}}$$

$$(\forall\, X \in \mathcal{T}, x \in X) \quad X = \lfloor b, d, x, X, x, c, a \rfloor \xLeftrightarrow{\text{by (6.10.ii)}} (\forall\, X \in \mathcal{T}, x \in X) \quad \lfloor b, d, x \rfloor, \lfloor x, c, a \rfloor \in X. \tag{2}$$

If $a \neq c$, then by (5.2.i) implies that there exists $C \in \mathcal{T}$ such that $c \in C$ and $a \notin C$. If $a \circledcirc_{\mathcal{T}} b = c \circledcirc_{\mathcal{T}} d$, then (2) implies that there exists $y \in C$ such that

$$\lfloor c, c, a \rfloor = y \implies a = y \in C : \text{an absurdity.}$$

It follows that $a \circledcirc_{\mathcal{T}} b \neq c \circledcirc_{\mathcal{T}} d$. An analogous argument shows that if $b \neq c$, then $a \circledcirc_{\mathcal{T}} b \neq c \circledcirc_{\mathcal{T}} d$. QED

(6.13) Theorem The following are equivalent assertions for an aggregate $\mathcal{T}$ of balanced subsets of a libra L:

(i) $(\forall\, B \in \mathcal{T} \text{ and } x, y \in L) \quad [x, B, y] = B \iff x, y \in B$;
(ii) $(\exists\, B \in \mathcal{T}) \quad (\forall\, x, y \in L)\ \lfloor x, B, y \rfloor = B \iff x, y \in B$;
(iii) $(\exists\, B \in \mathcal{T}) \quad \| B \| \cap \overline{\overline{\underline{\underline{B}}}} = \{B\}$;
(iv) $(\forall\, B \in \mathcal{T}) \quad \| B \| \cap \overline{\overline{\underline{\underline{B}}}} = \{B\}$.

Proof. That (i) implies (ii) is trivial.

Suppose that (ii) holds for $B \in \mathcal{T}$. Suppose that $\lfloor r, s, B \rfloor = \lfloor B, t, u \rfloor$ for $r, s, t, u \in L$. Let $b \in B$. Then, letting $x \equiv \lfloor r, s, b \rfloor$ and $y \equiv \lfloor b, t, u \rfloor$, we have

$$\lfloor x, b, B \rfloor = \lfloor r, s, b, b, B \rfloor = \lfloor r, s, B \rfloor = \lfloor B, t, u \rfloor = \lfloor B, b, y \rfloor.$$

Thus

$$B = \lfloor b, x, x, b, B \rfloor = \lfloor b, x, \lfloor x, b, B \rfloor \rfloor = \lfloor b, x, \lfloor B, b, y \rfloor \rfloor \xlongequal{\text{by (5.2.i)}} \lfloor b, x, \lfloor \lfloor b, B, b \rfloor, b, y \rfloor \rfloor =$$

$$\lfloor b, x, b, B, b, b, y \rfloor = \lfloor \lfloor b, x, b \rfloor, B, y \rfloor.$$

By (ii) we have $y, \lfloor b, x, b \rfloor \in B$. Consequently $\lfloor r, s, B \rfloor = \lfloor x, b, B \rfloor = B$, and so $\| B \| \cup \overline{\overline{\underline{\underline{B}}}} = \{B\}$. Hence (iii) holds.

Now suppose that (iii) holds for $B\in\mathcal{T}$ and that C is any other element of $\mathcal{T}$. Then there exist $x,y\in L$ such that $C=\lfloor x,B,y\rfloor$. Suppose that $\lfloor r,s,C\rfloor=\lfloor C,t,u\rfloor$ for $r,s,t,u\in L$. We have

$$\lfloor B,\lfloor r,s,x\rfloor,x\rfloor = \lfloor y,y,B,x,s,r,x\rfloor = \lfloor y,\lfloor r,s,\lfloor x,B,y\rfloor\rfloor,x\rfloor = \lfloor y,\lfloor r,s,C\rfloor,x\rfloor =$$

$$\lfloor y,\lfloor C,u,t\rfloor,x\rfloor = \lfloor y,\lfloor\lfloor x,B,y\rfloor,u,t\rfloor,x\rfloor = \lfloor y,t,u,y,B,x,x\rfloor = \lfloor y,\lfloor y,u,t\rfloor,B\rfloor.$$

It follows from (iii) that $\lfloor B,\lfloor r,s,x\rfloor,x\rfloor=B$. Thus

$$C=\lfloor x,B,y\rfloor = \lfloor x,\lfloor B,\lfloor r,s,x\rfloor,x\rfloor,y\rfloor = \lfloor x,x,r,s,x,B,y = \lfloor r,s,\lfloor x,B,y\rfloor\rfloor = \lfloor r,s,C\rfloor.$$

This implies (iv).

Finally, we suppose that (iv) holds and let B be any element of $\mathcal{T}$. Suppose that $B=\lfloor x,B,y\rfloor$ for $x,y\in L$. For $b\in B$ we have

$$\lfloor b,x,B\rfloor = \lfloor b,x,\lfloor x,B,y\rfloor\rfloor = \lfloor\lfloor b,x,x\rfloor,B,y\rfloor\rfloor \xlongequal{\text{by (5.2.i)}} \lfloor\lfloor b,x,x\rfloor,\lfloor b,B,b\rfloor,y\rfloor = \lfloor b,x,x,b,B,b,y\rfloor = \lfloor B,b,y\rfloor.$$

From (iv) follows that $\lfloor b,x,B\rfloor=B$. Hence $\lfloor b,x,b\rfloor=d$ for some $d\in B$, whence $x=\lfloor b,d,b\rfloor\in B$. It follows that $B=\lfloor b,x,B\rfloor=\lfloor B,b,y\rfloor$ which implies that $y=\lfloor b,b,y\rfloor\in\lfloor B,b,y\rfloor=B$. This means that (i) holds. QED

(6.14) Notation If two sets R and S have a singleton for their intersection $R\cap S$, we shall denote the element of the singleton by $R\wedge S$:

$$R\cap S=\{R\wedge S\}. \tag{1}$$

(6.15) Theorem Let $\mathcal{T}$ be a cartesian aggregate on the libra L. Let $\mathcal{A},\mathcal{B}\in ⦀$ and $\mathcal{C},\mathcal{D}\in\equiv$. Then

$$\mathcal{A}\wedge\mathcal{C}\text{ exists} \tag{1}$$

and

$$\mathcal{A}\wedge\mathcal{C}=\mathcal{B}\wedge\mathcal{D}\iff\lceil\mathcal{A},\mathcal{C}\rceil=\lceil\mathcal{B},\mathcal{D}\rceil. \tag{2}$$

Proof. Let A be in $\mathcal{A}$ and c in $\mathcal{C}$. Since $\mathcal{T}$ is an aggregate of balanced sets, there exists $x\in L$ such that $\lfloor x,A,x\rfloor=C$. For any $a\in A$ we have

$$C=\lfloor x,A,x\rfloor \xlongequal{\text{by (5.2.i)}} \lfloor x,\lfloor a,A,a\rfloor,x\rfloor = \lfloor\lfloor x,a,A\rfloor,a,x\rfloor \Longrightarrow \lfloor C,x,a\rfloor=\lfloor x,a,A\rfloor\Longrightarrow\mathcal{A}\cap\mathcal{C}\neq\emptyset.$$

That $\mathcal{A}\cap\mathcal{C}$ is a singleton follows from (6.13.iv). This establishes (1).

$\xRightarrow{(2)}$: Let $A\equiv\mathcal{A}\wedge\mathcal{C}=\mathcal{B}\wedge\mathcal{D}$. Then A is in $\mathcal{A}$ and $\mathcal{B}$ so $\Vert A\Vert=\mathcal{A}=\mathcal{B}$. Similarly, $\overline{\overline{\underline{\underline{A}}}}=\mathcal{C}=\mathcal{D}$. Thus $\lceil\mathcal{A},\mathcal{C}\rceil=\lceil\mathcal{B},\mathcal{D}\rceil$.

$\xLeftarrow{(2)}$: Trivial. QED

(6.16) Theorem Let $\mathcal{T}$ be a cartesian aggregate on the libra L. Then the $\mathcal{T}$-inner representation is cartesian.

Proof. Let $x,y\in L$ be distinct. Choose $B\in\mathcal{T}$ such that $x\in B$ and $y\notin B$. Then $x^{\hat{\mathcal{T}}}(\Vert B\Vert)=\overline{\overline{\underline{\underline{\lfloor x,B,x\rfloor}}}}=\overline{\overline{\underline{\underline{B}}}}$ but, for $b\in B$,

$$y^{\hat{\mathcal{T}}}(\Vert B\Vert)=y^{\hat{\mathcal{T}}}(\Vert\lfloor y,b,B\rfloor\Vert)=\overline{\overline{\underline{\underline{\lfloor y,\lfloor y,b,B\rfloor,y\rfloor}}}}=\overline{\overline{\underline{\underline{\lfloor\lfloor y,B,b\rfloor,y,y\rfloor}}}}=\overline{\overline{\underline{\underline{\lfloor y,B,b\rfloor}}}}=\overline{\overline{\underline{\underline{\lfloor y,b,B\rfloor}}}}.$$

Since y is in $\lfloor y,b,B\rfloor$ but not in B, we know that $\lfloor y,b,B\rfloor\neq B$. It follows from Theorem (6.13.iv) that $\overline{\overline{\underline{\underline{B}}}}\neq\overline{\overline{\underline{\underline{\lfloor y,b,B\rfloor}}}}$. Thus $x^{\hat{\mathcal{T}}}\neq y^{\hat{\mathcal{T}}}$. It follows that the representation is faithful.

Let $\mathcal{A}$ and $\mathcal{B}$ be distinct elements of $\Vert\mathcal{T}\Vert$ and $\mathcal{C}$ and $\mathcal{D}$ distinct elements of $\overline{\overline{\underline{\underline{\mathcal{T}}}}}$. Choose a from $\mathcal{A}\wedge\mathcal{C}$ such that it is not in $\mathcal{B}\wedge\mathcal{D}$. Then $a^{\hat{\mathcal{T}}}(\mathcal{A})=\mathcal{C}$ but $a^{\hat{\mathcal{T}}}(\mathcal{B})\neq\mathcal{D}$. QED

(6.17) Discussion Theorems (6.9) and (6.16) imply that the cartesian aggregates of a libra L correspond exactly to the equivalence classes of cartesian representations of L. Along with the diagram of

(6.9.1), we have its obverse[25]:

$$(\forall\, a \in L) \quad \begin{tikzcd} X \arrow[r, "\rho_a"] \arrow[d, "\mu"'] & Y \arrow[d, "\nu"] \\ ||| (\mathcal{T}_\rho) \arrow[r, "a^{\widehat{\textcircled{T}}}"] & \equiv (\mathcal{T}_\rho) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[d, "\mu"'] & Y \arrow[l, "\tilde{\rho}_a"'] \arrow[d, "\nu"] \\ ||| (\mathcal{T}_\rho) & \equiv (\mathcal{T}_\rho) \arrow[l, "a^{\breve{\textcircled{T}}}"'] \end{tikzcd} \quad . \tag{1}$$

Each column $\mathcal{C}$ of $\mathcal{T}$ intersects each row $\mathcal{R}$ of $\mathcal{T}$ in exactly one element:

$$\{\mathcal{C} \wedge \mathcal{R}\} = \mathcal{C} \cap \mathcal{R}. \tag{2}$$

Thus $\wedge$ may be viewed as a bijection from $\mathcal{C}\times\mathcal{R}$ onto $\mathcal{T}$. The operator $\textcircled{T}$ is actually a representation of the symmetrization $L\times L$ of L on $\mathcal{T}\times\mathcal{T}$, and it is equivalent to the symmetrization $\overset{\leftrightarrow}{\rho}$ of the representation ρ:

$$(\forall\, [a,b] \in L\times L) \quad \begin{tikzcd} X \times Y \arrow[rr, "\overset{\leftrightarrow}{\rho}_{[a,b]}"] \arrow[d, "\mu\times\nu"'] & & X \times Y \arrow[d, "\mu\times\nu"] \\ ||| (\mathcal{T}_\rho) \times \equiv (\mathcal{T}_\rho) \arrow[d, "\wedge"'] & & ||| (\mathcal{T}_\rho) \times \equiv (\mathcal{T}_\rho) \arrow[d, "\wedge"] \\ \mathcal{T} \arrow[rr, "a \textcircled{T} b"] & & \mathcal{T} \end{tikzcd} \tag{3}$$

(where $\mu\times\nu([x,y]) \equiv [\mu(x), \nu(y)]$). It is a corollary to Theorem (6.12) that

$$\text{the symmetrization representation } \overset{\leftrightarrow}{\rho} \text{ of } L\times L \text{ on } X\times Y \text{ is faithful.} \tag{4}$$

The cardinality of $|||$ is the same as the cardinality of $\equiv$: we define the **dimension of $\mathcal{T}$** to be this cardinal number. Thus the cardinality of $\mathcal{T}$ is the square of its dimension.

For $a \in L$ we define the **diagonal of $\mathcal{T}$ determined by a** as

$$\backslash\!\backslash a \backslash\!\backslash \equiv \{A \in \mathcal{T} : a \in A\}. \tag{5}$$

The cardinality of such a diagonal is the dimension of $\mathcal{T}$.

A diagonal $\backslash\!\backslash a \backslash\!\backslash$ can be used to give form to an aggregate in the sense that it associates to each column a row, and *vice versa*. A column $\mathcal{C}$ is a partition of L and so has exactly one element which contains a: this element is $\mathcal{C} \cap \backslash\!\backslash a \backslash\!\backslash$. Thus we have the bijections

$$||| \ni \mathcal{C} \hookrightarrow \overline{\overline{\underline{\underline{\mathcal{C} \cap \backslash\!\backslash a \backslash\!\backslash}}}} \in \equiv \ \text{and}\ \equiv \ni \mathcal{R} \hookrightarrow \|\mathcal{R} \cap \backslash\!\backslash a \backslash\!\backslash\| \in |||. \tag{6}$$

If $\{A_i\}_{i\in N}$ is a well ordering of $\backslash\!\backslash a \backslash\!\backslash$, then the aggregate $\mathcal{T}$ may be visualized as the elements of a matrix:

$$\begin{pmatrix} A_1 & \overline{\overline{\underline{\underline{A_1}}}} \wedge \|A_2\| & \dots & \overline{\overline{\underline{\underline{A_1}}}} \wedge \|A_n\| & \dots \\ \overline{\overline{\underline{\underline{A_2}}}} \wedge \|A_1\| & A_2 & \dots & \overline{\overline{\underline{\underline{A_2}}}} \wedge \|A_n\| & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ \overline{\overline{\underline{\underline{A_n}}}} \wedge \|A_1\| & \overline{\overline{\underline{\underline{A_n}}}} \wedge \|A_2\| & & A_n & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \tag{7}$$

Once an aggregate is visualized as a matrix, one can depict the actions of the operators $x \textcircled{T} y$ and $x^{\textcircled{T}}$ for

[25] *Cf.* (5.15).

$x, y \in L$. Let, for instance, B be an element of $\mathcal{T}$ and b an element of B. Then $x \in \lfloor x, b, B\rfloor$ and $y \in \lfloor B, b, y\rfloor$ so

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \lfloor B, b, y\rfloor & \dots & B & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & x\,\text{Ⓣ}\,y(B) & \dots & \lfloor x, b, B\rfloor & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \text{ and } \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \lfloor B, b, x\rfloor & \dots & B & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \overset{\text{Ⓣ}}{x}(B) & \dots & \lfloor x, b, B\rfloor & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{8}$$

In the case $x = a$, there exist indices $i, j \in N$ such that $\lfloor B, b, a\rfloor = A_i$ and $\lfloor a, b, B\rfloor = A_j$ and so the second of the above matrices becomes

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & A_i & \dots & \overset{\text{Ⓣ}}{a}(B) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & B & \dots & A_j & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{9}$$

We say that B and $\overset{\text{Ⓣ}}{a}(B)$ are symmetric with respect to $\backslash\!\backslash a \backslash\!\backslash$. More formally, two elements B and C of $\mathcal{T}$ are **symmetric relative to the diagonal of $\mathcal{T}$ determined by a** if $C = \lfloor a, B, a\rfloor$[26].

We shall say that B and C in $\mathcal{T}$ are **skew** provided that

$$\overline{\overline{\underline{\underline{B}}}} \neq \overline{\overline{\underline{\underline{C}}}} \text{ and } \|B\| \neq \|C\|; \tag{10}$$

and we shall say that B and C are **a-skew** provided that they are skew and that they are not symmetric with respect to a.

For each $x \in L$, the operator $\overset{\text{Ⓣ}}{x}$ permutes the elements of the matrix, sending columns to rows and rows to columns. The matrix in (9) suggests Theorem (6.19) *infra*.

(6.18) Example We return to the example of (4.21) and (2.42). The libra $\mathit{\Gamma}(\mathcal{M})$ is precisely the set of bijections of its four point domain domain $\mathcal{M} = \{A, B, C, D\}$. We denote by ρ the identity representation of $\mathit{\Gamma}(\mathcal{M})$ on $\mathcal{M}$. Evidently $\mathcal{T}$ has sixteen elements: $[A \overset{\rho}{=} A]$, $[A \overset{\rho}{=} B]$,. . . and $[D \overset{\rho}{=} D]$. We have, for instance,

$$\left\|[A \overset{\rho}{=} B]\right\| = \{[A \overset{\rho}{=} A], [A \overset{\rho}{=} B], [A \overset{\rho}{=} C], [A \overset{\rho}{=} D]\}$$

and

$$\overline{\overline{\underline{\underline{[A \overset{\rho}{=} B]}}}} = \{[A \overset{\rho}{=} B], [B \overset{\rho}{=} B], [C \overset{\rho}{=} B], [D \overset{\rho}{=} B]\}.$$

If we set $a \equiv \begin{array}{ccc} \cdot & \leftrightarrow & \cdot \\ & & \\ \cdot & \leftrightarrow & \cdot \end{array}$, then

$$\backslash\!\backslash a \backslash\!\backslash = \{[A \overset{\rho}{=} B], [B \overset{\rho}{=} A], [C \overset{\rho}{=} D], [D \overset{\rho}{=} C]\}$$

and a corresponding matrix is

$$\begin{pmatrix} [A \overset{\rho}{=} B] & [B \overset{\rho}{=} B] & [C \overset{\rho}{=} B] & [D \overset{\rho}{=} B] \\ [A \overset{\rho}{=} A] & [B \overset{\rho}{=} A] & [C \overset{\rho}{=} A] & [D \overset{\rho}{=} A] \\ [A \overset{\rho}{=} D] & [B \overset{\rho}{=} D] & [C \overset{\rho}{=} D] & [D \overset{\rho}{=} D] \\ [A \overset{\rho}{=} C] & [B \overset{\rho}{=} C] & [C \overset{\rho}{=} C] & [D \overset{\rho}{=} C] \end{pmatrix}.$$

Obviously the dimension is 4.

(6.19) Theorem Let x be in L and $\mathcal{T}$ be a cartesian aggregate for L. Then

(i) $(\forall\, A, B \in \mathcal{T}) \quad \overset{\text{Ⓣ}}{x}(A) = B \iff x \in (\|A\| \wedge \overline{\overline{\underline{\underline{B}}}}) \cap (\|B\| \wedge \overline{\overline{\underline{\underline{A}}}})$;

[26] Or, equivalently, $B = \lfloor a, C, a\rfloor$.

(ii) $(\forall\, A\in\mathcal{T})\quad x^{\textcircled{\mathcal{T}}}(A)=A \iff x\in A \iff x^{\widehat{\textcircled{\mathcal{T}}}}(\|A\|)=\overline{\overline{\underline{\underline{A}}}}$.

Proof. $\overset{\text{(i)}}{\Longrightarrow}$: Suppose that $x^{\textcircled{\mathcal{T}}}(A)=B$ for $A,B\in\mathcal{T}$. Let a be an element of A. Then

$$B = x^{\textcircled{\mathcal{T}}}(A) = \lfloor x,A,x\rfloor \xlongequal{\text{by (5.2.i)}} \lfloor x,\lfloor a,A,a\rfloor,x\rfloor = \lfloor\lfloor x,a,A\rfloor,a,x\rfloor \Longrightarrow \lfloor x,a,A\rfloor = \lfloor B,x,a\rfloor. \tag{1}$$

We have

$$x = \lfloor x,a,a\rfloor\in\lfloor x,a,A\rfloor \xlongequal{\text{by (1)}} \|A\|\wedge\overline{\overline{\underline{\underline{B}}}} \qquad (2).$$

Now let b be an element of B. Then

$$A = x^{\textcircled{\mathcal{T}}}\circ x^{\textcircled{\mathcal{T}}}(A) = x^{\textcircled{\mathcal{T}}}(B) = \lfloor x,B,x\rfloor \xlongequal{\text{by (5.2.i)}} \lfloor x,b,B,b,x\rfloor = \lfloor\lfloor x,b,B\rfloor,b,x\rfloor \Longrightarrow \lfloor A,x,b\rfloor = \lfloor x,b,B\rfloor.$$

This implies that $x=\lfloor x,b,b\rfloor\in\lfloor x,b,B\rfloor=\|B\|\wedge\overline{\overline{\underline{\underline{A}}}}$ which, with (2) yields $x\in(\|A\|\wedge\overline{\overline{\underline{\underline{B}}}})\cap(\|B\|\wedge\overline{\overline{\underline{\underline{A}}}})$.

$\overset{\text{(i)}}{\Longleftarrow}$: Suppose now that $x\in(\|A\|\wedge\overline{\overline{\underline{\underline{B}}}})\cap(\|B\|\wedge\overline{\overline{\underline{\underline{A}}}})$. Let a be in A and b be in B. That x is in $\|A\|\wedge\overline{\overline{\underline{\underline{B}}}}$ implies that $\lfloor x,a,A\rfloor=\lfloor B,b,x\rfloor$. That x is in $\|B\|\wedge\overline{\overline{\underline{\underline{A}}}}$ implies that $\lfloor A,a,x\rfloor=\lfloor x,b,B\rfloor$. We have

$$\lfloor B,b,\lfloor x,a,x\rfloor\rfloor = \lfloor\lfloor B,b,x\rfloor,a,x\rfloor = \lfloor\lfloor x,a,A\rfloor,a,x\rfloor = \lfloor x,a,\lfloor A,a,x\rfloor\rfloor = \lfloor x,a,\lfloor x,b,B\rfloor\rfloor = \lfloor\lfloor x,a,x\rfloor,b,B\rfloor.$$

The only right translate of B which is also a left translate of B is B itself. Thus

$$B = \lfloor B,b,\lfloor x,a,x\rfloor\rfloor \Longrightarrow (\exists\, c,d\in B)\quad c = \lfloor d,b,\lfloor x,a,x\rfloor\rfloor \Longrightarrow \lfloor x,a,x\rfloor = \lfloor b,d,c\rfloor\in B.$$

Consequently we have

$$x^{\textcircled{\mathcal{T}}}(A) = \lfloor x,A,x\rfloor\subset B \Longrightarrow x^{\textcircled{\mathcal{T}}}(A) = B.$$

which proves (i).

When $A=B$ we have $(\|A\|\wedge\overline{\overline{\underline{\underline{B}}}})\cap(\|B\|\wedge\overline{\overline{\underline{\underline{A}}}})=A\cap A=A$. Thus the first "$\iff$" of (ii) is a special case of (i).

If x is in A, then

$$x^{\widehat{\textcircled{\mathcal{T}}}}(\|A\|) = \overline{\overline{\underline{\underline{\lfloor x,A,x\rfloor}}}} = \overline{\overline{\underline{\underline{A}}}}.$$

Suppose, on the other hand, that $x^{\widehat{\textcircled{\mathcal{T}}}}(\|A\|)=\overline{\overline{\underline{\underline{A}}}}$, and let $a\in A$. Evidently $x\in\lfloor x,a,A\rfloor$ so

$$x = \lfloor x,x,x\rfloor\in\lfloor x,\lfloor x,a,A\rfloor,x\rfloor = \lfloor x,A,a,x,x\rfloor = \lfloor x,a,a,A,a\rfloor = \lfloor x,a,A\rfloor.$$

Thus $\lfloor x,\lfloor x,a,A\rfloor,x\rfloor = x^{\textcircled{\mathcal{T}}}(\lfloor x,a,A\rfloor)$ is both a right coset and a left coset of A. Hence it must be A. Thus $x\in\lfloor x,a,A\rfloor = A$. This finishes the proof of the second "$\iff$" of (ii). QED

(6.20) Definition Let $\mathcal{X}\in{\color{red}|||}\cup{\color{red}\equiv}$ and $x\in L$. Since $\mathcal{X}$ is a partition of L, we may define

$$x\wedge\mathcal{X}\equiv Y \text{ where } x\in Y\in\mathcal{X}.$$

(6.21) Theorem Let $\mathcal{T}$ be a cartesian aggregate for L and let $a,b\in L$. Then $a\,\widehat{\textcircled{\mathcal{T}}}\,b$ agrees with $a^{\widehat{\textcircled{\mathcal{T}}}}$ on ${\color{red}|||}$ and $a\,\check{\textcircled{\mathcal{T}}}\,b$ agrees with $b^{\check{\textcircled{\mathcal{T}}}}$ on ${\color{red}\equiv}$.

Proof. For $T\in\mathcal{T}$ and $t\in T$,

$$\begin{aligned} a^{\widehat{\textcircled{\mathcal{T}}}}(\|T\|) &= \{\lfloor a,\lfloor m,t,T\rfloor,a\rfloor : m\in L\} = \{\lfloor a,T,\lfloor t,m,a\rfloor\rfloor : m\in L\} = \{\lfloor a,T,n\rfloor : n\in L\} = \\ &\quad \{\lfloor a,T,\lfloor t,m,b\rfloor\rfloor : m\in L\} = \{\lfloor a,\lfloor m,t,T\rfloor,b\rfloor : m\in L\} = a\,\widehat{\textcircled{\mathcal{T}}}\,b(\|T\|)\end{aligned}$$

and

$$\begin{aligned} b^{\check{\textcircled{\mathcal{T}}}}(\overline{\overline{\underline{\underline{T}}}}) &= \{\lfloor b,\lfloor T,t,m\rfloor,b\rfloor : m\in L\} = \{\lfloor\lfloor b,m,t\rfloor,T,b\rfloor : m\in L\} = \{\lfloor n,T,b\rfloor : n\in L\} = \\ &\quad \{\lfloor\lfloor a,m,t\rfloor,T,b\rfloor : m\in L\} = \{\lfloor a,\lfloor T,t,m\rfloor,b\rfloor : m\in L\} = a\,\check{\textcircled{\mathcal{T}}}\,b(\overline{\overline{\underline{\underline{T}}}}).\end{aligned}$$

QED

7. Libra Polarity

(7.1) Definitions and Notation In examining the structure of a libra, the concept of a "polar" is sometimes of use. The polar of a subset S of a libra L is defined as follows:

$$S^\circ \equiv \{x \in L \colon (\forall\, s \in S) \quad \lfloor x, s, x \rfloor = s\}. \tag{1}$$

We shall usually abbreviate $\{i\}^\circ$ to $i^\square$ in the case of a singleton, and we shall abbreviate the polar $(S^\circ)^\circ$ of a polar to $S^{\circ\circ}$.

Polars are not necessarily balanced. In fact we have

(7.2) Theorem Let B be a balanced subset of a libra L and $A \subset L$. A condition both necessary and sufficient for $B \cap A^\circ$ to be balanced is for $B \cap A^\circ$ to be abelian.

Proof. Let $a, b, c \in B \cap A^\circ$. For $i \in A$ we have

$$\lfloor i, \lfloor a, b, c \rfloor, i \rfloor = \lfloor i, c, b, a, i \rfloor = \lfloor i, c, i, i, b, i, i, a, i \rfloor = \lfloor \lfloor i, c, i \rfloor, \lfloor i, b, i \rfloor, \lfloor i, a, i \rfloor \rfloor = \lfloor c, b, a \rfloor$$

which implies that $\lfloor a, b, c \rfloor$ is in $i^\square$ precisely if $\lfloor a, b, c \rfloor = \lfloor c, b, a \rfloor$. QED

(7.3) Theorem Let S be any subset of a libra L. Then

(i) $(\forall\, T \subset S) \quad S^\circ \subset T^\circ$;
(ii) $S^{\circ\circ}$ is the intersection of all polars containing S;
(iii) S is a polar $\iff S = S^{\circ\circ}$;
(iv) $S^{\circ\circ\circ} = S^\circ$;
(v) $(\forall\, T, S \subset L \colon T \subset S^\circ) \quad T^{\circ\circ} \subset S^\circ$.

Proof. $\overset{\text{(i)}}{\Longrightarrow}$: That (i) is true is follows directly from the definitions.

$\overset{\text{(iv)}}{\Longrightarrow}$: That $S \subset S^{\circ\circ}$ follows from the definition of polarity. Thus (i) implies that $S^{\circ\circ\circ} \subset S^\circ$. That $S^\circ \subset S^{\circ\circ\circ}$ follows directly from the definitions. Thus (iv) holds.

$\overset{\text{(ii)}}{\Longrightarrow}$: Suppose that $W \subset L$ and that $S \subset W^\circ$. From (i) then follows that $W^{\circ\circ} \subset S^\circ$. From (i) and (iv) follows

$$S^{\circ\circ} \subset W^{\circ\circ\circ} = W^\circ$$

which proves (ii).

$\overset{\text{(iii)}}{\Longrightarrow}$: That $S \subset S^{\circ\circ}$ follows from the definition. If S is a polar, then $S^{\circ\circ} \subset S$ by (ii), and so $S = S^{\circ\circ}$. That S is polar if $S = S^{\circ\circ}$ is trivial. This proves (iii).

$\overset{\text{(v)}}{\Longrightarrow}$: We have

$$T \subset S^\circ \xrightarrow{\text{by (i)}} S^{\circ\circ} \subset T^\circ \xrightarrow{\text{by (i)}} T^{\circ\circ} \subset S^{\circ\circ\circ} \xrightarrow{\text{by (iv)}} T^{\circ\circ} \subset S^\circ.$$

QED

(7.4) Theorem Let A and B be balanced polars. Then

$$A = B^\circ \iff B = A^\circ \implies A \cup B \text{ is balanced.}$$

Proof. Suppose that $A = B^\circ$. Then $A^\circ = B^{\circ\circ}$. Since B is a polar, (7.3.iii) implies that $B = B^{\circ\circ}$. Hence $B = A^\circ$.

That $B = A^\circ$ implies $A = B^\circ$ follows by an analogous argument.

Let $x, y, z \in A \cup B$. If $x, y, z \in A$ or $x, y, z \in B$, then $\lfloor x, y, z \rfloor$ would be in $A \cup B$ since both A and B are balanced. Thus, without loss of generality, we can and shall suppose that x and y are in A and that z is in B. We need to show that $\lfloor x, y, z \rfloor$, $\lfloor x, z, y \rfloor$ and $\lfloor z, x, y \rfloor$ are in $A \cup B$ – however, since z is in the polar of A, these are all the same. We have, for any $a \in A$ we have (since Theorem (7.2) implies that A is abelian)

$$\lfloor a, \lfloor x, y, z \rfloor, a \rfloor = \lfloor a, z, \lfloor y, x, a \rfloor \rfloor = \lfloor a, z, \lfloor a, x, y \rfloor \rfloor = \lfloor a, z, a, x, y \rfloor =$$

$$\lfloor a, a, z, x, y \rfloor = \lfloor a, a, x, z, y \rfloor = \lfloor a, a, x, y, z \rfloor = \lfloor x, y, z \rfloor.$$

It follows that $\lfloor x, y, z \rfloor$ is in $A^\circ = B \subset A \cup B$. QED

(7.5) Theorem Let A and B be balanced polars of one another. Let a be in A and b be in B. Then

$$B = \lfloor a, b, A \rfloor. \tag{1}$$

Proof. From (7.2) we know that

$$A \text{ is abelian.} \tag{2}$$

Let c and d be generic elements of A. Then of course

$$a, c, d \in B^\circ. \tag{3}$$

We have

$$\lfloor d, \lfloor a,b,c \rfloor, d \rfloor = \lfloor d,c,b,a,d \rfloor \overset{\text{by (3)}}{=\!=\!=} \lfloor d,c,a,d,b \rfloor \overset{\text{by (3)}}{=\!=\!=} \lfloor d,d,a,c,b \rfloor = \lfloor a,b,c \rfloor \Longrightarrow$$
$$d \in \lfloor a,b,c \rfloor^\circ = B \Longrightarrow A \subset B. \tag{4}$$

For $e \in B$, we have

$$e = \lfloor e,a,a \rfloor \overset{\text{by (3)}}{=\!=\!=} \lfloor a,e,a \rfloor \in \lfloor a,b,A \rfloor \Longrightarrow B \subset \lfloor a,b,A \rfloor. \tag{5}$$

Inclusions (4) and (5) imply (1). QED

(7.6) Definition We shall say that a libra L is **polar** provided that there exists $a \in L$ such that no proper balanced subset of L contains $a^\square$ (as a subset).

(7.7) Theorem Let L be a polar libra and let u and v be elements of L. Then there exists $n \in \mathbb{N}$ odd and $\{x_i\}_{i=1}^{i=n} \subset u^\square$ such that

$$\lfloor x_1, x_2, \ldots, x_n \rfloor = v.$$

Proof. Since no proper balanced subset of L contains $a^\square$ as a subset, and since

$$\{\lfloor t_1, t_2, \ldots, t_n \rfloor : n \in \mathbb{N} \text{ odd and } \{t_i\}_{i=1}^{i=n} \subset a^\square\}$$

is a balanced subset of L containing $a^\square$, there exists $n \in \mathbb{N}$ odd and $\{t_i\}_{i=1}^{i=n} \subset a^\square$ such that

$$\lfloor t_1, t_2, \ldots, t_n \rfloor = \lfloor a, u, v \rfloor. \tag{1}$$

For $i = 1, 2, \ldots, n$ let $x_i \equiv \lfloor u, a, t_i \rfloor$. Then

$$\lfloor x_1, x_2, x_3, \ldots, x_{n-1}, x_n \rfloor = \lfloor \lfloor u,a,t_1 \rfloor, \lfloor u,a,t_2 \rfloor, \lfloor u,a,t_3 \rfloor, \ldots, \lfloor u,a,t_{n-1} \rfloor, \lfloor u,a,t_n \rfloor \rfloor =$$
$$\lfloor u,a,t_1,t_2,a,u,u,a,t_3,\ldots,t_{n-1},a,u,u,a,t_n \rfloor = \lfloor u,a,\lfloor t_1,t_2,\ldots,t_n \rfloor \rfloor = \lfloor u,a,\lfloor a,u,v \rfloor \rfloor = v.$$

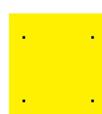
Furthermore, for each $i \in \{1, \ldots, n\}$,

$$\lfloor u, x_i, u \rfloor = \lfloor u, \lfloor u,a,t_i \rfloor, u \rfloor = \lfloor u,t_i,a,u,u \rfloor = \lfloor u,t_i,a \rfloor \overset{\text{by (1)}}{=\!=\!=} \lfloor u,a,t_i \rfloor = x$$

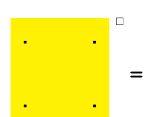
and so $\{x_i\}_{i=1}^{i=n} \subset u^\square$. QED

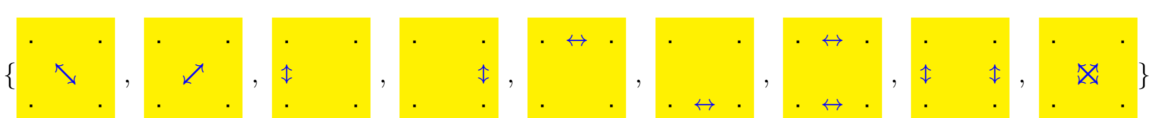
(7.8) Example We return again to the example of (2.42), (4.20) and (6.18). We shall compute some polars of subsets of $\Gamma(\mathcal{M})$ for this example.

First, we write

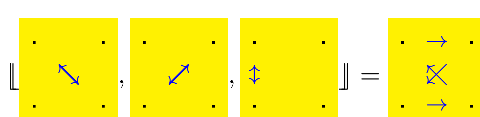


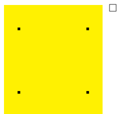
for the identity permutation of the four corner points. Direct calculation shows that its polar is the family of involutions of $\Gamma(\mathcal{M})$:

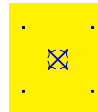


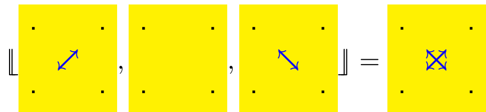
 (1)

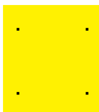
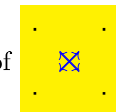
We have

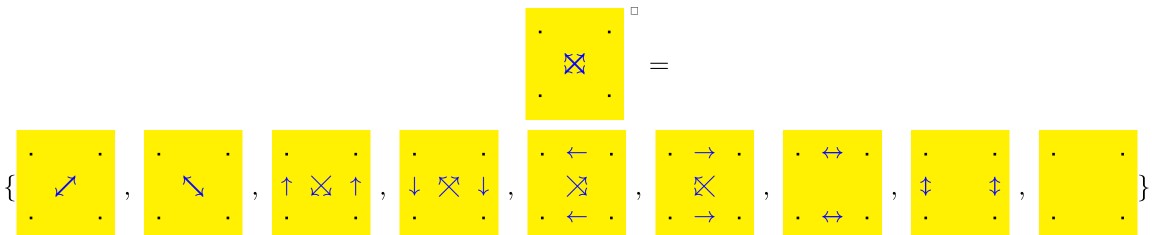


which shows that  is not balanced. From (7.2) we know that it cannot be abelian either.

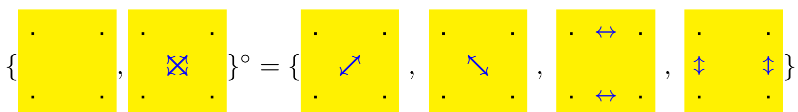
We choose one of the elements of this polar at random, say , and observe that

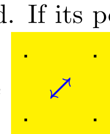
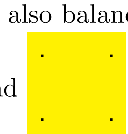
 (2)

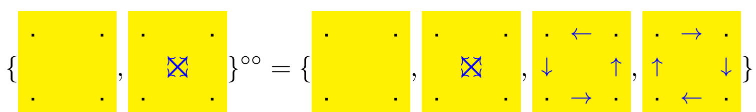
Replacing  serially by the elements of (1) in (2), we can compute the polar of :

 (3)

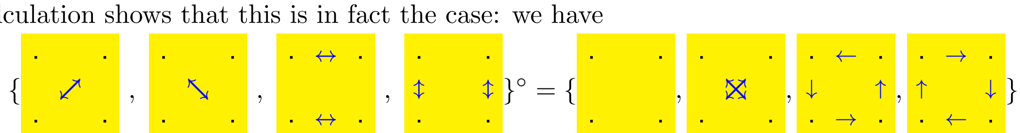
From (1) and (3) we have

 (4)

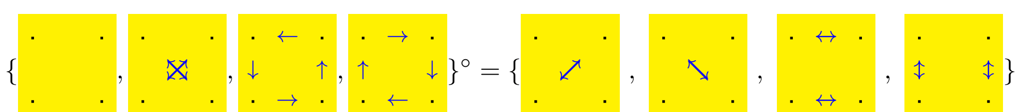
Direct calculation shows that this polar is balanced. If its polar were also balanced, then (7.3.iv) and (7.5) would show us how to calculated it: we could take  for A and  for B and use (7.5.1). We would obtain



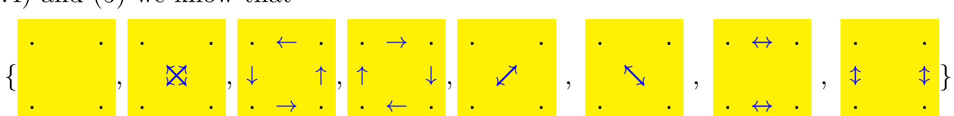
Direct calculation shows that this is in fact the case: we have



and

 (5)

From (7.4) and (5) we know that



is balanced.

8. Meridian Libras

(8.1) Discussion In the cross ratio definition of a meridian of Section (2), the set $\mathbf{Mor}(\mathfrak{M}, \mathcal{M})$[27] was seen to be a libra[28]. In general, the set of isomorphisms from one meridian onto another, if non-void, is a libra. Such libras have an intrinsic characterization, which is the subject of the present section.

(8.2) Notation . If $\mathcal{A}$ is a subfamily of an aggregate $\mathcal{T}$ of a libra L, and a and b are in L, we define

$$\lfloor a, \mathcal{A}, b \rfloor \equiv \{\lfloor a, A, b \rfloor : A \in \mathcal{A}\}. \tag{1}$$

(8.3) Theorem Let $\mathcal{T}$ be a cartesian aggregate of balanced subsets of a libra L such that $\mathcal{T}$ has dimension[29] at least 4. Suppose further that there exists $a \in L$ such that

(i) $(\forall\, B \in \mathcal{T})$ $B \cap a^{\square}$ is balanced and has more than one element;

(ii) $(\forall\, B, C \in \mathcal{T}$ distinct and a-skew$)$ $B \cap C \cap a^{\square}$ is a singleton.

Define

$$\Pi(\boxed{|||}) \equiv \{a^{\breve{\mathcal{T}}} \circ x^{\hat{\mathcal{T}}} : x \in a^{\square}\}. \tag{1}$$

Then $\Pi(\boxed{|||})$ is a meridian family of involutions on $\boxed{|||}$.

Proof. For $x, a \in L$ such that $\lfloor x, a, x \rfloor = a$, we have

$$(a^{\breve{\mathcal{T}}} \circ x^{\hat{\mathcal{T}}}) \circ (a^{\breve{\mathcal{T}}} \circ x^{\hat{\mathcal{T}}}) = a^{\breve{\mathcal{T}}} \circ (x^{\hat{\mathcal{T}}} \circ a^{\breve{\mathcal{T}}} \circ x^{\hat{\mathcal{T}}}) \overset{\text{by (6.6)}}{\equiv} a^{\breve{\mathcal{T}}} \circ \lfloor x, a, x \rfloor^{\hat{\mathcal{T}}} = a^{\breve{\mathcal{T}}} \circ a^{\hat{\mathcal{T}}}$$

which shows that $a^{\breve{\mathcal{T}}} \circ x^{\hat{\mathcal{T}}}$ is an involution.

Let $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E} \in \boxed{|||}$ with $\{\mathcal{A}, \mathcal{E}\} \cap \{\mathcal{B}, \mathcal{D}\} = \emptyset$. If $\lfloor a, \mathcal{A} \wedge a^{\hat{\mathcal{T}}}(\mathcal{E}), a \rfloor = \mathcal{B} \wedge a^{\hat{\mathcal{T}}}(\mathcal{D})$, then (6.15) implies that $\mathcal{E} = \mathcal{D}$, which is absurd. It follows that $\mathcal{A} \wedge a^{\hat{\mathcal{T}}}(\mathcal{E})$ and $\mathcal{B} \wedge a^{\hat{\mathcal{T}}}(\mathcal{D})$ are distinct and skew, and so (ii) implies that $(\mathcal{A} \wedge a^{\hat{\mathcal{T}}}(\mathcal{E})) \cap (\mathcal{B} \wedge a^{\hat{\mathcal{T}}}(\mathcal{D})) \cap a^{\square}$ is some singleton $\{j\}$. Thus $a^{\breve{\mathcal{T}}} \circ j^{\hat{\mathcal{T}}}$ sends $\mathcal{A}$ to $\mathcal{E}$ and $\mathcal{B}$ to $\mathcal{D}$. If any other $\phi \in \Pi(\boxed{|||})$ did the same, since ϕ can be written as $a^{\breve{\mathcal{T}}} \circ k^{\hat{\mathcal{T}}}$, we would have $k \in (\mathcal{A} \wedge a^{\hat{\mathcal{T}}}(\mathcal{E})) \cap (\mathcal{B} \wedge a^{\hat{\mathcal{T}}}(\mathcal{D})) \cap a^{\square}$, whence $k = j$ and $\phi = a^{\breve{\mathcal{T}}} \circ j^{\hat{\mathcal{T}}}$. This establishes (4.4.1).

Let $\mathcal{P}$ and $\mathcal{Q}$ be in $\boxed{|||}$. Suppose that $\beta, \gamma, \delta \in \Pi(\boxed{|||})$ and that $\beta(\mathcal{P}) = \gamma(\mathcal{P}) = \delta(\mathcal{P}) = \mathcal{Q}$. By (1) there exist $b, c, d \in L$ such that $a^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}} = \beta$, $a^{\breve{\mathcal{T}}} \circ c^{\hat{\mathcal{T}}} = \gamma$ and $a^{\breve{\mathcal{T}}} \circ d^{\hat{\mathcal{T}}} = \delta$. Let P be the element of $\mathcal{P}$ which contains a and let Q be in $\mathcal{Q}$. Let $R \equiv \|Q\| \wedge \overline{\overline{\underline{\underline{P}}}}$. We have

$$\|Q\| = \mathcal{Q} = \beta(\mathcal{P}) = a^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}}(\|P\|) = a^{\breve{\mathcal{T}}}(\overline{\overline{\underline{\underline{\lfloor b, P, b \rfloor}}}}) = \|\lfloor a, \lfloor b, P, b \rfloor, a \rfloor\| = \|\lfloor P, b, a \rfloor\| = \|\lfloor P, a, b \rfloor\|$$

$$\Longrightarrow \lfloor P, a, b \rfloor \in \|Q\| \cap \overline{\overline{\underline{\underline{P}}}} \Longrightarrow \lfloor P, a, b \rfloor = \|Q\| \wedge \overline{\overline{\underline{\underline{P}}}} = R.$$

We have $b = \lfloor a, a, b \rfloor \in \lfloor P, a, b \rfloor = R$. Similarly, c and d are in R as well. Thus

$$b, c, d \in R \cap a^{\square} \overset{\text{by (i)}}{\Longrightarrow} \lfloor b, c, d \rfloor \in R \cap a^{\square} \Im_{\text{by (7.2)}} \lfloor b, c, d \rfloor = \lfloor d, c, b \rfloor. \tag{2}$$

Hence

$$(a^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}}) \circ (a^{\breve{\mathcal{T}}} \circ c^{\hat{\mathcal{T}}})^{-1} \circ (a^{\breve{\mathcal{T}}} \circ d^{\hat{\mathcal{T}}})(\mathcal{P}) = (a^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}}) \circ (a^{\breve{\mathcal{T}}} \circ c^{\hat{\mathcal{T}}})^{-1} \circ (a^{\breve{\mathcal{T}}} \circ d^{\hat{\mathcal{T}}})(\|P\|) =$$

$$\|\lfloor a, \lfloor b, \lfloor a \lfloor c, \lfloor a \lfloor d, P, d \rfloor, a \rfloor, c \rfloor, a \rfloor, b \rfloor, a \rfloor\| = \|\lfloor \lfloor \lfloor a, b, a, c, a \rfloor, d, P \rfloor, d, a, c, a, b, a \rfloor\| =$$

$$\|\lfloor b, c, d, a, a, a, P, a, a, a, d, c, b \rfloor\| \overset{\text{by (2)}}{\equiv} \|\lfloor \lfloor b, c, d \rfloor, a, P, a, \lfloor b, c, d \rfloor \rfloor\| \overset{\text{by (2)}}{\equiv} \|\lfloor a, \lfloor b, c, d \rfloor, P, , \lfloor b, c, d \rfloor, a \rfloor\| =$$

$$a^{\breve{\mathcal{T}}} \circ \lfloor a, b, c \rfloor^{\hat{\mathcal{T}}}(\|P\|) = a^{\breve{\mathcal{T}}} \circ \lfloor a, b, c \rfloor^{\hat{\mathcal{T}}}(\mathcal{P})$$

This shows that (4.4.2) holds.

Let β and γ be in $\Pi(\boxed{|||})$. Choose $b, c \in a^{\square}$ such that $\beta = a^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}}$ and $\gamma = a^{\breve{\mathcal{T}}} \circ c^{\hat{\mathcal{T}}}$. We have

$$\beta \circ \gamma^{-1} \circ \beta = a^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}} \circ c^{\breve{\mathcal{T}}} \circ a^{\hat{\mathcal{T}}} \circ a^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}} = a^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}} \circ c^{\breve{\mathcal{T}}} \circ b^{\hat{\mathcal{T}}} = a^{\breve{\mathcal{T}}} \circ \lfloor b, c, b \rfloor^{\hat{\mathcal{T}}}. \tag{3}$$

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[27] *Cf.* (2.8).

[28] *Cf.* (3.6).

[29] *Cf.* (6.17) for definitions of "dimension" and "a-skew".

Furthermore

$$\lfloor a, \lfloor b,c,b \rfloor, a \rfloor = \lfloor a,b,c,b,a \rfloor = \lfloor a,a,b,c,b \rfloor = \lfloor b,c,b \rfloor \Longrightarrow \lfloor b,c,b \rfloor \in a^{\square}$$

which, with (3), implies (4.4.3). QED

(8.4) Definition and Remarks A cartesian aggregate of balanced sets satisfying (i) and (ii) of Theorem 7.1, and of dimension at least 4, will be called a **meridian aggregate**.

It follows from Theorem (8.3) and Theorem (4.5) that Ⅲ is a meridian when $\mathcal{T}$ is a meridian aggregate. Evidently ☰ is as well, with meridian family of involutions $\{a^{\hat{\circledT}} \circ x^{\check{\circledT}} \colon x \in a^{\square}\}$.

(8.5) Example Let **S** be three dimensional real projective space, and let **Q** be a quadric surface in **S** in the sense of (1.7). We shall regard **Q** as a circular hyperboloid extending vertically as in Figure (8).[30] We write **L** for the complement of **Q** in **S**. We write M for the family of rules of **Q** which go up counter-clockwise, and we write N for the family of rules of **Q** which go up clockwise. Each element$a$ of **L** corresponds to a mapping ρ_a of M onto N as in Figure (1.11). We shall show in Section (10) that these mappings form a libra, which libra operation carries over to **L** in exactly one way so that ρ is a meridian representation. The elements of the associated meridian aggregate $\mathfrak{T}$ are then intersections with of **L** with planes in **S** tangent to **Q**. Each such tangent plane intersects with **Q** in two rules, one from M and one from N.

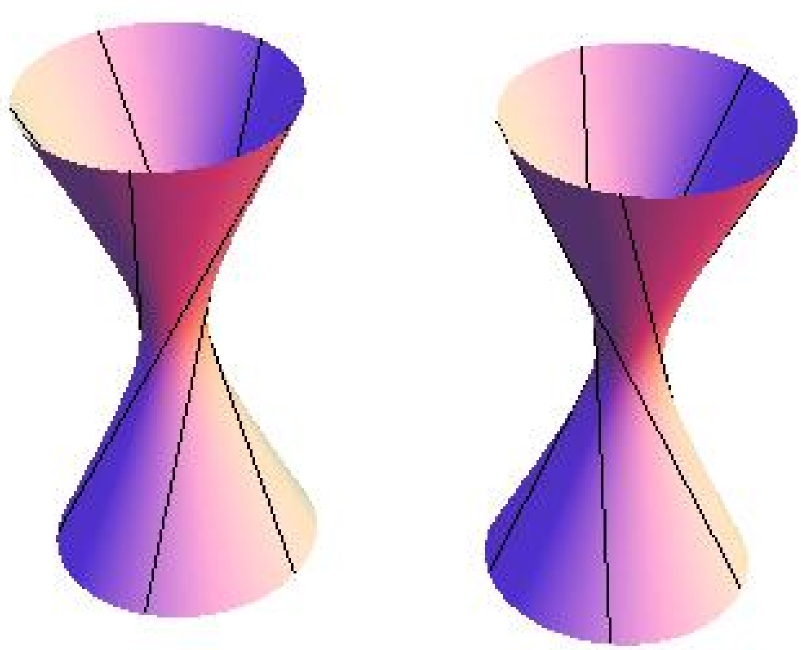

Fig. 16: Some elements of M and N

[30] If **S** carries homogeneous coordinates $[x, y, z, t]$, the solution to the equation $x^2 + y^2 = z^2 + t^2$ gives such a quadric surface.

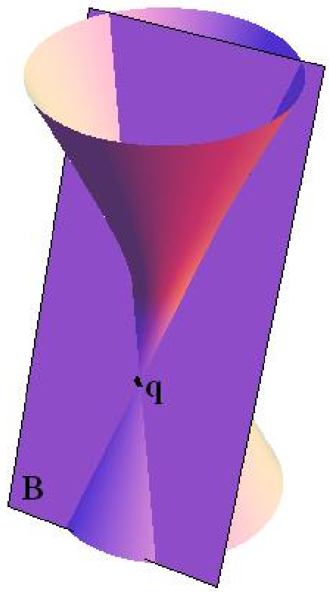

Fig. 17: Plane **B** Tangent to **Q** (element of $\mathfrak{T}$) at Point **q**

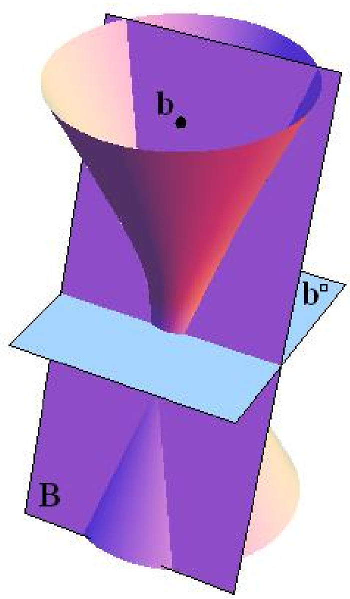

Fig. 18: Plane **B** Tangent to **Q** (element of $\mathfrak{T}$) cutting Polar of a Point **b**

(8.6) Theorem Let $\mathcal{T}$ be a meridian aggregate of balanced subsets of a libra L. Then, for all $b \in L$.

(i) $(\forall\, B \in \mathcal{T}) \quad B \cap b^{\square}$ is balanced and has more than one element;

(ii) $(\forall\, B, C \in \mathcal{T}$ b-skew$) \quad B \cap C \cap b^{\square}$ is a singleton.

Furthermore, if a and $\Pi(\boxed{\text{III}})$ are as in (8.3.1), then

(iii) $\Pi(\boxed{\text{III}}) = \{ y^{\breve{\textcircled{\scriptsize$\mathcal{T}$}}} \circ x^{\hat{\textcircled{\scriptsize$\mathcal{T}$}}} :\ x, y \in L \text{ and } x \in y^{\square} \}$.

If L is a polar libra, than we also have

(iv) $\Gamma(\boxed{\text{III}}) = \{ b^{\breve{\textcircled{\scriptsize$\mathcal{T}$}}} \circ x^{\hat{\textcircled{\scriptsize$\mathcal{T}$}}} :\ x \in L \}$.

Proof. $\overset{(i)}{\Longrightarrow}$: Let B be in $\mathcal{T}$. Let a be as in (8.3) and let b be a generic element of L. Since $\lfloor a,b,B\rfloor$ is in $\mathcal{T}$, (8.3.i) implies that $\lfloor a,b,B\rfloor \cap a^{\square}$ is balanced and has and least two elements. Thus there are two elements $r,s\in B$ such that $\lfloor a,b,r\rfloor$ and $\lfloor a,b,s\rfloor$ are in $a^{\square}$. We have

$$\lfloor b,r,b\rfloor = \lfloor b,r,b,a,a\rfloor = \lfloor b,\lfloor b,\lfloor a,b,r\rfloor,a\rfloor\rfloor = \lfloor b,a,\lfloor a,b,r\rfloor\rfloor = \lfloor b,a,a,b,r\rfloor = \lfloor b,b,r\rfloor = r \tag{1}$$

It follows that r, and by analogy s, are in $b^{\square}$: $B\cap b^{\square}$ has at least two elements. If t is any other element of $B\cap b^{\square}$ we have

$$\lfloor a,\lfloor a,b,t\rfloor,a\rfloor = \lfloor a,t,b,a,a\rfloor = \lfloor a,t,b\rfloor = \lfloor a,b,t\rfloor \Longrightarrow \lfloor a,b,t\rfloor\in a^{\square}.$$

By (8.3.i) we know that $\lfloor\lfloor a,b,r\rfloor,\lfloor a,b,s\rfloor,\lfloor a,b,t\rfloor\rfloor$ is in $\lfloor a,b,B\rfloor\cap a^{\square}$. Thus there exists $u\in B$ such that $\lfloor a,b,u\rfloor\in a^{\square}$ and

$$\lfloor a,b,u\rfloor = \lfloor\lfloor a,b,r\rfloor,\lfloor a,b,s\rfloor,\lfloor a,b,t\rfloor\rfloor = \lfloor a,b,r,s,b,a,a,b,t\rfloor = \lfloor a,b,r,s,b,b,t\rfloor = \lfloor a,b,\lfloor r,s,t\rfloor\rfloor.$$

Thus $u=\lfloor r,s,t\rfloor$ and, as with r in (1), we have u in $b^{\square}$. It follows the (i) holds.

$\overset{(ii)}{\Longrightarrow}$: Let now B and C be in $\mathcal{T}$ and b-skew. Then $\lfloor a,b,B\rfloor$ and $\lfloor a,b,C\rfloor$ are $\lfloor a,b,b\rfloor$-skew: *i.e* they are a-skew. By (8.3.ii) $\lfloor a,b,B\rfloor\cap\lfloor a,b,C\rfloor\cap a^{\square}$ is a singleton. Hence the intersection of $B=\lfloor b,a,\lfloor a,b,B\rfloor\rfloor$, $C=\lfloor b,a,\lfloor a,b,C\rfloor\rfloor$, and $b^{\square}=\lfloor b,a,a\rfloor^{\square}$ is a singleton. This proves (ii).

$\overset{(iii)}{\Longrightarrow}$: That $\Pi\subset\{y^{\check{\mathcal{T}}}\circ x^{\hat{\mathcal{T}}}\colon x,y\in L \text{ and } x\in y^{\square}\}$ is trivial. Suppose that $x,y\in L$ and $x\in y^{\square}$. As in the first paragraph of this proof, we can show that $\lfloor a,y,x\rfloor$ is in $a^{\square}$. We have

$$y^{\check{\mathcal{T}}}\circ x^{\hat{\mathcal{T}}} = a^{\check{\mathcal{T}}}\circ a^{\hat{\mathcal{T}}}\circ y^{\check{\mathcal{T}}}\circ x^{\hat{\mathcal{T}}} \overset{\text{by (6.6)}}{=\!=\!=} a^{\check{\mathcal{T}}}\circ {}_{\lfloor}a,y,x_{\rfloor}{}^{\hat{\mathcal{T}}} \Longrightarrow y^{\check{\mathcal{T}}}\circ x^{\hat{\mathcal{T}}}\in\Pi(\mathrm{III}),$$

which establishes (iii).

$\overset{(iv)}{\Longrightarrow}$: By Theorem (4.20) we know that $\Gamma(\mathrm{III})$ is the smallest group of bijections of III containing $\Pi(\mathrm{III})$. It follows immediately that $\Gamma(\mathrm{III})\subset\{b^{\check{\mathcal{T}}}\circ x^{\hat{\mathcal{T}}}\colon x\in L\}$, so to establish (iv), it will suffice to show, for any $x\in L$, that $b^{\check{\mathcal{T}}}\circ x^{\hat{\mathcal{T}}}$ is a composition of a finite sequence of elements of $\Pi(\mathrm{III})$. From Theorem (7.7) follows that there exists $n\in\mathbb{N}$ odd and $\{t_i\}_{i=1}^{i=n}\subset b^{\square}$ such that

$$x=\lfloor t_1,t_2,\ldots,t_n\rfloor.$$

We have

$$b^{\check{\mathcal{T}}}\circ x^{\hat{\mathcal{T}}} \overset{6.6}{=\!=} b^{\check{\mathcal{T}}}\circ t_1{}^{\hat{\mathcal{T}}}\circ t_2{}^{\check{\mathcal{T}}}\circ t_3{}^{\hat{\mathcal{T}}}\circ\ldots\circ t_{n-1}{}^{\check{\mathcal{T}}}\circ t_n{}^{\hat{\mathcal{T}}} =$$

$$(b^{\check{\mathcal{T}}}\circ t_1{}^{\hat{\mathcal{T}}})\circ(b^{\check{\mathcal{T}}}\circ t_2{}^{\hat{\mathcal{T}}})^{-1}\circ(b^{\check{\mathcal{T}}}\circ t_3{}^{\hat{\mathcal{T}}})\circ\ldots\circ(b^{\check{\mathcal{T}}}\circ t_{n-1}{}^{\hat{\mathcal{T}}})^{-1}\circ(b^{\check{\mathcal{T}}}\circ t_n{}^{\hat{\mathcal{T}}})$$

QED

(8.7) Theorem Let $\mathcal{M}$ be a meridian. Then $\Gamma(\mathcal{M})$ is a polar libra[31] relative to the canonical libra operator $[\![,,]\!]$.

Let ι be the identity representation of $\Gamma(\mathcal{M})$ on $\mathcal{M}\times\mathcal{M}$. Let $\mathcal{T}$ be an abbreviation for $\mathcal{T}_\iota$, as defined in (5.9.2). Then $\mathcal{T}$ is a meridian aggregate.

The identity representation is equivalent to the $\mathcal{T}$-inner representation[32] $\mathcal{T}^{\hat{\mathcal{T}}}$ of $\Gamma(\mathcal{M})$ on $\mathrm{III}\mathcal{T}\times\equiv\mathcal{T}$.

Proof. By Theorem (2.15) $\Gamma(\mathcal{M})$ is a group. Thus $\Gamma(\mathcal{M})$ is balanced relative to the libra operator $[\![,,]\!]$.

Define the bijections

$$\mu|\mathrm{III}\ni\left\|[m\overset{\iota}{=}n]\right\|\hookrightarrow n\in\mathcal{M} \text{ and } \nu|\equiv\ni\overline{\overline{\underline{\underline{[m\overset{\iota}{=}n]}}}}\hookrightarrow m\in\mathcal{M}. \tag{1}$$

Then, for $\alpha\in\Gamma(\mathcal{M})$ and $m\in\mathcal{M}$,

$$\nu\circ\alpha^{\hat{\mathcal{T}}}\circ\mu^{-1}(m) = \nu\circ\alpha^{\hat{\mathcal{T}}}(\left\|[m\overset{\iota}{=}m]\right\|) = \nu(\overline{\overline{\underline{\underline{[\alpha(m)\overset{\iota}{=}\alpha^{-1}(m)]}}}}) = \alpha(m) = \iota_\alpha(m)$$

which establishes the equivalence. QED

(8.8) Definition, Notation and Discussion Let L be a libra containing a meridian aggregate $\mathcal{T}$ of balanced subsets of L. It being somewhat cumbersome to deal with III and $\equiv$, we shall deal at times with

[31] *Cf.* (7.6).
[32] *Cf.* (6.8).

a representation ρ of L on a product $M \times N$ equivalent to the $\mathcal{T}$-inner representation. Such a representation will be said to be **characteristic**. We reserve $\mu | ||| \to M$ and $\nu | \equiv \to N$ to represent the bijections such that

$$\rho_a = \nu \circ a^{\widehat{(\mathcal{T})}} \circ \mu^{-1} \quad (\forall\, a \in L). \tag{1}$$

For a given $a \in L$ it is sometimes expedient to form the representation

$$\overset{\boxed{a}}{\rho} \equiv {\rho_a}^{-1} \circ \rho \tag{2}$$

of L on M. We shall say that $\overset{\boxed{a}}{\rho}$ is the **a-representation founded on ρ**. In this case we shall sometimes denote one element of M by ∞, write F for the complement of $\{\infty\}$ in M, and then choose two distinct elements 0 and 1 of F. We shall use the field operations of addition and multiplication defined in Theorem (4.11) as well as the matrix notation $\begin{pmatrix} & \\ & \end{pmatrix}$ for elements of the set $\Gamma(M)$ defined there, which by (8.6.iv), is evidently just the set $\{\overset{\boxed{a}}{\rho}_x : x \in L\}$. Such a choice of 0, 1, and ∞ will be called **choice of a basis for M**.

From (8.3) we know that M is a meridian where

$$\Pi(M) = \{\overset{\boxed{a}}{\rho}_x : x \in a^{\square}\} \quad \text{ and } \quad \Gamma(M) = \{\overset{\boxed{a}}{\rho}_x : x \in L\}. \tag{3}$$

(8.9) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L, let ρ be a characteristic representation on L on $M \times N$, and let $\overset{\boxed{a}}{\rho}$ be the corresponding a-representation. Then

(i) $(\forall\, a, b, c, u, v, w \in M : \#\{a, b, c\} = \#\{u, v, w\} = 3)(\exists!\, x \in L) \quad \overset{\boxed{a}}{\rho}_x(a) = u,\ \overset{\boxed{a}}{\rho}_x(b) = v$ and $\overset{\boxed{a}}{\rho}_x(c) = w$;

(ii) $(\forall\, a, b, c \in M, r, s, t \in N : \#\{a, b, c\} = \#\{r, s, t\} = 3)(\exists!\, x \in L) \quad \rho_x(a) = r,\ \rho_x(b) = s$ and $\rho_x(c) = t$;

(iii) $(\forall\, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in |||$ distinct$)(\forall\, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \in \equiv$ distinct$)(\exists!\, x \in L) \quad x^{\widehat{(\mathcal{T})}}(\mathcal{A}_i) = \mathcal{B}_i \quad$ for $i = 1, 2, 3$.

(iv) $(\forall\, A, B, C \in \mathcal{T}$ pairwise skew$) \quad A \cap B \cap C$ is a singleton.

Proof. $\overset{\text{(i)}}{\Longrightarrow}$: Follows from the fundamental theorem (2.12) applied to (8.3).

$\overset{\text{(ii)}}{\Longrightarrow}$: Let $[u, v, w] \equiv [r, s, t]$ and apply (i).

$\overset{\text{(iii)}}{\Longrightarrow}$: Follows from the fact that ρ and the left inner representation are equivalent.

$\overset{\text{(iv)}}{\Longrightarrow}$: Let $A, B, C \in \mathcal{T}$ be pairwise skew. Then $\|A\|$, $\|B\|$ and $\|C\|$ are distinct, as are $\overline{\overline{\underline{\underline{A}}}}$, $\overline{\overline{\underline{\underline{B}}}}$ and $\overline{\overline{\underline{\underline{C}}}}$. By (iii) there is a unique $x \in L$ such that $x^{\widehat{(\mathcal{T})}}(\|A\|) = \overline{\overline{\underline{\underline{A}}}}$, $x^{\widehat{(\mathcal{T})}}(\|B\|) = \overline{\overline{\underline{\underline{B}}}}$ and $x^{\widehat{(\mathcal{T})}}(\|C\|) = \overline{\overline{\underline{\underline{C}}}}$. By Theorem (6.19.ii) this implies that $x \in A \cap B \cap C$. If any other $y \in L$ were in $A \cap B \cap C$, then we would have $y^{\widehat{(\mathcal{T})}}(\|A\|) = \overline{\overline{\underline{\underline{A}}}}$, $y^{\widehat{(\mathcal{T})}}(\|B\|) = \overline{\overline{\underline{\underline{B}}}}$ and $y^{\widehat{(\mathcal{T})}}(\|C\|) = \overline{\overline{\underline{\underline{C}}}}$, which would violate the uniqueness of x in this respect. QED

(8.10) Lemma Let $\mathcal{T}$ be a meridian aggregate of a libra L, and let $\overset{\boxed{a}}{\rho}$ be an a-representation for $a \in L$. Let $b \in L$ be distinct from a. Then there exists a choice of basis and $q, r \in F$ such that, if

$$\mathcal{A} \equiv \{\begin{pmatrix} e & rd \\ qd & e \end{pmatrix} : e, d \in F \text{ and } e^2 \neq qrd^2\} \text{ and } \mathcal{B} \equiv \{\begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix} : e, d \in F \text{ and } e^2 \neq qrd^2\},$$

then

$$(\forall\, u, v \in \mathcal{A} \text{ distinct}) \quad \{u, v\}^{\circ} = \mathcal{B}, \tag{1}$$

$$(\forall\, g, h \in \mathcal{B} \text{ distinct}) \quad \{g, h\}^{\circ} = \mathcal{A}, \tag{2}$$

$$\mathcal{A}^{\circ} = \mathcal{B} \text{ and } \mathcal{B}^{\circ} = \mathcal{A}. \tag{3}$$

Furthermore, these choices can be made such that

$$\overset{\boxed{a}}{\rho}_b = \begin{pmatrix} 1 & r \\ q & 1 \end{pmatrix} \text{ if } b \notin a^{\square} \text{ and } \overset{\boxed{a}}{\rho}_b = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix} \text{ if } b \in a^{\square}. \tag{4}$$

Proof. Suppose first that $b \in a^{\square}$. Let 0 be any element of M and set $\infty \equiv \rho_b(0)$. Let 1 be any third element and define $q \equiv 1$ and $r \equiv \rho_b(1)$.

$\overset{(4)}{\Longrightarrow}$ Now suppose that $b \notin a^{\square}$. It follows from (4.20) that there exists $\pi \in \Pi(M)$ with fixed points and $\sigma \in \Pi(M)$ such that $\overset{\boxed{a}}{\rho}_b = \pi \circ \sigma$. Let 0 and ∞ be the fixed points of π, and let 1 be any third element of M.

Then $\pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since σ is self-inverse, there exist $i, j, k \in F$ such that $\sigma = \begin{pmatrix} i & j \\ k & -i \end{pmatrix}$. We have

$$\overset{\boxed{@}}{\rho}_b = \pi \circ \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circ \begin{pmatrix} i & j \\ k & -i \end{pmatrix} = \begin{pmatrix} i & j \\ -k & i \end{pmatrix}.$$

The hypothesis that $b \notin a^{\square}$ insures that i cannot be 0. We let $q \equiv \frac{-k}{i}$ and $r \equiv \frac{j}{i}$. It is now evident that (4) holds.

$\overset{(1)}{\Longrightarrow}$ Whenever $u, v \in \mathcal{A}$, direct calculation shows that

$$\mathcal{B} \subset \{u, v\}^{\circ}. \tag{5}$$

Let $u, v \in \mathcal{A}$ be distinct and choose $d, e, s, t \in F$ distinct such that

$$\overset{\boxed{@}}{\rho}_u = \begin{pmatrix} e & rd \\ qd & e \end{pmatrix} \quad \text{and} \quad e^2 \neq qrd^2, \tag{6}$$

$$\overset{\boxed{@}}{\rho}_v = \begin{pmatrix} t & rs \\ qs & t \end{pmatrix} \quad \text{and} \quad t^2 \neq qrs^2. \tag{7}$$

Let p be a member of $\{u, v\}^{\circ}$ and choose $w, x, y, z \in M$ such that

$$\overset{\boxed{@}}{\rho}_p = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad \text{and} \quad wz \neq xy. \tag{8}$$

We have

$$\lfloor u, p, u \rfloor = p \Longleftrightarrow \begin{pmatrix} e & rd \\ qd & e \end{pmatrix} \circ \begin{pmatrix} w & x \\ y & z \end{pmatrix}^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \circ \begin{pmatrix} e & rd \\ qd & e \end{pmatrix}^{-1} \Longleftrightarrow$$

$$\begin{pmatrix} ez - rdy & rdw - ex \\ qdz - ey & ew - qdx \end{pmatrix} = \begin{pmatrix} ew - qdx & -rdw + ex \\ -qdz + ey & ez - rdy \end{pmatrix}. \tag{9}$$

Equation (9) implies that the matrix $\begin{bmatrix} ez - rdy & rdw - ex \\ qdz - ey & ew - qdx \end{bmatrix}$ is a non-0 multiple k of the matrix $\begin{bmatrix} ew - qdx & -(rdw - ex) \\ -(qdz - ey) & ez - rdy \end{bmatrix}$. If $k = -1$, then

$$ez - rdy = -ew + qdx \tag{10}$$

and if $k \neq -1$, then

$$rdw - ex = 0 = qdz - ey \quad \text{and} \quad ez - rdy = ew - qdx. \tag{11}$$

If $e = 0$, then (11) and (6) imply that $w = 0 = z$ and $ry = qx$: thus

$$\overset{\boxed{@}}{\rho}_p = \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & \frac{ry}{q} \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & ry \\ qy & 0 \end{pmatrix} = \begin{pmatrix} e & rd \\ qd & e \end{pmatrix} = \overset{\boxed{@}}{\rho}_u. \tag{12}$$

If $e \neq 0$, then (11) and (6) imply

$$x = r\frac{d}{e}w,\ \ y = q\frac{d}{e}z,\ \ z = r\frac{d}{e}y + w - q\frac{d}{e}x \Longrightarrow z = rq\left(\frac{d}{e}\right)^2 z + w - qr\left(\frac{d}{e}\right)^2 w \Longrightarrow$$

$$\left(1 - \frac{d^2}{e^2}rq\right)z = \left(1 - \frac{d^2}{e^2}qr\right)w \overset{\text{by (6)}}{\Longrightarrow} z = w \Longrightarrow x = rdy \Longrightarrow$$

$$\overset{\boxed{@}}{\rho}_p = \begin{pmatrix} w & r\frac{d}{e}w \\ q\frac{d}{e}w & w \end{pmatrix} = \begin{pmatrix} e & rd \\ qd & e \end{pmatrix} = \overset{\boxed{@}}{\rho}_u. \tag{13}$$

Since u is not in $\{u, v\}^{\circ}$ and p is, it follows from (12) and (13) that (11) cannot hold. It follows that (10) must hold, and so

$$e(z + w) = d(qx + ry) \tag{14}$$

holds. An analogous argument, using v instead of u, yields

$$t(z + w) = s(qx + ry). \tag{15}$$

From (14) and (15) follows that either

$$z + w = 0 = qx + ry \tag{16}$$

or

$$\frac{e}{t} = \frac{d}{s} \tag{17}$$

If (17) held, then

$$\overset{\boxed{@}}{\rho}_u = \begin{pmatrix} e & rd \\ qd & e \end{pmatrix} = \begin{pmatrix} \frac{dt}{s} & rd \\ qd & \frac{dt}{s} \end{pmatrix} = \begin{pmatrix} t & rs \\ qs & t \end{pmatrix} = \overset{\boxed{@}}{\rho}_v$$

which is absurd since $\overset{\boxed{@}}{\rho}$ is faithful. It follows that (16) holds and so

$$\overset{\boxed{@}}{\rho}_p = \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & -\frac{ry}{q} \\ y & -w \end{pmatrix} = \begin{pmatrix} qw & -ry \\ qy & -qw \end{pmatrix},$$

which in turn implies that p is in $\mathcal{B}$. Thus $\{u, v\}^\circ \subset \mathcal{B}$. This, with (5), establishes (1).

$\overset{(2)}{\Longrightarrow}$ Whenever $g, h \in \mathcal{B}$, direct calculation shows that

$$\mathcal{A} \subset \{g, h\}^\circ. \tag{18}$$

Let $g, h \in \mathcal{B}$ be distinct and choose $d, e, s, t \in F$ distinct such that

$$\overset{\boxed{@}}{\rho}(g) = \begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix} \quad \text{and} \quad -e^2 \neq qrd^2, \tag{19}$$

$$\overset{\boxed{@}}{\rho}(h) = \begin{pmatrix} t & -rd \\ qs & -t \end{pmatrix} \quad \text{and} \quad -t^2 \neq qrs^2. \tag{20}$$

Let p be a member of $\{g, h\}^\circ$ and choose $w, x, y, z \in M$ such that

$$\overset{\boxed{@}}{\rho}_p = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad \text{and} \quad wz \neq xy. \tag{21}$$

$$\lfloor g, p, g \rfloor = p \iff \begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix} \circ \begin{pmatrix} w & x \\ y & z \end{pmatrix}^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \circ \begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix}^{-1} \iff$$

$$\begin{pmatrix} ez + rdy & -ex - rdw \\ qdz + ey & -qdx - ew \end{pmatrix} = \begin{pmatrix} qdx + ew & -ex - rdw \\ qdz + ey & -ez - rdy \end{pmatrix} \tag{22}$$

Equation (22) implies that the matrix $\begin{bmatrix} ez + rdy & -ex - rdw \\ qdz + ey & -qdx - ew \end{bmatrix}$ is a non-0 multiple k of the matrix $\begin{bmatrix} qdx + ew & -ex - rdw \\ qdz + ey & -ez - rdy \end{bmatrix}$. If $k = 1$, then

$$ez + rdy = qdx + ew \tag{23}$$

and if $k \neq 1$, then

$$-ex - rdw = 0 = qdz + ey \quad \text{and} \quad ez + rdy = -qdx - ew. \tag{24}$$

If $e = 0$, then (24) and (19) imply that $w = 0 = z$ and $ry = -qx$: thus

$$\overset{\boxed{@}}{\rho}_p = \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & -\frac{ry}{q} \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & -ry \\ qy & 0 \end{pmatrix} = \begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix} = \overset{\boxed{@}}{\rho}_g. \tag{25}$$

If $e \neq 0$, then (24) and (19) imply

$$x = -r\frac{d}{e}w,\ y = -q\frac{d}{e}z,\ z = -r\frac{d}{e}y - w - q\frac{d}{e}x \Longrightarrow z = rq(\frac{d}{e})^2 z - w + qr(\frac{d}{e})^2 w \Longrightarrow$$

$$(1 - \frac{d^2}{e^2}rq)z = -(1 - \frac{d^2}{e^2}qr)w \xrightarrow{\text{by (19)}} z = -w \Longrightarrow x = -rdy \Longrightarrow$$

$$\overset{\boxed{@}}{\rho}_p = \begin{pmatrix} w & -r\frac{d}{e}w \\ q\frac{d}{e}w & -w \end{pmatrix} = \begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix} = \overset{\boxed{@}}{\rho}_g. \tag{26}$$

Since u is not in $\{u, v\}^\circ$ and p is, it follows from (25) and (26) that (24) cannot hold. It follows that (23) must hold, and so

$$e(z - w) = d(qx - ry) \tag{27}$$

holds. An analogous argument, using h instead of g, yields

$$t(z-w)=s(qx-ry). \tag{28}$$

From (27) and (28) follows that either

$$z-w=0=qx-ry \tag{29}$$

or

$$\frac{e}{t}=\frac{d}{s} \tag{30}$$

If (30) held, then

$$\overset{\boxed{@}}{\rho}_g=\begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix}=\begin{pmatrix} \frac{dt}{s} & -rd \\ qd & -\frac{dt}{s} \end{pmatrix}=\begin{pmatrix} t & -rs \\ qs & -t \end{pmatrix}=\overset{\boxed{@}}{\rho}_h$$

which is absurd since $\overset{\boxed{@}}{\rho}$ is faithful. It follows that (29) holds and so

$$\overset{\boxed{@}}{\rho}_p=\begin{pmatrix} w & x \\ y & z \end{pmatrix}=\begin{pmatrix} w & \frac{ry}{q} \\ y & w \end{pmatrix}=\begin{pmatrix} qw & ry \\ qy & qw \end{pmatrix},$$

which in turn implies that p is in $\mathcal{B}$. Thus $\{g,h\}^\circ\subset\mathcal{A}$. This, with (18), establishes (2).

$\overset{(3)}{\Longrightarrow}$ That $\mathcal{B}\subset\mathcal{A}^\circ$ follows from direct computation. Let g be in $\mathcal{A}^\circ$. Since a and b are in $\mathcal{A}$, it follows that g is then in $\{a,b\}^\circ$, which by (1) is just $\mathcal{B}$. Thus $\mathcal{A}^\circ=\mathcal{B}$. That $\mathcal{B}^\circ=\mathcal{A}$ is proved analogously. QED

(8.11) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let a and b be distinct elements of L. Then

(i) $\{a,b\}^\circ$ has at least three elements;
(ii) $\{a,b\}^\circ$ is balanced;
(iii) $(\forall\, x,y\in\{a,b\}^\circ$ distinct$)\quad \{x,y\}^\circ=\{a,b\}^{\circ\circ}$;
(iv) $(\forall\, c,d\in\{a,b\}^{\circ\circ}$ distinct$)\quad \{c,d\}^\circ=\{a,b\}^\circ$.

Proof. $\overset{(i)}{\Longrightarrow}$: Let 0, 1, ∞, $\mathcal{A}$, $\mathcal{B}$, q and r be as in Lemma (8.10). Then both $\overset{\boxed{@}}{\rho}_a=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\overset{\boxed{@}}{\rho}_b$ are in $\mathcal{A}$ and distinct. It follows from (8.10.1) that

$$\{a,b\}^\circ=\mathcal{B}. \tag{1}$$

We may and shall presume that $-1\neq qr\neq 1$. It follows that $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -r \\ q & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & r \\ -q & 1 \end{pmatrix}$ are distinct elements of $\mathcal{B}$. This with (1) establishes (i).

$\overset{(ii)}{\Longrightarrow}$: Follows from (1) and direct computation.

$\overset{(iv)}{\Longrightarrow}$: Let x and y be distinct elements of $\{a,b\}^\circ$. We have

$$\{a,b\}^\circ \overset{\text{by (8.10.1)}}{=\!=\!=\!=} \mathcal{B}\Longrightarrow\{a,b\}^{\circ\circ}=\mathcal{B}^\circ \overset{\text{by (8.10.3)}}{=\!=\!=\!=} \mathcal{A} \overset{\text{by (8.10.2)}}{=\!=\!=\!=} \{x,y\}^\circ$$

which is (iii).

$\overset{(iv)}{\Longrightarrow}$: In (iii) we now replace x and y by c and d, and then replace a and b by x and y: which yields that $\{c,d\}^\circ=\{x,y\}^{\circ\circ}$. Hence $\{c,d\}^\circ=\{a,b\}^\circ$, which is (iv). QED

(8.12) Definition and Notation For distinct a and b in a meridian libra L we shall denote the set $\{a,b\}^{\circ\circ}$ by $\overline{a,b}$ and shall refer to it as a **line trace**. We adopt the notation

$$\mathcal{L}(L)\equiv\{\overline{a,b}:\ a,b\in L \text{ distinct}\}.$$

(8.13) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let K be an element of $\mathcal{L}(L)$. Then

(i) K° is in $\mathcal{L}(L)$;
(ii) $K^{\circ\circ}=K$;
(iii) $K\cap K^\circ=\emptyset$;
(iv) $K\cup K^\circ$ is balanced;
(v) if $x,y\in K$ are distinct, then $\overline{x,y}=K$.

Proof. $\overset{(i)}{\Longrightarrow}$: Let a and b be distinct and such that $\{a,b\}^{\circ\circ}=K$. By (8.11.i) and (8.11.iii) there exist distinct $x, y \in \{a,b\}^{\circ}$ such that $\{a,b\}^{\circ\circ} = \{x,y\}^{\circ}$. Thus

$$\overline{x,y} = \{x,y\}^{\circ\circ} = \{a,b\}^{\circ\circ\circ} = K^{\circ}$$

which establishes (i).

$\overset{\text{(ii)}}{\Longrightarrow}$: That (ii) holds follows from (7.3.iv).

$\overset{\text{(iii)}}{\Longrightarrow}$: If a were in $K \cap K^{\circ}$ there would be another element b of K and we could apply (8.10) to obtain disjoint line traces $\mathcal{A} = K$ and and $\mathcal{B} = K^{\circ\circ}$: which is absurd.

$\overset{\text{(iv)}}{\Longrightarrow}$: That (iv) holds follows from Theorem (7.4).

$\overset{\text{(v)}}{\Longrightarrow}$: Let x and y be distinct elements of K. We have

$$x, y \in K \overset{\text{by (ii)}}{=\!=\!=} K^{\circ\circ} \xrightarrow{\text{by (8.11.iv)}} \{x,y\}^{\circ} = K^{\circ} \Longrightarrow \overline{x,y} = \{x,y\}^{\circ\circ} = K^{\circ\circ} \overset{\text{by (8.11.ii)}}{=\!=\!=} K,$$

which establishes (v). QED

(8.14) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let a be an element of L and B an element of $\mathcal{T}$. Then

(i) $B \cap a^{\square} \in \mathcal{L}(L)$;

(ii) $a \in B \Longleftrightarrow (B \cap a^{\square})^{\circ} \subset B$.

Proof. There are two cases to consider: either $B = [0 \overset{\overset{\boxed{@}}{\rho}}{=} 0]$ for some $0 \in M$, or $B = [0 \overset{\overset{\boxed{@}}{\rho}}{=} \infty]$ for distinct 0 and ∞ in M.

Case I: $B = [0 \overset{\overset{\boxed{@}}{\rho}}{=} 0]$ for some $0 \in M$. Choose 1 and ∞ in M distinct from each other and from 0, and let $F \equiv \{x \in M : x \neq \infty\}$. Since $\overset{\boxed{@}}{\rho}_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\{\overset{\boxed{@}}{\rho}_x : x \in B \cap a^{\square}\} = \{\begin{pmatrix} 1 & 0 \\ r & -1 \end{pmatrix} : r \in F\}. \tag{1}$$

Let $u, v \in B \cap a^{\square}$ satisfy $\overset{\boxed{@}}{\rho}_u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\overset{\boxed{@}}{\rho}_v = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. For $w, x, y, z \in F$ we have

$$[\![\overset{\boxed{@}}{\rho}_u, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \overset{\boxed{@}}{\rho}_u]\!] = \begin{pmatrix} z & x \\ y & w \end{pmatrix}$$

which implies that

$$[\![\overset{\boxed{@}}{\rho}_u, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \overset{\boxed{@}}{\rho}_u]\!] = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \Longleftrightarrow (\exists\, k \in F : \ k \neq 0) \quad \begin{pmatrix} z & x \\ y & w \end{pmatrix} = k \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix};$$

and, similarly,

$$[\![\overset{\boxed{@}}{\rho}_v, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \overset{\boxed{@}}{\rho}_v]\!] = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \Longleftrightarrow (\exists\, k \in F : \ k \neq 0) \quad \begin{pmatrix} z - x & x \\ z + y - x - w & x + w \end{pmatrix} = k \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

which implies that

$$\{u, v\}^{\circ} = \{t \in L : (\exists\, r \in F) \quad \overset{\boxed{@}}{\rho}_t = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\}. \tag{2}$$

It follows from (1) and (2) that $\overline{u,v} = B \cap a^{\square}$. Furthermore a is in B and, from (2),

$$(B \cap a^{\square})^{\circ} = \{u, v\}^{\circ} \subset [0 \overset{\overset{\boxed{@}}{\rho}}{=} 0] = B.$$

Thus (i) and (ii) hold for Case I.

Case II: $B = [0 \overset{\overset{\boxed{@}}{\rho}}{=} \infty]$ for $0, \infty \in M$ distinct. Choose $1 \in M$ distinct from 0 and ∞ and let $F \equiv \{x \in M : x \neq \infty\}$. We have

$$\{\overset{\boxed{@}}{\rho}_x : x \in B \cap a^{\square}\} = \{\begin{pmatrix} 0 & r \\ 1 & 0 \end{pmatrix} : r \in F\}. \tag{3}$$

Let $u, v \in B \cap a^{\square}$ satisfy $\overset{\boxed{@}}{\rho}_u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\overset{\boxed{@}}{\rho}_v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For $w, x, y, z \in F$ we have

$$\llbracket \overset{\boxed{@}}{\rho}_u, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \overset{\boxed{@}}{\rho}_u \rrbracket = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \Longleftrightarrow (\exists\, k \in F: \ k \neq 0) \quad \begin{pmatrix} w & -y \\ -x & z \end{pmatrix} = k \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

and

$$\llbracket \overset{\boxed{@}}{\rho}_v, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \overset{\boxed{@}}{\rho}_v \rrbracket = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \Longleftrightarrow (\exists\, k \in F: \ k \neq 0) \quad \begin{pmatrix} -w & -y \\ -x & -z \end{pmatrix} = k \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

which implies

$$\{u, v\}^{\circ} = \{t \in L : (\exists\, r \in F) \quad \overset{\boxed{@}}{\rho}_t = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}\}. \tag{4}$$

It follows from (1) and (2) that $\overline{u, v} = B \cap a^{\square}$. Furthermore a is not in B but is in $(B \cap a^{\square})^{\circ}$. Thus (i) and (ii) hold for Case II. QED

(8.15) Corollary Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let K be an element of $\mathcal{L}(L)$ and let A be an element of $\mathcal{T}$. If $K \cap A$ has at least two elements, then $K \subset A$.

Proof. Let a be in K° and let b, c be distinct elements of $K \cap A$. From (8.14)(i) follows that $A \cap a^{\square}$ is in $\mathcal{L}(L)$, and from (8.13.v) follows that $K = \overline{b, c} = A \cap a^{\square}$. QED

(8.16) Corollary Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let A be an element of $\mathcal{T}$. Then $A^{\circ} = \emptyset$.

Proof. Assume that a is in A°. It follows that $A \cap a^{\square} = A$, and so (8.14.i) implies that A is in $\mathcal{L}(L)$. From (8.10.iii) follows that $a \notin A$. Let b be in A. Then $\lfloor a, b, A \rfloor$ contains a and so is not A. From (7.2) we know that A is abelian. For $c, d \in A$ we have

$$\lfloor d, \lfloor a, b, c \rfloor, d \rfloor = \lfloor \lfloor d, c, b \rfloor, a, d \rfloor = \lfloor \lfloor b, c, d \rfloor, a, d \rfloor = \lfloor b, c, d, d, a \rfloor = \lfloor b, c, a \rfloor = \lfloor a, b, c \rfloor$$

which means that $d \in \lfloor a, b, A \rfloor^{\circ}$. This implies that $A \subset \lfloor a, b, A \rfloor^{\circ}$. Since left translates are pairwise disjoint, it follows that $A = \lfloor a, b, A \rfloor$, which is absurd. QED

(8.17) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let A and B be distinct skew elements of $\mathcal{T}$. Then

(i) $A \cap B$ is in $\mathcal{L}(L)$;

(ii) $(A \cap B)^{\circ} = (\|A\| \wedge \overline{\overline{\underline{\underline{B}}}}) \cap (\|B\| \wedge \overline{\overline{\underline{\underline{A}}}})$.

Proof. Let a be in A. There exist $\infty \in M$ and $n \in N$ such that $A = [\infty \overset{\rho}{=} n]$. We have $\rho_a(\infty) = n$ so

$$\overset{\boxed{@}}{\rho}_a(\infty) = ({\rho_a}^{-1} \circ \rho_a)(\infty) = ({\rho_a}^{-1})(n) = \infty \Longrightarrow A = [\infty \overset{\overset{\boxed{@}}{\rho}}{=} \infty].$$

There exist $0, 1 \in M$ such that $B = [0 \overset{\overset{\boxed{@}}{\rho}}{=} 1]$. Since A and B are skew, neither 0 nor 1 can be ∞. We have

$$(\forall\, x \in A \cap B)(\exists\, r \in F) \quad \overset{\boxed{@}}{\rho}(x) = \begin{pmatrix} r & 1 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Let r and s in L satisfy

$$\overset{\boxed{@}}{\rho}_r = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \overset{\boxed{@}}{\rho}_s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Direct calculations show that a necessary and sufficient condition for $x \in L$ to be in $r^{\square} \cap s^{\square}$ is for there to be $u \in F$ such that

$$\overset{\boxed{@}}{\rho}_x = \begin{pmatrix} 1 & u \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Thus $\overline{r, s} = \{x \in L : (1) \text{ holds}\}$ and

$$\overline{r, s}^{\circ} = \{x \in L : (2) \text{ holds}\}. \tag{3}$$

The first of these two equalities implies that $\overline{r, s} = A \cap B \in \mathcal{L}(L)$. The set of $x \in L$ satisfying equation (2) is just $[\infty \overset{\overset{\boxed{@}}{\rho}}{=} 1] \cap [0 \overset{\overset{\boxed{@}}{\rho}}{=} \infty] = (\|A\| \wedge \overline{\overline{\underline{\underline{B}}}}) \cap (\|B\| \wedge \overline{\overline{\underline{\underline{A}}}})$. From this and equation (3) follows that

$$(A \cap B)^{\circ} = (\|A\| \wedge \overline{\overline{\underline{\underline{B}}}}) \cap (\|B\| \wedge \overline{\overline{\underline{\underline{A}}}}).$$

QED

(8.18) Corollary Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let A and B be skew elements of $\mathcal{T}$. Then

$$\{x \in L\colon\, B = \lfloor x, A, x\rfloor\} = (A \cap B)^{\circ}. \tag{1}$$

Proof. This follows from (6.19.i) and (8.17.ii). QED

(8.19) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let K be an element of $\mathcal{L}(L)$. Then the following four statements are pairwise equivalent:

(i) $(\exists\, A \in \mathcal{T})\quad K \cup K^{\circ} \subset A$;

(ii) $(\exists\, A \in \mathcal{T})\quad K \subset A$ and $K^{\circ} \cap A \neq \emptyset$;

(iii) $(\exists!\, B \in \mathcal{T})\quad K \subset B$;

(iv) $(\exists\, A \in \mathcal{T})\quad \{E \in \mathcal{T}\colon\, K \cap E = \emptyset\} = \{D \in \overline{\overline{\underline{\underline{A}}}} \cup \|A\|\colon\, D \neq A\}$.

If these four statements hold, then

(v) $(\forall\, n \in K)\quad K \cap n^{\square} = \emptyset$.

Proof. (i)$\Longrightarrow$(ii): Trivial.

(ii)$\Longrightarrow$(iii): Suppose that (ii) holds and that a is in $K^{\circ} \cap A$. We have $K \subset a^{\square} \cap A$ and so by Theorem (8.14.i), $K = a^{\square} \cap A$. If $\lfloor a, A, a\rfloor$ were not A, they obviously would be a skew pair and so Theorem (8.17.i) would imply that $K = A \cap \lfloor a, A, a\rfloor$, which would imply that $a \in K \cap K^{\circ}$, which would violate (8.13.iii). It follows that $A = \lfloor a, A, a\rfloor$. Thus, if B were any element of $\mathcal{T}_K$ distinct from A, then B and A would be a-skew, and so Theorem (8.6)(ii) would imply that $A \cap B \cap a^{\square}$ were a singleton: an absurdity. Thus (iii) holds.

(iii)$\Longrightarrow$(iv) and (i): Suppose that (iii) holds. Let 0, 1, ∞, q and r be as in Lemma (8.10). Since a is in B and $\overset{\boxed{@}}{\rho}(a)$ is the identity mapping, there exists $m \in M$ such that $B = [m \overset{\overset{\boxed{@}}{\rho}}{=} m]$. If m were in F, then for all $e, d \in M$ such that $e^2 \neq qrd^2$, $\begin{pmatrix} e & rd \\ qd & e \end{pmatrix}(m) = m$. This is evidently impossible, so m must be ∞. It follows that $q = 0 \neq r$. That

$$\{D \in \overline{\overline{\underline{\underline{B}}}} \cup \|B\|\colon\, D \neq B\} \subset \{E \in \mathcal{T}\colon\, K \cap E = \emptyset\}$$

follows from he fact that $\overline{\overline{\underline{\underline{B}}}}$ and $\|B\|$ are partitions of L. Let $E \in \mathcal{T}$ have void intersection with K, and assume that E is in neither $\|B\|$ nor $\overline{\overline{\underline{\underline{B}}}}$. Then there exist $j, k \in F$ such that $E = [j \overset{\overset{\boxed{@}}{\rho}}{=} k]$. But

$$\begin{pmatrix} 1 & \frac{k-j}{r} r \\ \frac{k-j}{r} q & 1 \end{pmatrix}(j) = \begin{pmatrix} 1 & k-j \\ 0 & 1 \end{pmatrix}(j) = k \Longrightarrow K \cap E \neq \emptyset :\ \text{a contradiction.}$$

It follows that (iv) holds. It follows from Lemma (8.10) that the image by $\overset{\boxed{@}}{\rho}$ of anything in K° is of the form $\begin{pmatrix} e & -dr \\ 0 & -e \end{pmatrix}$ and so it is evidently in $[\infty \overset{\overset{\boxed{@}}{\rho}}{=} \infty] = B$. Thus (i) holds.

(iv)$\Longrightarrow$(iii): trivial.

(i)$\Longrightarrow$(v): Suppose that (i) holds. Since $\|A\|$ and $\overline{\overline{\underline{\underline{A}}}}$ are partitions of L, it is trivial that

$$\{D \in \overline{\overline{\underline{\underline{A}}}} \cup \|A\|\colon\, D \neq A\} \subset \{E \in \mathcal{T}\colon\, K \cap E = \emptyset\} \qquad (1).$$

Suppose that $B \in \mathcal{T}$ is neither a left nor a right translate of A. Let a be an element of K°. Since a is in A, we have $A = \lfloor a, A, a\rfloor$ which implies that A and B are not a-skew. By (8.6.ii) we know that $a^{\square} \cap A \cap B$ is a singleton. But $a^{\square} \cap A = K$, which implies that $K \cap B$ is a singleton. Thus the set containment symbol in (1) can be replaced by an equals symbol. Thus (v) holds. QED

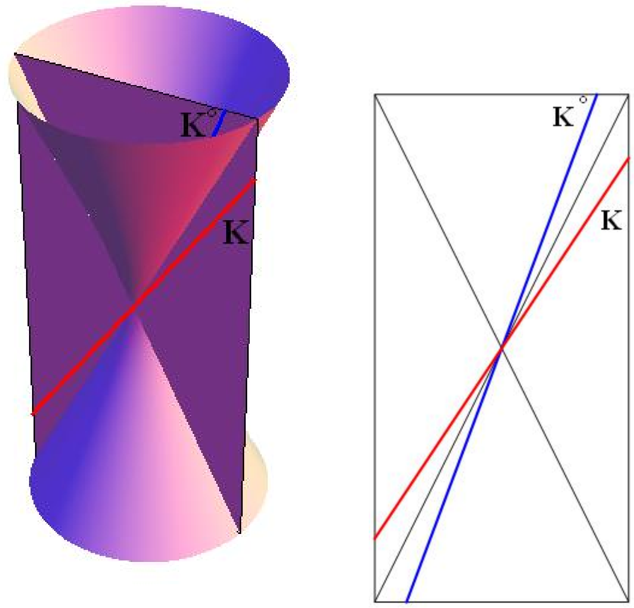

Fig. 19: A Parabolic Line **K** and its Polar Showing the Rules of their Plane

(8.20) Definition We shall say than an element of $\mathcal{L}(L)$ which satisfies (8.19.iii) is **parabolic**. If $K \in \mathcal{L}(L)$ is contained in more than one element of $\mathcal{T}$ we shall say that it is **elliptic**. If $K \in \mathcal{L}(L)$ is neither parabolic nor elliptic, we shall say that it is **hyperbolic**.

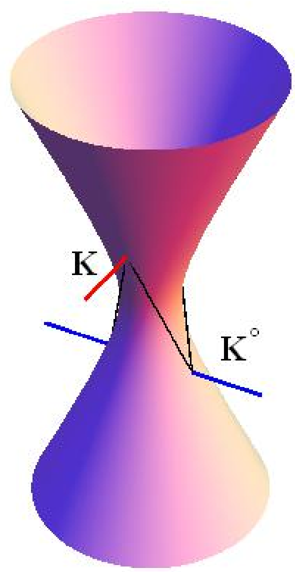

Fig. 20: An Elliptic Line **K** and its Polar

(8.21) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. For $K \in \mathcal{L}(L)$, the following are pairwise equivalent statements:

(i) K is elliptic;

(ii) $\#\{A \in \mathcal{T} : K \subset A\} = 2$;

(iii) $(\exists\, A, B \in \mathcal{T}) \quad K = A \cap B$;
(iv) K° is elliptic;
(v) $(\exists\, A, B \in \mathcal{T}$ skew$) \quad K = \{x \in L : \lfloor x, A, x \rfloor = B\}$.

Proof. (i)$\Longleftrightarrow$(ii): Suppose that (i) holds. Assume that $A, B, C \in \mathcal{T}$ are distinct and that K is a subset of each. Since $\|A\|$ is a partition of L, A is its only member with L as a subset. Consequently neither B nor C is in $\|A\|$, whence follows that $\|A\|$ cannot equal $\|B\|$ nor can it equal $\|C\|$. Similarly, $\|B\|$ cannot equal $\|C\|$. Since $\boxed{\;}\mathcal{L}$ is a meridian, it follows from the Fundamental Theorem that there exists exactly one element x of L such that $x^{\widehat{\textcircled{$\scriptstyle\mathcal{T}$}}}(\|A\|) = \overline{\overline{\underline{\underline{A}}}}$, $x^{\widehat{\textcircled{$\scriptstyle\mathcal{T}$}}}(\|B\|) = \overline{\overline{\underline{\underline{B}}}}$, and $x^{\widehat{\textcircled{$\scriptstyle\mathcal{T}$}}}(\|C\|) = \overline{\overline{\underline{\underline{C}}}}$. Evidently each $k \in K$ can serve for x in these last three equalities, and by (8.11.i) K has at least three elements: an absurdity. Thus (ii) holds. That (ii) implies (i) is trivial.

(ii)$\Longrightarrow$(iii): Suppose that (ii) holds and that A and B are the elements of $\mathcal{T}$ of which K is a subset. From Theorem (8.17.i) we know that $A \cap B$ is a line trace and so, by (8.11.iv), it must be K. Thus (iii) holds.

(iii)$\Longrightarrow$(i): trivial.

(i)$\Longleftrightarrow$(iv): Suppose again that (i) holds, and let A and B be the elements of $\mathcal{T}_K$. From Theorem (8.17) we know that $K^\circ = (\|A\| \cap \overline{\overline{\underline{\underline{B}}}}) \cap (\|B\| \cap \overline{\overline{\underline{\underline{A}}}})$. This implies that $\{\|A\| \cap \overline{\overline{\underline{\underline{B}}}}, \|B\| \cap \overline{\overline{\underline{\underline{A}}}}\} \subset \mathcal{T}K^\circ$. It follows that (iv) holds. Since $K = K^{\circ\circ}$, we also have that (iv) implies (i).

(i)$\Longrightarrow$(v): Suppose that (i) holds and that A and B are the elements of $\mathcal{T}_K$. Assume $K \cap C = \emptyset$ for $C \in \mathcal{T}$ but that $C \notin \mathcal{T}_K$. Let a be in K°. We saw above that $\lfloor a, A, a \rfloor = B$ and so the pair A and C are a-skew. It follows from Theorem (8.6.ii) that $a^{\square} \cap A \cap C$ is a singleton. But $a^{\square} \cap A$ contains K, and so by (8.14.i) is precisely K. It follows that the singleton $a^{\square} \cap A \cap C$ must be in K, which is absurd. We thus have

$$\{C \in \mathcal{T} : K \cap C = \emptyset\} \subset \{D \in \mathcal{T} : K^\circ \subset D\}. \tag{1}$$

On the other hand, if D in $\mathcal{T}_{K^\circ}$ contained an element of K, Theorem (8.19.ii) would imply that K were parabolic. It thus follows that the containment symbol in (1) may be replaced by an equality symbol. Consequently, (v) holds.

(iv)$\Longleftrightarrow$(v): From Theorem (6.19.ii) and Theorem (8.16.ii) follows that, for any skew $A, B \in \mathcal{T}$

$$\{x \in L : \lfloor x, A, x \rfloor = B\} = (\|A\| \wedge \overline{\overline{\underline{\underline{B}}}}) \cap (\|B\| \wedge \overline{\overline{\underline{\underline{A}}}}) = (A \cap B)^\circ.$$

Thus (v) is equivalent to the statement that $K = (A \cap B)^\circ$ for some $A, B \in \mathcal{T}$. If K° is elliptic, then $K^\circ = A \cap B$ for some some $A, B \in \mathcal{T}$ by (iii), which in turn implies that $K = K^{\circ\circ} = (A \cap B)^\circ$. On the other hand, if (vii) holds, then $K^\circ = (A \cap B)^{\circ\circ} = A \cap B$ and so K° is elliptic. QED

(8.22) Discussion Let K be an elliptic line trace. Then K° is elliptic too and so there are precisely two elements A and B of $\mathcal{T}$ containing K° as a subset: in fact we have $K^\circ = A \cap B$. Let $C \equiv \|A\| \wedge \overline{\overline{\underline{\underline{B}}}}$ and $D \equiv \|B\| \wedge \overline{\overline{\underline{\underline{A}}}}$. Then C and D are the two elements containing K as a subset: $K = C \cap D$. We have

$$\{X \in \mathcal{T} : K \cap X = \emptyset\} = \{X \in \|C\| \cup \overline{\overline{\underline{\underline{C}}}} \cup \|D\| \cup \overline{\overline{\underline{\underline{D}}}} : X \notin \{C, D\}\} = \{X \in \|A\| \cup \overline{\overline{\underline{\underline{B}}}} \cup \|C\| \cup \overline{\overline{\underline{\underline{C}}}} : X \notin \{C, D\}\}.$$

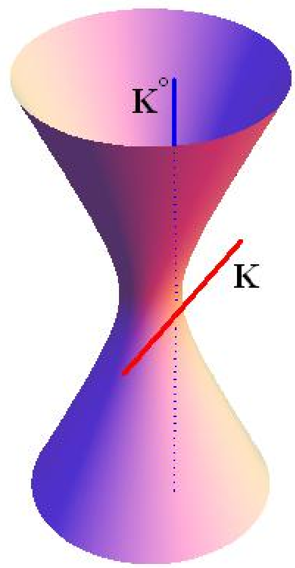

Fig. 21: A Hyperbolic Line **K** and its Polar

(8.23) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. For $K \in \mathcal{L}(L)$, the following are pairwise equivalent statements:

(i) K is hyperbolic;

(ii) K° is hyperbolic;

(iii) $(\forall\, A \in \mathcal{T}) \quad K \cap A \neq \emptyset$;

(iv) $(\forall\, A \in \mathcal{T}) \quad K \cap A$ is a singleton.

Proof. (i)$\Longleftrightarrow$(ii): Suppose the (i) holds. From (8.19.i) follows that K° cannot be parabolic. From (8.21.iii), we know that K° cannot be elliptic. Thus (ii) holds. That (ii) implies (i) follows now by interchanging the roles of K with K°.

(iii)$\Longrightarrow$(i). Suppose that (iii) holds. That K cannot be parabolic follows from (8.19.iv). That K cannot be elliptic follows from (8.21.iv). Thus (i) holds.

(i)$\Longrightarrow$(iv). Suppose that (i) holds. Let a and b be distinct points of K. Let 0 be any element of M and let $\infty \equiv \overset{@}{\rho}_b(0)$. Then there exist $r, s, l \in F$ such that $\overset{@}{\rho}_b = \begin{pmatrix} r & l \\ s & 0 \end{pmatrix}$. It follows that

$$K^\circ = \{x \in L : (\exists\, p, q \in F) \quad \overset{@}{\rho}_x = \begin{pmatrix} p & q \\ -qs - pr & -p \end{pmatrix}\}.$$

From this follows that

$$K = K^{\circ\circ} = \{x \in L : (\exists\, w, z \in F) \quad \overset{@}{\rho}_x = \begin{pmatrix} w & z \\ sz & w - rz \end{pmatrix}\}.$$

Let A be any element of $\mathcal{T}$. Then there exist $m, n \in M$ such that $A = [m \overset{\overset{@}{\rho}}{=} n]$. Evidently a is in $[m \overset{\overset{@}{\rho}}{=} n]$ whenever $m = n$. If any other $k \in K$ were in $[m \overset{\overset{@}{\rho}}{=} n]$ for $m = n$, then all of K would be as well, and so K would not be hyperbolic. Thus we may and shall suppose that $m \neq n$. If $m = \infty$ we set $z \equiv l$ and $w \equiv l + rn$ to obtain $x \in [m \overset{\overset{@}{\rho}}{=} n]$. If $n = \infty$ we set $w \equiv l$ and $z \equiv -m$ to obtain $x \in [m \overset{\overset{@}{\rho}}{=} n]$. If neither m nor n is ∞, we set $z \equiv m - n$ and $w \equiv sm - rn - l$ to obtain $x \in [m \overset{\overset{@}{\rho}}{=} n]$. It follows that (iv) holds.

(iv)$\Longrightarrow$(iii): Trivial. QED

(8.24) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let a be in L and b be in $a^{\square}$. Then
(i) $b^{\square} \cap a^{\square} \in \mathcal{L}(L)$;
(ii) $(\forall\, c \in b^{\square} \cap a^{\square}) \quad c^{\square} \cap b^{\square} \cap a^{\square}$ is a singleton $\{d\}$ and $b = \lfloor c, a, d \rfloor$;
(iii) if $\overline{a,b}$ is elliptic, $(\exists\, c \in b^{\square} \cap a^{\square})(\exists\, d \in c^{\square} \cap b^{\square} \cap a^{\square}) \quad \overline{a,c}$ is elliptic, $\overline{c,d}$ is elliptic and $b = \lfloor c, a, d \rfloor$;
(iv) if $\overline{a,b}$ is hyperbolic, $(\exists\, c \in b^{\square} \cap a^{\square})(\exists\, d \in c^{\square} \cap b^{\square} \cap a^{\square}) \quad \overline{a,c}$ is elliptic, $\overline{c,d}$ is hyperbolic and $b = \lfloor c, a, d \rfloor$.

Proof. Let $0 \in M$ be such that $\overset{\boxed{a}}{\rho}_b$ does not fix it. Let $\infty \equiv \overset{\boxed{a}}{\rho}_b(0)$ and choose $1 \in M$ distinct from 0 and ∞. Then there exists $e \in M$ such that
$$\overset{\boxed{a}}{\rho}_b = \begin{pmatrix} 0 & 1 \\ e & 0 \end{pmatrix} \tag{1}$$
It follows that
$$\{\overset{\boxed{a}}{\rho}_x : x \in a^{\square} \cap b^{\square}\} = \{\begin{pmatrix} x & y \\ -ey & -x \end{pmatrix} : x, y \in F\}. \tag{2}$$
This implies that
$$b^{\square} \cap a^{\square} = \overline{c,d} \quad \text{where } \overset{\boxed{a}}{\rho}_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \overset{\boxed{a}}{\rho}_d = \begin{pmatrix} 0 & 1 \\ -e & 0 \end{pmatrix}$$
which proves (i). That (ii) holds now follows from direct calculation.

If $\overline{a,b}$ is elliptic, then it is contained in two elements A and B of $\mathcal{T}$ and so in particular $a, b \in A \cap B$: there equal m and 1 distinct in M such that $\overset{\boxed{a}}{\rho}_b(1) = 1$ and $\overset{\boxed{a}}{\rho}_b(m) = m$. Thus
$$\overset{\boxed{a}}{\rho}_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \overset{\boxed{a}}{\rho}_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \overset{\boxed{a}}{\rho}_d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Evidently $\overline{a,c} \subset [0 \overset{\overset{\boxed{a}}{\rho}}{=} 0] \cap [\infty \overset{\overset{\boxed{a}}{\rho}}{=} \infty]$ and so $\overline{a,c}$ is elliptic. Evidently $\overline{c,d} \subset [1 \overset{\overset{\boxed{a}}{\rho}}{=} -1] \cap [-1 \overset{\overset{\boxed{a}}{\rho}}{=} 1]$, which implies that $\overline{c,d}$ is elliptic. Thus (iii) holds.

If $\overline{a,b}$ is hyperbolic, then it is contained in no element of $\mathcal{T}$. In particular, the equation $t = \overset{\boxed{a}}{\rho}_b(t) = \frac{1}{e \cdot t}$ has no solution for t. From equation (2) we know that $\overset{\boxed{a}}{\rho}_c$ is of the form $\begin{pmatrix} x & y \\ -ey & -x \end{pmatrix}$ and $\overset{\boxed{a}}{\rho}_d$ is of the form $\begin{pmatrix} r & s \\ -es & -r \end{pmatrix}$. The equation $\overset{\boxed{a}}{\rho}_d(t) = \overset{\boxed{a}}{\rho}_c(t)$ resolves into $e \cdot t^2 = -1$. Hence the $\overline{c,d}$ is hyperbolic as well. If we choose c to be such that $\overset{\boxed{a}}{\rho}_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then evidently $\overline{a,c} \subset [0 \overset{\overset{\boxed{a}}{\rho}}{=} 0] \cap [\infty \overset{\overset{\boxed{a}}{\rho}}{=} \infty]$ and so $\overline{a,c}$ is elliptic. This establishes (iv) QED

(8.25) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L. Let K be a line trace and a an element of K. Then
$$(\exists\, k \in K \text{ distinct from } a : \ k \in a^{\square}) \iff K \text{ is not parabolic.} \tag{1}$$

Proof. $\Longrightarrow$ This follows from (8.19.v).

$\Longleftarrow$ Let $b \in K$ be distinct from a, and suppose that K is not parabolic. If b is in $a^{\square}$, we are done, so we shall presume that b is not in $a^{\square}$. By Lemma (8.10) there exists an a-representation $\overset{\boxed{a}}{\rho}$, a choice of basis, and $q, r \in F$ such that
$$K = \{\begin{pmatrix} e & rd \\ qd & e \end{pmatrix} : e, d \in F\}, \quad \text{and} \quad \overset{\boxed{a}}{\rho}_b = \begin{pmatrix} 1 & r \\ q & 1 \end{pmatrix}.$$
Evidently an element of K is in $a^{\square}$ if and only if it is of the form $\begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}$. For this it is necessary and sufficient that $q \neq 0 \neq r$. But one easily checks that
$$q = 0 \Longrightarrow K \subset [\infty \overset{\overset{\boxed{a}}{\rho}}{=} \infty] \quad \text{and} \quad r = 0 \Longrightarrow K \subset [0 \overset{\overset{\boxed{a}}{\rho}}{=} 0]$$
both of which are absurd, since K is not parabolic. QED

(8.26) Theorem Let a and b be distinct elements of L. Then
(i) $\overline{a,b}$ is hyperbolic $\iff (\exists\, c, d \in a^{\square}) \quad \overline{a,c}$ is elliptic, $\overline{c,d}$ is hyperbolic, and $b = \lfloor c, a, d \rfloor$;

(ii) $\overline{a,b}$ is elliptic$\Longleftrightarrow$ $(\exists\, c,d\in a^{\square})$ $\quad\overline{a,c}$ is elliptic, $\overline{c,d}$ is elliptic, and $b=\lfloor c,a,d\rfloor$;
(iii) $\overline{a,b}$ is parabolic$\Longleftrightarrow$ $(\exists\, c,d\in a^{\square})$ $\quad\overline{a,c}$ is elliptic, $\overline{c,d}$ is parabolic, and $b=\lfloor c,a,d\rfloor$;

Proof. Suppose first that b is in $a^{\square}$. If $\overline{a,b}$ were parabolic, then Theorem (8.19.v) would imply that $b\in a^{\square}\cap\overline{a,b}=\emptyset$: an absurdity. Thus $\overline{a,b}$ is either hyperbolic or elliptic, and so (ii) and (iii) follow from Theorem (8.24.iv) and (8.24.iii) respectively.

In the remainder of this proof we shall suppose that b is not in $a^{\square}$. Suppose that $\overline{a,b}$ is elliptic. Then $\overline{a,b}$ is contained in two distinct elements of $\mathcal{T}$, which means that $\overset{\boxed{@}}{\rho}_a$ and $\overset{\boxed{@}}{\rho}_b$ agree on two distinct elements of M. This means that $\overset{\boxed{@}}{\rho}_b$ has two fixed points, and so is a dilation. By (4.20.iii) there exist $\pi,\sigma\in\Pi(M)$ agreeing on two distinct points of M and such that π is a dilation and such that $\overset{\boxed{@}}{\rho}_b=\pi\circ\sigma$. Choose $c,d\in a^{\square}$ such that $\overset{\boxed{@}}{\rho}_c=\pi$ and $\overset{\boxed{@}}{\rho}_d=\sigma$. We have

$$\llbracket\overset{\boxed{@}}{\rho}_c,\overset{\boxed{@}}{\rho}_a,\overset{\boxed{@}}{\rho}_d\rrbracket=\overset{\boxed{@}}{\rho}_c\circ\overset{\boxed{@}}{\rho}_d=\pi\circ\sigma=\overset{\boxed{@}}{\rho}_b\Longrightarrow b=\lfloor c,a,d\rfloor. \tag{1}$$

Since $\overset{\boxed{@}}{\rho}_c=\pi$ has two fixed points, it agrees with $\overset{\boxed{@}}{\rho}$ at two points – hence the line $\overline{a,c}$ is elliptic. Since $\overset{\boxed{@}}{\rho}_c$ and $\overset{\boxed{@}}{\rho}_d$ agree on two distinct points, it follows that $\overline{c,d}$ is elliptic. This proves (ii).

Now suppose that $\overline{a,b}$ is hyperbolic. This means that a and b are in no common element of $\mathcal{T}$: that $\overset{\boxed{@}}{\rho}_b$ leaves not point fixed, and so is a rotation. By (4.20.iv) there exist $\pi,\sigma\in\Pi(M)$ agreeing on no point of M, such that π is a dilation, and such that $\overset{\boxed{@}}{\rho}_b=\pi\circ\sigma$. Choose $c,d\in a^{\square}$ such that $\overset{\boxed{@}}{\rho}_c=\pi$ and $\overset{\boxed{@}}{\rho}_d=\sigma$. As before, it follows that (1) holds and that $\overline{a,c}$ is elliptic. This time however, $\overline{c,d}$ is hyperbolic since $\overset{\boxed{@}}{\rho}_c$ and $\overset{\boxed{@}}{\rho}_d$ agree on no point of M. This proves (i).

Finally we suppose that $\overline{a,b}$ is parabolic. This means that a and b are in a single element of $\mathcal{T}$: that $\overset{\boxed{@}}{\rho}_a$ and $\overset{\boxed{@}}{\rho}_b$ agree on at a single point of M: that $\overset{\boxed{@}}{\rho}_b$ has exactly one fixed point: that $\overset{\boxed{@}}{\rho}_b$ is a translation. By (4.20.ii) there exist $\pi,\sigma\in\Pi(M)$ with a single common fixed point of M, such that π and σ are dilations, and such that $\overset{\boxed{@}}{\rho}_b=\pi\circ\sigma$. Choose $c,d\in a^{\square}$ such that $\overset{\boxed{@}}{\rho}_c=\pi$ and $\overset{\boxed{@}}{\rho}_d=\sigma$. As before, it follows that (1) holds and that $\overline{a,c}$ is elliptic. This time however, $\overline{c,d}$ is parabolic since $\overset{\boxed{@}}{\rho}_c$ and $\overset{\boxed{@}}{\rho}_d$ agree on a single point of M. This proves (iii). QED

(8.27) Corollary Let a be an element of L. Then

$$L=\{\lfloor x,y,z\rfloor:\ x,y,z\in a^{\square}\}.$$

In particular, L is a polar libra.[33].

Proof. By Theorem (8.24) there exist $b,c,d\in a^{\square}$ such that $b=\lfloor c,a,d\rfloor$. It follows that $a=\lfloor d,b,c\rfloor$. Corollary (8.27) now follows from Theorem (8.26). QED

(8.28) Discussion The final three theorems of this section will be needed *infra* in discussing the connection of L with three dimensional projective space.[34]

(8.29) Theorem Let A be in $\mathcal{T}$ and let K and V be distinct line traces in A. Then either K intersects V or

$$(\exists\, X\in\|A\|\cup\underline{\overline{\overline{A}}})\quad K^{\circ}\cup V^{\circ}\subset X. \tag{1}$$

Proof. Let a be in K. Referring to the notation of (8.8) we let $\infty\equiv\mu(\|A\|)$. We have

$$\overset{\boxed{@}}{\rho}(\infty)=\nu(a^{\hat{\textcircled{\tiny T}}}(\mu^{-1}(\infty)))=\nu(a^{\hat{\textcircled{\tiny T}}}(\|A\|))=\nu(\underline{\overline{\overline{A}}})$$

and so

$$[\infty\overset{\overset{\boxed{@}}{\rho}}{=}\infty]=(\mu^{-1}(\infty))\wedge(\nu^{-1}(\rho_a(\infty)))=\|A\|\wedge\underline{\overline{\overline{A}}}=A.$$

Let b be an element of K distinct from a and let c be an element of V not in K. Then $c\neq a$ and so there exists some point $0\in F$ such that $1\equiv\overset{\boxed{@}}{\rho}_c(0)$ is not 0. Let $s\equiv\overset{\boxed{@}}{\rho}_b(0)$ and $r\equiv\overset{\boxed{@}}{\rho}_b(1)-s$. Then we have

$$\overset{\boxed{@}}{\rho}_a=\begin{pmatrix}1&0\\0&1\end{pmatrix}\text{ and }\overset{\boxed{@}}{\rho}_b=\begin{pmatrix}r&s\\0&1\end{pmatrix}.$$

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33 *Cf.* (7.6).
34 *Cf.* Section (10) *infra*.

If $r=1$, then $\overset{\boxed{@}}{\rho}_a$ and $\overset{\boxed{@}}{\rho}_b$ agree only at the one point ∞ of M, which implies that K is parabolic. Direct calculation shows that

$$\{a,b\}^\circ = \{x\in L\colon (\exists\, t\in F)\quad \overset{\boxed{@}}{\rho}(x) = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}\} \text{ and } K = \overline{a,b} = \{x\in L\colon (\exists\, t\in F)\quad \overset{\boxed{@}}{\rho}(x) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\}. \tag{2}$$

If $r\neq 1$, then $\overset{\boxed{@}}{\rho}_a$ and $\overset{\boxed{@}}{\rho}_b$ agree also at the point $\frac{s}{1-r}$ and so K is elliptic with

$$K = \{x\in L\colon (\exists\, t\in F)\quad \overset{\boxed{@}}{\rho}_x = \begin{pmatrix} t+(1-t)r & (1-t)s \\ 0 & 1 \end{pmatrix}\}. \tag{3}$$

Recalling that $1 = \overset{\boxed{@}}{\rho}_c(0)$ we choose $q\in F$ such that $\overset{\boxed{@}}{\rho}_c = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$. Let d be an element of V distinct from 0,and choose $u, v\in F$ such that $\overset{\boxed{@}}{\rho}_d = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}$. If $q=u$, arguing as above we find that V is parabolic and

$$\overline{c,d} = \{x\in L\colon (\exists\, t\in F)\quad \overset{\boxed{@}}{\rho}_x = \begin{pmatrix} q & t \\ 0 & 1 \end{pmatrix}\}. \tag{4}$$

If $q\neq u$, then $\overset{\boxed{@}}{\rho}_c$ and $\overset{\boxed{@}}{\rho}_d$ agree also at $\frac{v-1}{q-u}$ and so V is elliptic and

$$\overline{c,d} = \{x\in L\colon (\exists\, t\in F)\quad \overset{\boxed{@}}{\rho}(x) = \begin{pmatrix} tq+(1-t)u & t+(1-t)v \\ 0 & 1 \end{pmatrix}\}. \tag{5}$$

Case I: (2) and (4) hold. Here (1) holds with $X\equiv A$.

Case II: (2) and (5) hold. If $y\in L$ satisfies $\overset{\boxed{@}}{\rho}_y = \begin{pmatrix} 1 & \frac{1-u}{q-u} \\ 0 & 1 \end{pmatrix}$, then $y\in K\cap V$.

Case III: (3) and (4) hold. We have $\begin{pmatrix} q & \frac{1-q}{1-r}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t+(1-t)r & (1-t)s \\ 0 & 1 \end{pmatrix}$ when $t\equiv\frac{q-r}{1-r}$.

Case IV: (3) and (5) hold. Suppose first that $\frac{s}{1-r} = \frac{v-1}{q-u}$. If $\overset{\boxed{@}}{\rho}_c(\frac{v-1}{q-u})$ equals $\frac{v-1}{q-u}$, then

$$K = [\infty \overset{\overset{\boxed{@}}{\rho}}{=} \infty] \cap [\frac{s}{1-r} \overset{\overset{\boxed{@}}{\rho}}{=} \frac{s}{1-r}] = [\infty \overset{\overset{\boxed{@}}{\rho}}{=} \infty] \cap [\frac{v-1}{q-u} \overset{\overset{\boxed{@}}{\rho}}{=} \frac{v-1}{q-u}] = V$$

which is absurd. Thus $e\neq f$ where $f\equiv\overset{\boxed{@}}{\rho}_c(\frac{v-1}{q-u})$ and $e\equiv\frac{v-1}{q-u}$. We have

$$K = [\infty \overset{\overset{\boxed{@}}{\rho}}{=} \infty] \cap [e \overset{\overset{\boxed{@}}{\rho}}{=} e] \text{ and } V = [\infty \overset{\overset{\boxed{@}}{\rho}}{=} \infty] \cap [e \overset{\overset{\boxed{@}}{\rho}}{=} f]$$

which implies

$$K^\circ = [\infty \overset{\overset{\boxed{@}}{\rho}}{=} e] \cap [e \overset{\overset{\boxed{@}}{\rho}}{=} \infty] \text{ and } V^\circ = [\infty \overset{\overset{\boxed{@}}{\rho}}{=} f] \cap [e \overset{\overset{\boxed{@}}{\rho}}{=} \infty]$$

which in turn implies (5) where $X\equiv[e \overset{\overset{\boxed{@}}{\rho}}{=} \infty]$.

Now we suppose that $\frac{s}{1-r}\neq\frac{v-1}{q-u}$. Then $e\equiv i - v + \frac{q-u}{1-r}s$ is not 0 and we may define $t_2\equiv\frac{\frac{u-r}{1-r}s+s-v}{e}$ and $t_1\equiv\frac{t_2(q-u)+u-r}{q-u}$ to obtain

$$\begin{pmatrix} t_1+(1-t_1)r & (1-t_1)s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_2q+(1-t_2)u & t_2+(1-t_2)v \\ 0 & 1 \end{pmatrix}.$$

QED

(8.30) Theorem Let a be in L and let K and V be distinct lines in in $a^\square$. Then either K intersects V or

$$(\exists\, X\in\mathcal{T})\quad K^\circ\cup V^\circ\subset X. \tag{1}$$

Proof. The line trace K° contains a and so by (8.10) there exists a choice $\{0,1,\infty\}$ of basis for M and $q, r\in F$ such that

$$\{\overset{\boxed{@}}{\rho}_x\colon x\in K\} = \{\begin{pmatrix} x & -ry \\ qy & -x \end{pmatrix}\colon x, y\in F \text{ and } x^2\neq qry^2\}. \tag{2}$$

Let c be an element of L° distinct from a, and choose $t, u, v, w \in F$ such that $\overset{\boxed{@}}{\rho}_c = \begin{pmatrix} t & u \\ v & w \end{pmatrix}$. Since $V = \{a, c\}^\circ$, direct computation shows that if $t \neq w$,

$$\{\overset{\boxed{@}}{\rho}_x : x\in V\} = \{\begin{pmatrix} uy+vx & (t-w)x \\ (t-w)y & -uy-vx \end{pmatrix} : x, y\in F \text{ and } (uy+vx)^2 + (t-w)^2xy \neq 0\}. \tag{3}$$

If $(uq+vr)^2 \neq rq(w-t)^2$ then there exists $d\in L$ such that $\overset{\boxed{@}}{\rho}_d = \begin{pmatrix} uq+vr & -r(w-t) \\ q(w-t) & -(uq+vr) \end{pmatrix}$. Direct computation with (2) and (3) shows that d is in $K \cap V$. If $(uq+vr)^2 = rq(w-t)^2$, we have the following cases:

Case $q=0$: Since $(uq+vr)^2 = rq(w-t)^2$, we have $v=0$ as well. This implies that $\overset{\boxed{@}}{\rho}_c(\infty) = \infty$, whence follows that $V^\circ \subset [\infty \overset{\overset{\boxed{@}}{\rho}}{=} \infty]$. If $c' \in L$ is such that $\overset{\boxed{@}}{\rho}(c') = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, then the fact that $q=0$ implies that c' is in K°. Since $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}(\infty) = \infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \overset{\boxed{@}}{\rho}_a$, it follows that $K \subset [\infty \overset{\overset{\boxed{@}}{\rho}}{=} \infty]$.

Case $r=0$: Since $(uq+vr)^2 = rq(w-t)^2$, we have $u=0$ as well. This implies that $\overset{\boxed{@}}{\rho}_c(0) = 0$, whence follows that $V^\circ \subset [0 \overset{\overset{\boxed{@}}{\rho}}{=} 0]$. If $k\in L$ is such that $\overset{\boxed{@}}{\rho}_k = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$, then the fact that $r=0$ implies that k is in K°. Since $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}(0) = 0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \overset{\boxed{@}}{\rho}_a$, it follows that $K \subset [0 \overset{\overset{\boxed{@}}{\rho}}{=} 0]$.

Case $rq \neq 0$: From $(uq+vr)^2 = rq(w-t)^2$ follows that $\frac{r}{q} = (q\frac{uq+vr}{w-t})^2$. Let $e\in L$ satisfy $\overset{\boxed{@}}{\rho}_e = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}$. The equation $\overset{\boxed{@}}{\rho}_e(x) = x$ for $x\in F$ is the same as $x^2 = \frac{r}{q}$. Thus it has the solutions $x = \pm\frac{1}{q}\frac{uq+vr}{w-t}$. In particular we have $e \in [\frac{1}{q}\frac{uq+vr}{w-t} \overset{\overset{\boxed{@}}{\rho}}{=} \frac{1}{q}\frac{uq+vr}{w-t}]$, which since a is there as well, implies $V^\circ \subset [\frac{1}{q}\frac{uq+vr}{w-t} \overset{\overset{\boxed{@}}{\rho}}{=} \frac{1}{q}\frac{uq+vr}{w-t}]$. On the other hand we have

$$\begin{pmatrix} t & u \\ v & w \end{pmatrix}(\frac{1}{q}\frac{uq+vr}{w-t}) = \frac{1}{q}\frac{uq+vr}{w-t} \iff \frac{t}{q}\frac{uq+vr}{w-t} + u = (\frac{v}{q}\frac{uq+vr}{w-t})^2 + \frac{w}{q}\frac{uq+vr}{w-t} \iff$$

$$v\frac{r}{q} = \frac{w}{q}\frac{uq+vr}{w-t} - \frac{t}{q}\frac{uq+vr}{w-t} - u \iff v\frac{r}{q} = \frac{uq+vr}{q} - \frac{uq}{q}$$

which last is obviously true. It follows that $\overset{\boxed{@}}{\rho}_c$ is in $[\frac{l}{q}\frac{uq+vr}{w-t} \overset{\overset{\boxed{@}}{\rho}}{=} \frac{l}{q}\frac{uq+vr}{w-t}]$. Since $\overset{\boxed{@}}{\rho}_a$ is as well, we have $V^\circ \subset [\frac{1}{q}\frac{uq+vr}{w-t} \overset{\overset{\boxed{@}}{\rho}}{=} \frac{1}{q}\frac{uq+vr}{w-t}]$. We have established (8.30) for the case $t \neq w$.

Suppose that $t = w$. Choose $d\in L$ such that $\overset{\boxed{@}}{\rho}_d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. From (2) we know that d is in K. Direct computation shows that d is in $c^\square$, and so, since it evidently is in $a^\square$, d is in V. QED

(8.31) Theorem For each $a\in L$, we have

$$a^{\square\circ} = \{a\}. \tag{1}$$

Proof. That $a \in a^{\square\circ}$ is trivial. Assume that there existed b in $a^{\square\circ}$ distinct from a. Then

$$a, b\in a^{\square\circ} \xRightarrow{\text{by (7.3.i) and by (7.3.iv)}} a^\square \subset \{a, b\}^\circ = \{a, b\}^{\circ\circ\circ} \xRightarrow{\text{by (7.3.i) and by (7.3.iv)}} \{a, b\}^{\circ\circ} \subset \{a\}$$

which by (8.13.i) implies that the singleton $\{a\}$ has a line as a subset: an absurdity. QED

9. Product Libras

(9.1) Discussion, Definitions and Notation Let L be a libra. The injection

$$L \ni x \hookrightarrow [x,x] \in L\times L$$

of L into its symmetrization libra[35] $L\times L$ is only an isomorphism if L is abelian. When L is non-abelian, the subset $\{[x,x] : x\in L\}$ is not balanced, but one may consider the intersection of all balanced subsets of $L\times L$ containing it:

$$L\boxtimes L \equiv \{\vdash[x_1,x_1],\ldots,[x_{2k-1},x_{2k-1}]\dashv : \{x_i\}_{i=1}^{2k-1}\subset L, k\in\mathbb{N}\}. \tag{1}$$

From (5.15.3) we have

$$L\boxtimes L = \{[\lfloor x_1,\ldots,x_{2k-1}\rfloor, \lfloor x_{2k-1},\ldots,x_1\rfloor] : \{x_i\}_{i=1}^{2k-1}\subset L, k\in\mathbb{N}\}. \tag{2}$$

Reversing the order of the elements of the sequence in (2), one sees that

$$L\boxtimes L = \{[x,y] : [y,x] \in L\boxtimes L\}. \tag{3}$$

Related to $L\boxtimes L$ is the subgroup of $\mathfrak{J}(L,L)$ generated by $\{{}_x\pi_x : x\in L\}$:[36]

$$L\circledast L \equiv \{{}_{x_1}\pi_{x_1}\circ\ldots\circ{}_{x_{2k-1}}\pi_{x_{2k-1}} : \{x_i\}_{i=1}^{2k-1}\subset L, k\in\mathbb{N}\}. \tag{4}$$

Expanding, one also has

$$L\circledast L = \{{}_{\lfloor x_1,\ldots,x_{2k-1}\rfloor}\pi_{\lfloor x_{2k-1},\ldots,x_1\rfloor} : \{x_i\}_{i=1}^{2k-1}\subset L, k\in\mathbb{N}\}. \tag{5}$$

(9.2) Theorem For any libra L, $L\boxtimes L$ is a normal balanced subset of $L\times L$.

Proof. For $m,n,l\in L$,

$$\vdash[m,n],[l,l],L\boxtimes L\dashv = \{\vdash[m,n],[l,l],[x_1,x_1],\ldots,[x_{2n-1},x_{2n-1}]\dashv : \{x_i\}_{i=1}^{2n-1}, n\in\mathbb{N}\} =$$
$$\{[\lfloor m,l,x_1,\ldots,x_{2n-1}\rfloor], \lfloor x_{2n-1},\ldots,x_1,l,n\rfloor : \{x_i\}_{i=1}^{2n-1}, n\in\mathbb{N}\} =$$
$$\{[\lfloor m,l,x_1,l,m,m,l,\ldots,m,l,x_{2n-1},l,m,m,l\rfloor], \lfloor l,n,n,l,x_{2n-1},l,n,\ldots,l,n,n,l,x_1,l,n\rfloor : \{x_i\}_{i=1}^{2n-1}, n\in\mathbb{N}\} =$$
$$\{[\lfloor\lfloor m,y_1,m\rfloor,\ldots,\lfloor m,y_{2n-1},m\rfloor,m,l\rfloor, \lfloor l,n,\lfloor n,y_{2n-1},n\rfloor,\ldots,\lfloor n,y_1,n\rfloor\rfloor] : \{y_i\}_{i=1}^{2n-1}, n\in\mathbb{N}\} =$$
$$\{[\lfloor z_l,\ldots,z_{2n-1},m,l\rfloor, \lfloor l,n,z_{2n-1},\ldots,z_1\rfloor] : \{z_i\}_{i=1}^{2n-1}, n\in\mathbb{N}\} =$$
$$\{\vdash[x_1,x_1],\ldots,[x_{2n-1},x_{2n-1}],[m,n],[l,l]\dashv : \{z_i\}_{i=1}^{2n-1}, n\in\mathbb{N}\} = \vdash L\boxtimes L,[m,n],[l,l]\dashv.$$

We have shown that every left coset of $L\boxtimes L$ is a right coset of $L\boxtimes L$. QED

(9.3) Example We return to the example of (2.27): $M \equiv \{a,b,c\}$ for distinct points a, b, and c, and $L\equiv\Gamma(M)$ is the family of permutations of M. We abbreviate:

$$\iota\equiv\begin{bmatrix} a\ b\ c \\ a\ b\ c\end{bmatrix},\quad \alpha\equiv\begin{bmatrix} a\ c\ b \\ a\ b\ c\end{bmatrix},\quad \beta\equiv\begin{bmatrix} c\ b\ a \\ a\ b\ c\end{bmatrix},\quad \gamma\equiv\begin{bmatrix} b\ a\ c \\ a\ b\ c\end{bmatrix},\quad \rho\equiv\begin{bmatrix} b\ c\ a \\ a\ b\ c\end{bmatrix},\text{ and } \lambda\equiv\begin{bmatrix} c\ a\ b \\ a\ b\ c\end{bmatrix}.$$

There are two non-trivial balanced subsets of L: $\Delta\equiv\{\alpha,\beta,\gamma\}$ and $\Theta\equiv\{\iota,\lambda,\rho\}$. Direct calculation shows that

$$\{[\chi,\chi] : \chi\in L\}\cup(\Delta\times\Delta)\cup(\Theta\times\Theta) \quad \text{is a balanced subset of } L\times L$$

and that

$$\{[\chi,\chi] : \chi\in L\}\cup(\Delta\times\Delta)\cup(\Theta\times\Theta) = L\boxtimes L \neq L\times L.$$

(9.4) Theorem If $L\boxtimes L = L\times L$, then

$$(\forall\, x,y\in L)(\exists\,\phi\in L\circledast L)\quad \phi(x) = y. \tag{1}$$

Proof. Let m and n be in L. By hypothesis and (9.1.4) there exists $k\in\mathbb{N}$ and $\{x_i\}_{i=1}^{2k-1}\subset L$ such that

$$[m,n] = [\lfloor x_1,\ldots,x_k\rfloor, \lfloor x_k,\ldots,x_1\rfloor].$$

Consequently

$${}_{\lfloor x_1,\ldots,x_k\rfloor}\pi_{\lfloor x_k,\ldots,x_1\rfloor}(n) = \lfloor x_1,\ldots,x_k,\lfloor x_k,\ldots,x_1\rfloor,x_k,\ldots,x_1\rfloor =$$

[35] *Cf.* (5.15).

[36] *Cf.* (3.12) for the definition of ${}_x\pi_y$.

$$\lfloor x_1,\ldots,x_k,x_1,\ldots,x_k,x_k,\ldots,x_1\rfloor = \lfloor x_1,\ldots,x_k\rfloor = m.$$

From (9.1.5) follows that $\phi \equiv {}_{\lfloor x_1,\ldots,x_k\rfloor}\pi_{\lfloor x_k,\ldots,x_1\rfloor}$ is in $L\circledast L$. QED

(9.5) Lemma Let L be a meridian libra and let $s,x,y\in L$ be such that

$$ {}_s\pi_s(y) = x. \tag{1}$$

Then there exist $r,t\in L$ such that

$$\left\lfloor [r,r],[s,s],[t,t]\right\rceil = [x,y]. \tag{2}$$

Proof. By (8.26) there exist

$$t,r\in w^{\square} \tag{3}$$

such that

$$\lfloor t,s,r\rfloor = y. \tag{4}$$

Then

$$x \overset{\text{by (1)}}{=\!=\!=} \lfloor s,y,s\rfloor \overset{\text{by (4)}}{=\!=\!=} \lfloor s,\lfloor t,s,r\rfloor,s\rfloor = \lfloor s,r,s,t,s\rfloor \overset{\text{by (3)}}{=\!=\!=} \lfloor s,s,r,s,r\rfloor = \lfloor r,s,t\rfloor.$$

Thus

$$[x,y] = [\lfloor r,s,t\rfloor,\lfloor t,s,r\rfloor] = \left\lfloor [r,r],[s,s],[t,t]\right\rceil.$$

QED

(9.6) Theorem Let L be a meridian libra. Then

$$L\boxtimes L = L\times L \iff (\forall\, x,y\in L)(\exists\,\phi\in L\circledast L)\quad \phi(x) = y. \tag{1}$$

Proof. In view of (9.4) we need only show that $L\boxtimes L = L$ holds if (9.4.1) holds.

We presume then that (9.4.1) holds, that m and n are generic elements of L, and proceed to deduce that $[m,n]$ is in $L\boxtimes L$. Towards this end we select ϕ as in (9.4.1) and then apply (9.1.5) to obtain $k\in\mathbb{N}$ and $\{x_i\}_{i=1}^{2k-1}\subset L$ such that

$$n = {}_{x_1}\pi_{x_1}\circ\ldots\circ{}_{x_{2k-1}}\pi_{x_{2k-1}}(m) \Longrightarrow m = {}_{x_{2k-1}}\pi_{x_{2k-1}}\circ\ldots\circ{}_{x_1}\pi_{x_1}(n). \tag{2}$$

We define $b_0\equiv n$ and, for $i=1,\ldots,2k-1$ we shall abbreviate ${}_{x_i}\pi_{x_i}\circ\ldots\circ{}_{x_{2k-1}}\pi_{x_{2k-1}}$ to b_i. From Lemma (9.5) we know that

$$(\forall\, i=1,\ldots,2k-1)\quad [b_i,b_{i-1}]\in L\boxtimes L. \tag{3}$$

It follows from (9.1.5) that

$$(\forall\, i=1,\ldots,2k-1)\quad [b_{i-1},b_i]\in L\boxtimes L. \tag{4}$$

Since $L\boxtimes L$ is balanced, it follows that $\left\lfloor [b_0,b_1],[b_2,b_1],\ldots,[b_{2k-2},b_{2k-3}],[b_{2k-2},b_{2k-1}]\right\rceil$ is in $L\boxtimes L$. Furthermore

$$\left\lfloor [b_0,b_1],[b_2,b_1],\ldots,[b_{2k-2},b_{2k-3}],[b_{2k-2},b_2k-1]\right\rceil \overset{\text{by (9.1.2)}}{=\!=\!=\!=}$$
$$[\lfloor b_0,b_2,\ldots,b_{2k-2},b_{2k-2}\rfloor,\lfloor b_{2k-1},b_{2k-3},\ldots,b_1,b_1\rfloor] = [b_0,b_{2k-1}] = [n,m]$$

and so $[n,m]$ is in $L\boxtimes L$. From (9.1.5) follows that $[m,n]$ is in $L\boxtimes L$. QED

(9.7) Example Let M be the circle meridian and let $L\equiv\Gamma(M)$. Each projectivity in $\Gamma(M)$ either preserves or reverses the orientation of the circle M. Involutions which have two fixed points reverse the orientation. Let m be a projectivity which preserves orientation and let n be a projectivity which reverses it. Assume that $[m,n]$ were in $L\boxtimes L$. By (9.1.3) there would then exist $k\in\mathbb{N}$ and $\{x_i\}_{i=1}^{2k-1}\subset L$ such that

$$m = \lfloor x_1,\ldots,x_{2k-1}\rfloor \text{ and } n = \lfloor x_{2k-1},\ldots,x_1\rfloor.$$

But each of the compositions $\lfloor x_1,\ldots,x_{2k-1}\rfloor = x_1\circ\ldots\circ x_{2k-1}$ and $\lfloor x_{2k-1},\ldots,x_1\rfloor = x_{2k-1}\circ\ldots\circ x_1$ has the same number of preservation and reversal components as the other. Hence m and n either both preserve the orientation of M or reverse it: an absurdity. It follows that

$$L\boxtimes L \neq L\times L. \tag{1}$$

We shall see *infra* that in this example $L\boxtimes L$ has exactly one coset.

(9.8) Example Let M be the sphere meridian and let $L\equiv\Gamma(M)$. Let m and n and be distinct generic elements of $\Gamma(M)$. Then $\overline{m,n}$ is either elliptic or parabolic, there being no hyperbolic lines in this example.

Suppose that $\overline{m,n}$ is elliptic. Then there are two elements 0 and ∞ of M which are fixed by all members of $\overline{m,n}$. Letting $1\in M$ be distinct from 0 and ∞ we have a basis for M. Let $r\equiv m(1)$ and $s\equiv n(1)$ Relative to this basis,

$$m=\begin{pmatrix}0 & r\\ 1 & 0\end{pmatrix} \quad \text{and} \quad n=\begin{pmatrix}0 & s\\ 1 & 0\end{pmatrix}.$$

Let $\cdot$ be the multiplicative operation relative to the basis 0, 1, and ∞ and choose $w\in F$ such that $w\cdot w{=}r\cdot s$. Let a be the element of $\Gamma(M)$ with basis matrix $\begin{pmatrix}0 & w\\ 1 & 0\end{pmatrix}$. Then

$$\lfloor a,n,a\rfloor=\begin{pmatrix}0 & w\\ 1 & 0\end{pmatrix}\circ\begin{pmatrix}0 & s\\ 1 & 0\end{pmatrix}\circ\begin{pmatrix}0 & w\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0 & w\cdot w\\ s & 0\end{pmatrix}=\begin{pmatrix}0 & r\\ 1 & 0\end{pmatrix}=m. \tag{1}$$

Suppose now that $\overline{m,n}$ is parabolic. Then there is exactly one element ∞ fixed by all members of $\overline{m,n}$. This time we shall let 0 be any element of M distinct from ∞, $1\equiv m(0)$ and $s\equiv n(0)$. Relative to the basis 0, 1 and ∞

$$m=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad n=\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}.$$

Let $a\equiv\begin{pmatrix}1 & \frac{1-s}{2}\\ 0 & 1\end{pmatrix}$. Then

$$\lfloor a,n,a\rfloor=\begin{pmatrix}1 & \frac{1-s}{2}\\ 0 & 1\end{pmatrix}\circ\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\circ\begin{pmatrix}1 & \frac{1-s}{2}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=m. \tag{2}$$

From (1) and (2) follows that $\lfloor a,n,a\rfloor=m$. Theorem (2.7) now implies that

$$L\boxtimes L=L\times L. \tag{3}$$

(9.9) Notation and Definitions Let $\mathcal{T}$ be a cartesian aggregate of a libra L, and let ρ be any representation of L on a cartesian product $X\times Y$ equivalent to the $\mathcal{T}$-inner representation. For $b\in Y$, the set $X_b\equiv\{[x,b]:x\in X\}$ is an **X-cross section**, and for each $a\in X$, the set $Y_a\equiv\{[a,y]:y\in Y\}$ is a **Y-cross section**. We write

$$(\forall\,[a,b]\in X\times Y)\quad L[a;\rho;b]\equiv\{\rho_x|_{X_a}:x\in L \text{ and } \rho_x(X_a)=Y_b\}. \tag{1}$$

It is trivial that each mapping[37] $\overset{\leftrightarrow}{\rho}_{[r,s]}$, $r,s\in L$, maps X-cross sections to Y-cross sections and, if it sends X_b onto Y_a, its restriction to X_b is in $L[a;\rho;b]$. Bijections from $X\times Y$ to $X\times Y$ satisfying both these properties will be called **ρ-contravariant mappings on $X\times Y$**.

For $a,m\in X$ and $n,b\in Y$ we write

$$L[a,m;\rho]\equiv\{{\rho_u}^{-1}\circ\rho_v|_{X_a}:u,v\in L \text{ and } {\rho_u}^{-1}\circ\rho_v(X_a)=X_m\} \tag{2}$$

and

$$L[\rho;n,b]\equiv\{\rho_u\circ{\rho_v}^{-1}|_{Y_n}:u,v\in L \text{ and } \rho_u\circ{\rho_v}^{-1}(Y_n)=Y_b\} \tag{3}$$

A function from $X\times Y$ to $X\times Y$ which sends X-cross sections onto X-cross sections, sends Y-cross sections onto Y-cross sections, and such that if it sends X_a to X_m is in $L[a,m;\rho]$, and if it sends Y_n to Y_b is in $L[\rho;n,b]$, will be called a **ρ-covariant mapping on $X\times Y$**. If f and g are ρ-contravariant mappings on $X\times Y$, evidently $f\circ g^{-1}$ is a ρ-covariant mapping. In particular, for $r,s\in L$, the function ${\rho_s}^{-1}\circ\rho_r$ is always a ρ-covariant mapping.

(9.10) Theorem Let $\mathcal{T}$ be a cartesian aggregate of a libra L, and let ρ be any representation of L on a cartesian product $X\times Y$ equivalent to the $\mathcal{T}$-inner representation. Let ϕ be a ρ-contravariant mapping. Then

$$(\exists\,r,s\in L)\quad \overset{\leftrightarrow}{\rho}_{[r,s]}=\phi. \tag{1}$$

Proof. Define $f|X\to Y$ and $g|Y\to X$ by

$$(\forall\,x\in X)\quad \phi(Y_x)=X_{f(x)} \quad \text{and} \quad (\forall\,y\in Y)\quad \phi(X_y)=Y_{g(y)}. \tag{2}$$

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[37] *Cf.* (6.17). By definition $\rho_{[r,s]}([x,y])=[{\rho_s}^{-1}(y),\rho_r(x)]$.

Define $\theta|X\times Y \to X\times Y$ by

$$(\forall\,[x,y]\in X\times Y)\quad \{\theta([x,y])\} \equiv X_{f(x)} \cap Y_{g(y)}. \tag{3}$$

For $[x,y]\in X\times Y$ we have

$$\{\phi([x,y])\} = \phi(Y_x)\cap\phi(X_y) \xRightarrow{\text{by (2) and (3)}} \{\theta([x,y])\}. \tag{4}$$

For $b\in Y$ we have by hypothesis that ϕ equals $\overset{\leftrightarrow}{\rho}_{[r,s]}$ on X_b: to wit

$$(\forall\,x\in X)\quad \phi([x,b]) = \overset{\leftrightarrow}{\rho}_{[r,s]}([x,b]) = [\rho_s(b),\rho_r(x)] \xRightarrow{\text{by (2), (3) and (4)}}$$

$$\{[\rho_s(b),\rho_r(x)]\} = X_{f(x)}\cap Y_{g(b)} = \{[g(b),f(x)]\} \Longrightarrow [\rho_s(b),\rho_r(x)] = [g(b),f(x)]. \tag{5}$$

A priori the heritage of s and r depended on $b\in Y$, but it follows from (5) that $\rho_r(x)$ does not. Since ρ is faithful, it follows that r does not. By an argument symmetric and analogous to the preceding, we see that s is completely determined by as well, as so (1) holds. QED

(9.11) Corollary Let $\mathcal{T}$ be a cartesian aggregate of a libra L, and let ρ be any representation of L on a cartesian product $X\times Y$ equivalent to the $\mathcal{T}$-inner representation. Let γ be a ρ-covariant mapping. Then

$$(\exists\,r,s,u,v\in L)\quad \overset{\leftrightarrow}{\rho}_{[r,s]}\circ\overset{\leftrightarrow}{\rho}_{[u,v]} = \gamma. \tag{1}$$

Proof. Let $u,v\in L$ and define $\phi\equiv\gamma\circ\overset{\leftrightarrow}{\rho}_{[u,v]}{}^{-1}$. Then ϕ is ρ-contravariant and so Theorem (9.10) implies that there exist $s,r\in L$ such that (9.10.1) holds. If follows that (9.11.1) holds. QED

(9.12) Lemma Let $\mathcal{T}$ be a meridian aggregate of a libra L, and let ρ be any representation of L on a cartesian product $X\times Y$ equivalent to the $\mathcal{T}$-inner representation. Let $\phi\in\mathit{\Pi}(X)$ – that is, let ϕ be an involution of the meridian X. Then

$$(\forall\,a\in L)(\exists\,z\in a^{\square})\quad \phi = \rho_a{}^{-1}\circ\rho_z. \tag{1}$$

Proof. Let a be an element of L. By definition of the meridian structure on X we have $\mathit{\Gamma}(X) = \{\rho_a{}^{-1}\circ\rho_z : z\in L\}$. Thus $\phi=\rho_a{}^{-1}\circ\rho_z$ for some $z\in L$. Since ϕ is an involution, we have

$$\rho_a{}^{-1}\circ\rho_z = (\rho_a{}^{-1}\circ\rho_z)^{-1} = \rho_z{}^{-1}\circ\rho_a \Longrightarrow \rho_z = \rho_a\circ\rho_z{}^{-1}\circ\rho_a.$$

QED

(9.13) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L, and let ρ be any representation of L on a cartesian product $X\times Y$ equivalent to the $\mathcal{T}$-inner representation. Let γ be a ρ-covariant mapping, and let a be an element of L. Then

$$(\exists\,b,c,d,e\in L)\quad a=\lfloor b,a,b\rfloor=\lfloor c,a,c\rfloor=\lfloor d,a,d\rfloor=\lfloor e,a,e\rfloor \quad\text{and}\quad \overset{\leftrightarrow}{\rho}_{[a,a]}\circ\overset{\leftrightarrow}{\rho}_{[\lfloor e,a,c\rfloor,\lfloor d,a,b\rfloor]} = \gamma. \tag{1}$$

Proof. By (9.11) there exist $r,s,u,v\in L$ such that

$$\gamma = \overset{\leftrightarrow}{\rho}_{[r,s]}\circ\overset{\leftrightarrow}{\rho}_{[u,v]}. \tag{2}$$

The mapping $\rho_s{}^{-1}\circ\rho_u$ is in $\mathit{\Gamma}(X)$ and so by (4.19) $\rho_s{}^{-1}\circ\rho_u$ is either the identity function, an involution, or a product of involutions. In the first case we set $e\equiv c\equiv a$. In the second case we set $e\equiv a$ and $c\equiv\lfloor a,s,u\rfloor$. In the third case we apply Lemma (9.12) to obtain $c,e\in a^{\square}$ such that $\rho_s = \rho_a{}^{-1}\circ\rho_e$ and $\rho_u = \rho_a{}^{-1}\circ\rho_c$. For each of these cases,

$$\rho_s = \rho_e{}^{-1}\circ\rho_a \Longrightarrow \rho_s{}^{-1}\circ\rho_u = \rho_a{}^{-1}\circ\rho_e\circ\rho_a{}^{-1}\circ\rho_c = \rho_a{}^{-1}\circ\rho_{\lfloor e,a,c\rfloor}.$$

Reasoning similarly, we can find $b,d\in L$ such that

$$a=\lfloor b,a,b\rfloor=\lfloor d,a,d\rfloor \quad\text{and}\quad \rho_r\circ\rho_v{}^{-1} = \rho_a\circ\rho_{\lfloor d,a,b\rfloor}{}^{-1}.$$

For any $[x,y]\in X\times Y$ we have

$$\overset{\leftrightarrow}{\rho}_{[a,a]}\circ\overset{\leftrightarrow}{\rho}_{[\lfloor e,a,c\rfloor,\lfloor d,a,b\rfloor]}([x,y]) = [\rho_a{}^{-1}\circ\rho_{\lfloor e,a,c\rfloor}(x),\rho_a\circ\rho_{\lfloor d,a,b\rfloor}{}^{-1}(y)] = [\rho_s{}^{-1}\circ\rho_u(x),\rho_r\circ\rho_v{}^{-1}(y)] = \overset{\leftrightarrow}{\rho}_{[r,s]}\circ\overset{\leftrightarrow}{\rho}_{[u,v]}([x,y])$$

which, with (2) implies (1). QED

(9.14) Corollary Let $\mathcal{T}$ be a meridian aggregate of a libra L, and let ρ be any representation of L on a cartesian product $X\times Y$ equivalent to the $\mathcal{T}$-inner representation. Let δ be a ρ-covariant mapping, and let a be an element of L. Then

$$(\exists\, b,c,d,e\in L)\quad a=\lfloor b,a,b\rfloor=\lfloor c,a,c\rfloor=\lfloor d,a,d\rfloor=\lfloor e,a,e\rfloor \quad\text{and}\quad \overset{\leftrightarrow}{\rho}_{[\lfloor d,a,b\rfloor,\lfloor e,a,c\rfloor]}\circ\overset{\leftrightarrow}{\rho}_{[a,a]}=\delta. \tag{1}$$

Proof. Let $\delta{\equiv}\gamma^{-1}$ in Theorem (9.13) and observe that $\overset{\leftrightarrow}{\rho}_{[\lfloor d,a,b\rfloor,\lfloor e,a,c\rfloor]}$ is the inverse of $\overset{\leftrightarrow}{\rho}_{[\lfloor e,a,c\rfloor,\lfloor d,a,b\rfloor]}$. QED

(9.15) Corollary Let L be a libra with a meridian aggregate $\mathcal{T}$. Let a be any element of L. Then[38]

$$\mathfrak{Group}(\mathcal{T})=\{\lfloor u,a\circledT a,v\rfloor:\ u,v\in L\}. \tag{1}$$

Proof. By definition,

$$\mathfrak{Group}(\mathcal{T})=\{(t\circledT m)\circ(n\circledT w):\ t,m,n,w\in L\}. \tag{2}$$

In view of (6.17.3) and (9.14), we can find $u,v\in L$ such that $(t\circledT m)\circ(n\circledT w)=(u\circledT v)\circ(a\circledT a)$. Thus (9.15.2) implies

$$\mathfrak{Group}(\mathcal{T})=\{(u\circledT v)\circ(a\circledT a):\ u,v\in L\}\ \overset{\text{by (6.7.2)}}{=\!=\!=}\ \{\lfloor u,a\circledT a,v\rfloor:\ u,v\in L\}.$$

QED

(9.16) Example We return to the example of (8.5). We have a three-dimensional real projective space $\mathbf{S}$, a quadric surface $\mathbf{Q}$, the family M of counter-clockwise upward rules in $\mathbf{Q}$, and the family N of clockwise upward rules in $\mathbf{Q}$. The complement $\mathbf{L}$ of $\mathbf{Q}$ in $\mathbf{S}$ will be denoted $\mathbf{L}$. Each element $\mathfrak{a}$ of $\mathbf{L}$ inherits two natural actions. One sends M to N as in Figure (11) of Section (1): a rule X in M, along with $\mathfrak{a}$ determines a plane, and this plane intersects $\mathbf{Q}$ in exactly one rule $\mathbf{Y}$ in N: we define the function $\rho_{\mathfrak{a}}$ by

$$\rho_{\mathfrak{a}}(X)\equiv\mathbf{Y}.$$

The other action sends $\mathbf{Q}$ to itself as in Figure (10) of Section (1): a point $\mathfrak{x}$ of $\mathbf{Q}$, along with $\mathfrak{a}$, determines a line which is either tangent to $\mathbf{Q}$ at $\mathfrak{y}{\equiv}\mathfrak{a}$ or cuts through $\mathbf{Q}$ in one other point $\mathfrak{y}$: we define the function

$$\widehat{\mathfrak{a}}(\mathfrak{x})\equiv\mathfrak{y}.$$

Each plane in $\mathbf{S}$ not tangent to $\mathbf{Q}$ intersects $\mathbf{Q}$ in a circle. Since each rule in M cuts that circle at exactly one point, this creates a bijection from M onto the circle, which in turn induces a meridian operator on M. Furthermore the induced operator is the same for each such plane (or circle). Consequently we can regard M as a meridian. Similarly, we may regard N as a meridian. We shall show in Section (10) that the family $\{\rho_{\mathfrak{a}}:\ \mathfrak{a}\in\mathbf{L}\}$ constitutes the libra $\Gamma(M,N)$. Thus we may define a libra operator $\lfloor,,\rfloor$ on $\mathbf{L}$ by

$$(\forall\,\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathbf{L})\quad\lfloor\mathfrak{a},\mathfrak{b},\mathfrak{c}\rfloor\equiv\rho^{-1}(\rho_{\mathfrak{a}}\circ{\rho_{\mathfrak{b}}}^{-1}\circ\rho_{\mathfrak{c}}).$$

By definition, ρ is a libra representation. If we define $\mathcal{T}$ for this example by

$$\mathcal{T}\equiv\{[\mathbf{X}\overset{\rho}{=}\mathbf{Y}]:\ [\mathbf{X},\mathbf{Y}]\in M\times N\},$$

then each element $[\mathbf{X}\overset{\rho}{=}\mathbf{Y}]$ is the plane of $\mathbf{S}$ containing $\mathbf{X}$ and $\mathbf{Y}$, and so is the plane in $\mathbf{S}$ tangent to $\mathbf{Q}$ at the point $\mathbf{X}\wedge\mathbf{Y}$ of intersection of $\mathbf{X}$ and $\mathbf{Y}$. It is evident that ρ is equivalent to the left inner representation of $\mathbf{L}$ relative to $\mathcal{T}$. For $\mathfrak{q}\in\mathbf{Q}$ we define

$$/\mathfrak{q}/\equiv\text{the element of } M \text{ containing } \mathfrak{q},$$

$$\backslash\mathfrak{q}\backslash\equiv\text{the element of } N \text{ containing } \mathfrak{q}.$$

For $\mathfrak{a},\mathfrak{b}\in\mathbf{L}$, we define

$$(\forall\,\mathfrak{x}\in\mathbf{Q})\quad\widehat{\mathfrak{a}\mathfrak{b}}(\mathfrak{x})\equiv/\widehat{\mathfrak{b}}(\mathfrak{x})/\wedge\backslash\widehat{\mathfrak{a}}(\mathfrak{x})\backslash$$

as illustrated in Figure (22).

[38] *Cf.* (6.7).

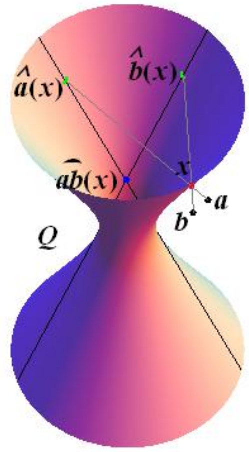

Fig. 22: The Value of $\widehat{\mathfrak{ab}}$ at a Point $\mathfrak{x}$ of $\mathbf{Q}$.

From Theorem (9.10) we know that these mappings $\widehat{\mathfrak{ab}}$, for $\mathfrak{a}, \mathfrak{b} \in \mathbf{L}$, are precisely the family of functions ϕ which send $\mathbf{Q}$ to itself such that rules in M are sent to rules in N in such a way that, for each line $N \in M$, the restriction of ϕ to N is a projective mapping.

From Corollary (9.14) we know that the mappings $\widehat{\mathfrak{uv}} \circ \widehat{\mathfrak{a}}$, for $\mathfrak{a}, \mathfrak{u}, \mathfrak{v} \in \mathbf{L}$, are precisely the family of functions δ which send $\mathbf{Q}$ to itself such that rules in M are sent to rules in M, rules in N are sent to rules in N and the restriction of δ to any rule is a projective mapping.

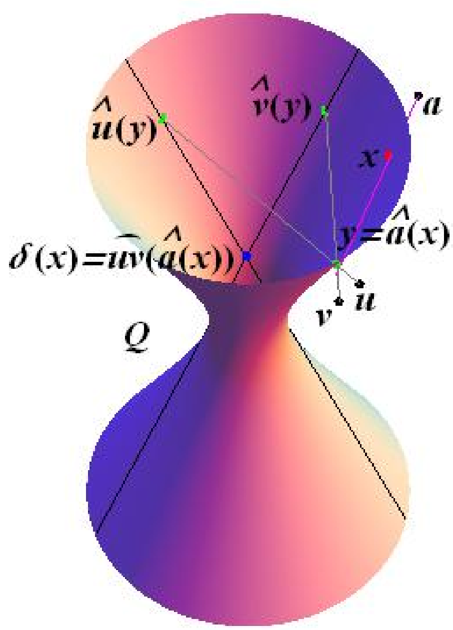

Fig. 23: The Value of $\widehat{\mathfrak{uv}} \circ \widehat{\mathfrak{a}}$ at a Point $\mathfrak{x}$ of $\mathbf{Q}$.

10. Meridian Space

(10.1) Discussion, Notation, and Definitions We purpose in the present section to show, how on the one hand, a meridian libra together with its meridian aggregate forms a three dimensional projective space and how, on the other hand, quadric surfaces in three dimensional projective spaces may be viewed as meridian aggregates.

Toward the first end we introduce the notion of a "space polar", which will provide us with a three dimensional projective space. Let $^{\blacksquare}$ be a function sending each element $\mathbf{s}$ of a non-void set $\mathbf{S}$ to a subset $\mathbf{s}^{\blacksquare}$ of $\mathbf{S}$. We call $^{\blacksquare}$ a **pre-polar operator** and adopt the following notation:

(i) $(\forall\, \mathbf{X}\subset\mathbf{S})\quad \mathbf{X}^{\bullet} \equiv \bigcap\limits_{\mathbf{s}\in\mathbf{X}} \mathbf{s}^{\blacksquare};$

(ii) $(\forall\, \mathbf{X}\subset\mathbf{S})\quad \mathbf{X}^{\bullet\bullet} \equiv (\mathbf{X}^{\bullet})^{\bullet}.$

The function $^{\bullet}$ will be called the **polar operator induced by** $^{\blacksquare}$.[39]

We shall say that $^{\blacksquare}$ is a **definitive pre-polar operator** if

$$(\forall\, \mathbf{s}\in\mathbf{S})\quad \mathbf{s}^{\blacksquare\bullet} \equiv \{\mathbf{s}\}, \tag{1}$$

in which case $^{\bullet}$ will be called a **definitive polar operator**.[40] We adopt the notation

(iii) $\mathfrak{P} \equiv \{\mathbf{s}^{\blacksquare} : \mathbf{s}\in\mathbf{S}\};$

(iv) $\mathfrak{L} \equiv \{\{\mathbf{x},\mathbf{y}\}^{\bullet\bullet} : \mathbf{x},\mathbf{y}\in\mathbf{S} \text{ distinct}\};$

Elements of $\mathfrak{P}$ will be called **planes** and elements of $\mathfrak{L}$ will be called **lines**. Two subsets of $\mathbf{S}$ will be said to **cross** if they are lines and have a single point in common. A **triangular triple** is an ordered triple $[\mathbf{a},\mathbf{b},\mathbf{c}]$ of distinct points such that $\mathbf{c}\notin\{\mathbf{a},\mathbf{b}\}^{\bullet\bullet}$. A definitive pre-polar operator $^{\blacksquare}$ for which the following three requirements are met will be called **space pre-polar operator**, in which case $^{\bullet}$ will be called a **space polar operator**:

$$\text{two lines of } \mathbf{S} \text{ cross if and only if their polars are lines which cross,} \tag{2}$$

$$(\forall\, [\mathbf{a},\mathbf{b},\mathbf{c}] \text{ a triangular triple})\quad \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet\bullet}\in\mathfrak{P}, \tag{3}$$

and

$$\text{each line in } \mathbf{S} \text{ is a proper subset of } \mathbf{S}, \text{ with at least three points.} \tag{4}$$

(10.2) Theorem Let $^{\blacksquare}$ be a pre-polar operator on a set $\mathbf{S}$ containing more than one element. Then

(i) $(\forall\, \mathbf{X}\subset\mathbf{Y}\subset\mathbf{S})\quad \mathbf{Y}^{\bullet}\subset\mathbf{X}^{\bullet}.$

(ii) $(\forall\, \mathbf{x}\in\mathbf{X})\quad \mathbf{x}^{\blacksquare} = \{\mathbf{x}\}^{\bullet};$

If $^{\blacksquare}$ is a definitive pre-polar operator, then

(iii) $(\forall\, \mathbf{X}\subset\mathbf{S})\quad \mathbf{X}\subset\mathbf{X}^{\bullet\bullet};$

(iv) $(\forall\, \mathbf{X}\subset\mathbf{S})\quad \mathbf{X}^{\bullet\bullet\bullet} = \mathbf{X}^{\bullet};$

(v) $\mathbf{S}^{\bullet} = \emptyset;$

(vi) $(\forall\, \mathbf{X}\subset\mathbf{Y}^{\bullet}\subset\mathbf{S})\quad \mathbf{X}^{\bullet\bullet}\subset\mathbf{Y}^{\bullet};$

(vii) $(\forall\, \mathbf{p},\mathbf{q}\in\mathbf{S})\quad \mathbf{p}\in\mathbf{q}^{\blacksquare} \iff \mathbf{q}\in\mathbf{p}^{\blacksquare}.$

Proof. $\xRightarrow{\text{(i) and (ii)}}$ These follow directly from the definition (10.1.i).

$\xRightarrow{\text{(iii)}}$ Let $\mathbf{x}$ be in $\mathbf{X}$ and $\mathbf{c}$ be in $\mathbf{X}^{\bullet}$. We have

$$\mathbf{c}\in\mathbf{X}^{\bullet} \xRightarrow{\text{by (i)}} \mathbf{c}\in\mathbf{x}^{\blacksquare} \overset{\text{by (ii)}}{=\!=} \{\mathbf{x}\}^{\bullet} \xRightarrow{\text{by (i)}} \{\mathbf{x}\}^{\bullet\bullet}\subset\{\mathbf{c}\}^{\bullet} \xRightarrow{\text{by (10.1.1) and (ii)}} \{\mathbf{x}\}\subset\mathbf{c}^{\blacksquare}$$

Since $\mathbf{c}$ was a generic element of $\mathbf{X}^{\bullet}$, it follows that $\mathbf{x}\in\mathbf{X}^{\bullet}$. Since $\mathbf{x}$ was a generic element of $\mathbf{X}$, it follows that (iii) holds.

$\xRightarrow{\text{(iv)}}$ From (iii) and (i) follows that $\mathbf{X}^{\bullet\bullet\bullet}\subset\mathbf{X}^{\bullet}$. Substituting $\mathbf{X}^{\bullet}$ for $\mathbf{X}$ in (iii), we have $\mathbf{X}^{\bullet}\subset\mathbf{X}^{\bullet\bullet\bullet}$. It follows that (iv) holds.

$\xRightarrow{\text{(v)}}$ Assume that there did exist $\mathbf{x}\in\mathbf{S}\cap\mathbf{S}^{\bullet}$. Let $\mathbf{y}\in\mathbf{S}$ be distinct from $\mathbf{x}$. By (10.1.1) there exists $\mathbf{s}$ in $\mathbf{y}^{\blacksquare}$ such that $\mathbf{x}\notin\mathbf{s}^{\blacksquare}$. By (ii) this means that $\{\mathbf{x}\}\not\subset\{\mathbf{s}\}^{\bullet}$. But $\mathbf{x}$ being in $\mathbf{S}^{\bullet}$ means that $\mathbf{S}=\mathbf{x}^{\blacksquare}$ and so

$$\{\mathbf{s}\}\subset\mathbf{S}=\mathbf{x}^{\blacksquare} \xRightarrow{\text{by (i) and (ii)}} \{\mathbf{x}\}=\mathbf{x}^{\bullet\bullet}\subset\{\mathbf{s}\}^{\bullet},$$

[39] The function $^{\square}$ defined on the points of a libra, and the function $^{\circ}$ defined on the subsets of a libra, respectively, are examples, respectively, of a pre-polar operator, and of a polar operator: *cf.* (7.1).

[40] When L is a meridian libra, the polar operator $^{\circ}$ is definitive: *cf.* (8.31).

which is absurd.

$\overset{\text{(vi)}}{\Longrightarrow}$ We have

$$\mathsf{X}\subset\mathsf{Y}^{\bullet}\xRightarrow{\text{by (i)}}\mathsf{Y}^{\bullet\bullet}\subset\mathsf{X}^{\bullet}\xRightarrow{\text{by (i)}}\mathsf{X}^{\bullet\bullet}\subset\mathsf{Y}^{\bullet\bullet\bullet}\overset{\text{by (iv)}}{=\!=\!=}\mathsf{Y}^{\bullet}.$$

$\overset{\text{(vii)}}{\Longrightarrow}$ We have

$$\mathfrak{p}\in\mathfrak{q}^{\blacksquare}\xRightarrow{\text{by (ii)}}\{\mathfrak{p}\}\subset\{\mathfrak{q}\}^{\bullet}\xRightarrow{\text{by (i)}}\{\mathfrak{q}\}^{\bullet\bullet}\subset\{\mathfrak{p}\}^{\bullet}\xRightarrow{\text{by (ii)}}\mathfrak{q}\in\mathfrak{p}^{\blacksquare}$$

and the reverse implication is proven analogously. QED

(10.3) Theorem Let $^{\blacksquare}$ be a space pre-polar operator on a set S with at least two points. Then S, along with the family $\mathfrak{L}$ (of lines), is a three dimensional synthetic projective space.

Proof. For a point $\mathfrak{x}$ disjoint from a line K we shall adopt the notation

$$\boxed{\mathfrak{x},\mathsf{K}}\equiv\bigcup_{\mathfrak{y}\in\mathsf{K}}\{\mathfrak{x},\mathfrak{y}\}^{\bullet\bullet}. \tag{1}$$

We must show that the following axioms are satisfied[41]:

(i) there exists at least one point and one line in S;
(ii) each couple $\mathfrak{a}$ and $\mathfrak{b}$ of distinct points in S lie on some line;
(iii) there is not more than one line through any two distinct points of S;
(iv) if $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ are distinct points of S such that $\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}$ and $\{\mathfrak{c},\mathfrak{d}\}^{\bullet\bullet}$ intersect, then $\{\mathfrak{a},\mathfrak{c}\}^{\bullet\bullet}$ and $\{\mathfrak{b},\mathfrak{d}\}^{\bullet\bullet}$ intersect;
(v) each line has at least three distinct points of S;
(vi) the whole space S is not a line;
(vii) if $\mathfrak{a}$ is a point not on a line N, then $\mathsf{S}\neq\boxed{\mathfrak{a},\mathsf{N}}$;
(viii) there exist points $\mathfrak{a}$ and $\mathfrak{c}$ and a line N such that $\mathfrak{a}\notin\boxed{\mathfrak{c},\mathsf{N}}$ and $\mathsf{S}=\bigcup_{\mathfrak{y}\in\boxed{\mathfrak{c},\mathsf{N}}}\{\mathfrak{a},\mathfrak{y}\}^{\bullet\bullet}$.

$\overset{\text{(i)}}{\Longrightarrow}$: By hypothesis S has at least two points, and therefor a line.

$\overset{\text{(ii)}}{\Longrightarrow}$: Evidently $\mathfrak{a},\mathfrak{b}\in\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}$.

$\overset{\text{(iii)}}{\Longrightarrow}$: Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are distinct points in S, K and N distinct lines in S, and $\mathfrak{a},\mathfrak{b}\in\mathsf{K}\cap\mathsf{N}$. We know that $\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}$ is a line, and without loss of generality we may assume that $\mathsf{N}\neq\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}$. There exist $\mathfrak{u},\mathfrak{v}\in\mathsf{S}$ such that $\mathsf{N}=\{\mathfrak{u},\mathfrak{v}\}^{\bullet\bullet}$. We have

$$\mathfrak{a},\mathfrak{b}\in\mathsf{N}\xRightarrow{\text{by (10.2.i)}}\mathsf{N}^{\bullet}\subset\{\mathfrak{a},\mathfrak{b}\}^{\bullet}\xRightarrow{\text{by (10.2.i)}}\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}\subset\mathsf{N}^{\bullet\bullet}\overset{\text{by (10.2.iv)}}{=\!=\!=\!=}\mathsf{N},$$

whence follows that there exists $\mathfrak{c}\in\mathsf{N}$ which is not in $\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}$. Thus $[\mathfrak{a},\mathfrak{b},\mathfrak{c}]$ is a triangular triple and (10.1.3) implies that $\{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}^{\bullet\bullet}=\mathfrak{d}^{\blacksquare}$ for some $\mathfrak{d}\in\mathsf{S}$. Consequently

$$\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathsf{N}\xRightarrow{\text{by (10.2.i)}}\mathsf{N}^{\bullet}\subset\{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}^{\bullet}\xRightarrow{\text{by (10.2.i)}}\mathfrak{d}^{\blacksquare}=\{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}^{\bullet\bullet}\subset\mathsf{N}^{\bullet\bullet}\xRightarrow{\text{by (10.2.i, ii and iv)}}$$
$$\mathsf{N}=\mathsf{N}^{\bullet\bullet\bullet}\subset\mathfrak{d}^{\blacksquare\bullet}=\{\mathfrak{d}\},$$

which is absurd.

$\overset{\text{(iv)}}{\Longrightarrow}$: If $\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}$ equals $\{\mathfrak{c},\mathfrak{d}\}^{\bullet\bullet}$, the conclusion of (iv) is trivial, so we shall presume that they are distinct. The lines $\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}$ and $\{\mathfrak{c},\mathfrak{d}\}^{\bullet\bullet}$ have a point $\mathfrak{e}$ in common. By (iii), $[\mathfrak{a},\mathfrak{e},\mathfrak{c}]$ forms a triangular triple. Thus (10.1.3) implies that there exists $\mathfrak{p}\in\mathsf{S}$ such that $\{\mathfrak{a},\mathfrak{e},\mathfrak{c}\}\subset\mathfrak{p}^{\blacksquare}$. From (10.2.vi) follows

$$(\{\mathfrak{a},\mathfrak{e}\}^{\bullet\bullet}\cup\{\mathfrak{c},\mathfrak{e}\}^{\bullet\bullet})\subset\mathfrak{p}^{\blacksquare}\xRightarrow{\text{by (iii)}}(\{\mathfrak{a},\mathfrak{b}\}^{\bullet\bullet}\cup\{\mathfrak{c},\mathfrak{d}\}^{\bullet\bullet})\subset\mathfrak{p}^{\blacksquare}\Longrightarrow\mathfrak{a},\mathfrak{d},\mathfrak{b},\mathfrak{c}\in\mathfrak{p}^{\blacksquare}\Longrightarrow$$
$$(\{\mathfrak{a},\mathfrak{c}\}^{\bullet\bullet}\cup\{\mathfrak{b},\mathfrak{d}\}^{\bullet\bullet})\subset\mathfrak{p}^{\blacksquare}\Longrightarrow\mathfrak{p}\in(\{\mathfrak{a},\mathfrak{c}\}^{\bullet}\cap\{\mathfrak{b},\mathfrak{d}\}^{\bullet})$$

(where we have used (10.2) freely). It follows from (10.2.iv) that the polars of the lines $\{\mathfrak{a},\mathfrak{c}\}^{\bullet\bullet}$ and $\{\mathfrak{b},\mathfrak{d}\}^{\bullet\bullet}$ cross, and so (10.1.2) implies that the lines themselves cross.

$\overset{\text{(v)}}{\Longrightarrow}$: This follows directly from (10.1.4).

$\overset{\text{(vi)}}{\Longrightarrow}$: By hypothesis S has at least two elements, and so it has a line, but by (10.1.4) this line is proper.

[41] *Cf.* [A. Seidenberg] Chapter V, Section 1.

$\xRightarrow{\text{(vii)}}$: Choose u and v in S such that $N = \{u,v\}^{\bullet\bullet}$. Then $[u,v,a]$ is a triangular triple and so by (10.1.3) there exists $p\in S$ such that $\{u,v,a\}^{\bullet\bullet} = p^{\blacksquare}$. We have

$$\{u,v\}\subset\{u,v,a\} \implies \{u,v,a\}^{\bullet}\subset\{u,v\}^{\bullet} \implies N = \{u,v\}^{\bullet\bullet}\subset\{u,v,a\}^{\bullet\bullet} = p^{\blacksquare}$$

and so, for each $n\in N$,

$$n\in p^{\blacksquare} \implies \{n,a\}\subset p^{\blacksquare} \implies p^{\blacksquare\bullet}\subset\{n,a\}^{\bullet} \implies \{n,a\}^{\bullet\bullet}\subset p^{\blacksquare\bullet\bullet} = p^{\blacksquare} \implies \boxed{a, N} = \subset p^{\blacksquare}.$$

If $p^{\blacksquare}$ were equal to S, then

$$\emptyset \xlongequal{\text{by (10.2.v)}} S^{\bullet} = p^{\blacksquare\bullet} = \{p\}^{\bullet\bullet} = \{p\},$$

which is absurd. It follows that $\boxed{a, N}\neq S$.

$\xRightarrow{\text{(viii)}}$: By hypothesis there exists a line $N = \{u,v\}^{\bullet\bullet}$ not equal to S and so there exists $c\in S$ not in $\{u,v\}^{\bullet\bullet}$. The triple $[u,v,c]$ is triangular and so (10.1.3) implies that there exists $p\in S$ such that

$$\{u,v,c\}^{\bullet\bullet}{=}p^{\blacksquare}. \tag{2}$$

Assume that there were an $s\in S$ not in $\bigcup_{y\in\boxed{c,\,N}} \{a,y\}^{\bullet\bullet}$. If $\{s,a\}^{\bullet\bullet}$ intersected $\{u,v,c\}^{\bullet\bullet}$ at some point x, then

$$u,v,x,c\in\{u,v,c\}^{\bullet\bullet} = p^{\blacksquare} \implies p\in\{u,v,x,c\}^{\bullet} = \{u,v\}^{\bullet}\cap\{c,x\}^{\bullet}$$

which means that $\{u,v\}^{\bullet}$ would cross $\{c,x\}^{\bullet}$. Then (10.1.2) would imply that $\{u,v\}^{\bullet\bullet}$ and $\{c,x\}^{\bullet\bullet}$ would cross. If t were that crossing point, then

$$x\in\{c,x\}^{\bullet\bullet} = \{t,c\}^{\bullet\bullet} \text{ and } t\in\{u,v\}^{\bullet\bullet} = N \implies x\in\boxed{c, N}$$

which, since $s\in\{s,a\}^{\bullet\bullet} = \{x,a\}^{\bullet\bullet}$, would imply

$$s\in \bigcup_{y\in\boxed{c,\,N}} \{a,y\}^{\bullet\bullet}\text{: an absurdity.}$$

It follows that $\{s,a\}^{\bullet\bullet}\cap\{u,v,c\}^{\bullet\bullet} = \emptyset$. Consequently

$$\emptyset = \{s,a\}^{\bullet\bullet}\cap\{u,v,c\}^{\bullet\bullet} = (\{s,a\}^{\bullet}\cup\{u,v,c\}^{\bullet})^{\bullet} \xlongequal{\text{by (2)}} (\{s,a\}^{\bullet}\cup\{p\}^{\bullet\bullet})^{\bullet} = (\{s,a\}^{\bullet}\cup\{p\})^{\bullet}. \tag{3}$$

Now p is either in $\{s,a\}^{\bullet}$ or not. In the first case, (2) would imply that $\emptyset = \{s,a\}^{\bullet\bullet}$, which is absurd. Thus

$$p\notin\{s,a\}^{\bullet}. \tag{4}$$

The lines $\{s,a\}^{\bullet\bullet}$ and $\{v,a\}^{\bullet\bullet}$ cross so (10.1.2) implies in particular that the polar $\{s,a\}^{\bullet}$ of $\{s,a\}^{\bullet\bullet}$ is a line. Thus there exist $w,z\in S$ such that

$$\{s,a\}^{\bullet} = \{w,z\}^{\bullet\bullet}. \tag{5}$$

The triple (w,z,p) is triangular by (4), and so (10.1.3) implies that there exists $q\in S$ such that

$$\{w,z,p\}^{\bullet\bullet} = q^{\blacksquare}. \tag{6}$$

Now

$$\{w,z,p\}^{\bullet} = (\{w,z\}\cup\{p\})^{\bullet} = \{w,z\}^{\bullet}\cap p^{\blacksquare} = \{w,z\}^{\bullet\bullet\bullet}\cap p^{\blacksquare} \xlongequal{\text{by (5)}} \{s,a\}^{\bullet\bullet}\cap p^{\blacksquare} \tag{7}$$

and so

$$\{q\} \xlongequal{\text{by (6)}} (\{w,z,p\}^{\bullet\bullet})^{\bullet} = \{w,z,p\}^{\bullet} \xlongequal{\text{by (7)}} \{s,a\}^{\bullet\bullet}\cap p^{\blacksquare} = (\{s,a\}^{\bullet}\cup\{p\})^{\bullet} \xlongequal{\text{by (3)}} \emptyset$$

which is absurd. QED

(10.4) Discussion Let S be any three dimensional projective space. We write $\mathfrak{L}$ for the family of lines in S and $\mathfrak{P}$ for the family of planes in S. A **collineation** of S is an incidence preserving bijection from $S\cup\mathfrak{P}$ to $S\cup\mathfrak{P}$ which sends points to points and planes to planes. Since a line is the intersection of all planes containing it, a collineation also sends lines to lines. A **correlation** of S is an incidence preserving bijection from $S\cup\mathfrak{P}$ to $S\cup\mathfrak{P}$ which sends points to planes and planes to points.[42] A correlation Φ such that

$$(\forall x\in S)\quad \Phi(\Phi(x)) = x$$

[42] Collineations also send lines to lines.

is called a **polarity**.

The space polar operator defined in (10.1) evidently induces a correlation of the induced space $\mathbf{S}$. Conversely, if Φ is any correlation polarity on $\mathbf{S}$, the restriction of Φ to $\mathbf{S}$ is a space pre-polar $^{\blacksquare}$ and

$$(\forall\, \mathbf{P}\in\mathfrak{P}) \quad \{\Phi(\mathbf{P})\} = \mathbf{P}^{\bullet}.$$

A **quadric surface** $\mathbf{Q}$ is a subset of $\mathbf{S}$ of the form $\{\mathfrak{x}\in\mathbf{S} : \mathfrak{x}\in\Phi(\mathfrak{x})\}$, where Φ is a correlation polarity, or equivalently, a set of the form

$$\{\mathfrak{x}\in\mathbf{S} : \mathfrak{x}\in\mathfrak{x}^{\blacksquare}\}$$

for a space pre-polar operator $^{\blacksquare}$. Some quadric surfaces contain lines, and some don't. For instance, a sphere in real projective space does not, and an elliptic hyperboloid does.

Those which contain lines are said to be **ruled**. These lines come in two families $\mathfrak{C}$ and $\mathfrak{R}$, called **reguli**. Each of $\mathfrak{C}$ and $\mathfrak{R}$ is a partition of $\mathbf{Q}$, and each element of $\mathfrak{C}$ intersects each element of $\mathfrak{R}$ in exactly one point. Thus a ruled surface may be identified with the product $\mathfrak{C}\times\mathfrak{R}$.

There are many ruled quadric surfaces. In fact, for any triple of mutually disjoint lines, there is exactly one quadric surface which contains that triple in one of its reguli.

Each ruled quadric surface $\mathbf{Q}$ induces a libra operator $\lfloor , , \rfloor$ on its complement in $\mathbf{S}$, which we shall now describe. For $\mathfrak{a}\in\mathbf{S}$ not in $\mathbf{Q}$ and each $\mathbf{C}\in\mathfrak{C}$, the plane determined by $\mathfrak{a}$ and $\mathbf{C}$ intersects $\mathbf{Q}$ in exactly one element $\hat{\mathfrak{a}}(\mathbf{C})$ of $\mathfrak{R}$. Furthermore, for any three $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\in\mathbf{S}$ not in $\mathbf{Q}$, there exists exactly one element $\mathfrak{d}$ of $\mathbf{S}$ (not in $\mathbf{Q}$) such that $\hat{\mathfrak{d}} = \hat{\mathfrak{a}}\circ(\hat{\mathfrak{b}})^{-1}\circ\hat{\mathfrak{c}}$. We set $\lfloor\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\rfloor\equiv\mathfrak{d}$. The libra thus defined is a meridian libra and it induces meridian operators on $\mathfrak{C}$ and $\mathfrak{R}$. Rather than proving these statements here, we shall instead show how a meridian libra *per se*, together with its meridian aggregate of cosets, generates a projective three dimensional space.

(10.5) Definitions and Notation Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. We saw in (6.17) that the members of $\mathcal{T}$ could be viewed as elements of a matrix, the columns of which being composed of the members of $\boxed{|||}$ and the rows of which being composed of the members of $\boxed{\equiv}$. This suggests forming the cartesian product

$$\mathbf{Q} \equiv \boxed{|||}\times\boxed{\equiv}. \tag{1}$$

It turns out that the pre-polar operator $^{\square}$ on $\mathbf{L}$ has a felicitous extension $^{\blacksquare}$ to the union

$$\mathbf{S} \equiv \mathbf{L}\cup\mathbf{Q} \tag{2}$$

We shall require some notation from (9.9):

$$(\forall\, \mathcal{C}\in\boxed{|||}, \mathcal{R}\in\boxed{\equiv}) \quad \boxed{\equiv}_{\mathcal{C}} = \{[\mathcal{C},\mathcal{Y}] : \mathcal{Y}\in\boxed{\equiv}\} \quad \text{and} \quad \boxed{|||}_{\mathcal{R}} = \{[\mathcal{X},\mathcal{R}] : \mathcal{X}\in\boxed{|||}\}. \tag{3}$$

We begin our extension of the pre-polar operator by, for $\mathbf{p} = [\mathcal{X},\mathcal{Y}]\in\mathbf{Q}$, letting $\mathbf{p}^{\square}$ be the element of $\mathcal{T}$ such that $\{\mathbf{p}^{\square}\}\equiv\mathcal{X}\cap\mathcal{Y}$:

$$\mathbf{p}^{\square} \equiv \mathcal{X}\wedge\mathcal{Y}. \tag{4}$$

Thus

$$(\forall\, \mathbf{p}\in\mathbf{Q}) \quad \mathbf{p} = [\|\mathbf{p}^{\square}\|, \underline{\underline{\overline{\overline{\mathbf{p}^{\square}}}}}]. \tag{5}$$

Recalling the notation from (6.17), we also have

$$(\forall\, \mathbf{p}\in\mathbf{Q}) \quad [\|\mathbf{p}^{\square}\| \overset{\hat{\mathcal{T}}}{=} \underline{\underline{\overline{\overline{\mathbf{p}^{\square}}}}}] = \mathbf{p}^{\square}. \tag{6}$$

For $\mathbf{p}\in\mathbf{L}$ we let

$$\mathbf{p}^{\boxminus} \equiv \{\mathbf{q}\in\mathbf{Q} : \mathbf{p}\in\mathbf{q}^{\square}\} \tag{7}$$

and, for $\mathbf{p} = [\mathcal{X},\mathcal{Y}]\in\mathbf{Q}$, we let

$$\mathbf{p}^{\boxminus} \equiv \boxed{|||}_{\underline{\underline{\overline{\overline{\mathbf{p}^{\square}}}}}} \cup \boxed{\equiv}_{\|\mathbf{p}^{\square}\|} = \{\mathbf{q}\in\mathbf{Q} : q^{\square}\in\left\|\underline{\underline{\overline{\overline{\mathbf{p}^{\square}}}}}\right\|\}. \tag{8}$$

For general $\mathbf{p}\in\mathbf{S}$ we let

$$\mathbf{p}^{\blacksquare} \equiv \mathbf{p}^{\square}\cup\mathbf{p}^{\boxminus}. \tag{9}$$

For general subsets $\mathfrak{U}$ of $\mathbf{S}$ we define

$$\mathfrak{U}^{\bullet} \equiv \bigcap_{\mathfrak{p}\in\mathfrak{U}} \mathfrak{p}^{\blacksquare}, \quad \mathfrak{U}^{\circ} \equiv \mathfrak{U}^{\bullet} \cap \mathbf{L} \quad \text{and} \quad \mathfrak{U}^{\ominus} \equiv \mathfrak{U}^{\bullet} \cap \mathbf{Q}. \tag{10}$$

Thus, $^{\blacksquare}$ is a pre-polar operator on $\mathbf{S}$ with associated polar operator $^{\bullet}$.

(10.6) Lemma Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. For $\mathfrak{p}\in\mathbf{L}$ we have

$$(\mathfrak{p}^{\square})^{\ominus} = \emptyset. \tag{1}$$

Proof. For $\mathfrak{x}\in p^{\square}$ we have $\mathfrak{x}^{\blacksquare} = \mathfrak{x}^{\square} \cup \{\mathfrak{q}\in\mathbf{Q} : \mathfrak{x}\in\mathfrak{q}^{\square}\}$ and so

$$(\mathfrak{p}^{\square})^{\bullet} \cap \mathbf{Q} = \{\mathfrak{q}\in\mathbf{Q} : \mathfrak{p}^{\square} \subset \mathfrak{q}^{\square}\}.$$

If $\mathfrak{q}$ were any element of $\mathbf{Q}$ such that $\mathfrak{p}^{\square}\subset\mathfrak{q}^{\square}$, it would follow from (10.5.4) and (8.17.i) that $\mathfrak{p}^{\square} \cap \mathfrak{q}^{\square} = \mathfrak{q}^{\square}$ were a line trace, which by (8.13.i) and (8.16) is impossible. It follows that (1) holds. QED

(10.7) Theorem Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. Let $\mathfrak{p}$ be an element of $\mathbf{S}$. Then

$$\mathfrak{p}^{\blacksquare\bullet} = \{\mathfrak{p}\}. \tag{1}$$

In other words, the pre-polar $^{\blacksquare}$ is definitive.

Proof. Suppose first that $\mathfrak{p}$ is in $\mathbf{L}$. If $\mathfrak{q}$ is any element of $\mathfrak{p}^{\boxminus}$, then $\mathfrak{p}$ is in $\mathfrak{q}^{\square}$ and so, by (10.5.7), we have

$$\mathfrak{p}\in\mathfrak{p}^{\boxminus\circ}. \tag{2}$$

From (10.5.10) we have

$$\mathfrak{p}^{\blacksquare\bullet} = (\mathfrak{p}^{\square} \cup \mathfrak{p}^{\boxminus})^{\bullet} = (\mathfrak{p}^{\square})^{\bullet} \cap (\mathfrak{p}^{\boxminus})^{\bullet} =$$

$$(\mathfrak{p}^{\square\circ} \cup \mathfrak{p}^{\square\ominus}) \cap (\mathfrak{p}^{\boxminus\circ} \cup \mathfrak{p}^{\boxminus\ominus}) \xlongequal{\text{by (8.31) , (10.6.1) and (2)}} \{\mathfrak{p}\}. \tag{3}$$

Now suppose that $\mathfrak{p}$ is in $\mathfrak{q}$. We first note that from the definition (10.5.8) follows that

$$\{\mathfrak{p}\} = \mathfrak{p}^{\boxminus\ominus}. \tag{4}$$

From (8.16) follows that

$$\mathfrak{p}^{\square\circ} = \emptyset. \tag{5}$$

From (10.5.10) we have

$$\mathfrak{p}^{\square\ominus} = \bigcap_{\mathfrak{x}\in\mathfrak{p}^{\square}} \mathfrak{x}^{\boxminus} \xlongequal{\text{by (10.5.7)}} \bigcap_{\mathfrak{x}\in\mathfrak{p}^{\square}} \{\mathfrak{q}\in\mathbf{Q} : \mathfrak{x}\in\mathfrak{q}^{\square}\} = \{\mathfrak{q}\in\mathbf{Q} : \mathfrak{p}^{\square} \subset \mathfrak{q}^{\square}\} = \{\mathfrak{p}\}. \tag{6}$$

Consequently

$$\mathfrak{p}^{\blacksquare\bullet} = (\mathfrak{p}^{\square} \cup \mathfrak{p}^{\boxminus})^{\bullet} = (\mathfrak{p}^{\square})^{\bullet} \cap (\mathfrak{p}^{\boxminus})^{\bullet} = (\mathfrak{p}^{\square\circ} \cup \mathfrak{p}^{\square\ominus}) \cap (\mathfrak{p}^{\boxminus\circ} \cup \mathfrak{p}^{\boxminus\ominus}) \xlongequal{\text{by (4) and (5)}}$$

$$\{\mathfrak{p}\} \cap (\mathfrak{p}^{\boxminus\circ} \cup \mathfrak{p}^{\boxminus\ominus}) \xlongequal{\text{by (6)}} \{\mathfrak{p}\}.$$

From this last and (3) we have (1). QED

(10.8) Theorem Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. Let $\mathfrak{p}$ be an element of $\mathbf{Q}$. Then

$$(\forall\, \mathfrak{q}\in {\color{red}\equiv}_{\|\mathfrak{p}^{\square}\|}) \quad \{\mathfrak{q},\mathfrak{p}\}^{\bullet} = {\color{red}\equiv}_{\|\mathfrak{p}^{\square}\|} = ({\color{red}\equiv}_{\|\mathfrak{p}^{\square}\|})^{\bullet} = \{\mathfrak{q},\mathfrak{p}\}^{\bullet\bullet} \tag{1}$$

and

$$(\forall\, \mathfrak{q}\in {\color{red}\mathrm{III}}_{\underline{\underline{\overline{\overline{\mathfrak{p}^{\square}}}}}}) \quad \{\mathfrak{q},\mathfrak{p}\}^{\bullet} = {\color{red}\mathrm{III}}_{\underline{\underline{\overline{\overline{\mathfrak{p}^{\square}}}}}} = ({\color{red}\mathrm{III}}_{\underline{\underline{\overline{\overline{\mathfrak{p}^{\square}}}}}})^{\bullet} = \{\mathfrak{q},\mathfrak{p}\}^{\bullet\bullet}. \tag{2}$$

Proof. For $\mathfrak{q}\in {\color{red}\equiv}_{\|\mathfrak{p}^{\square}\|}$ distinct from $\mathfrak{p}$ we have

$$\{\mathfrak{q},\mathfrak{p}\}\subset{\color{red}\equiv}_{\|\mathfrak{p}^{\square}\|} \Longrightarrow ({\color{red}\equiv}_{\|\mathfrak{p}^{\square}\|})^{\bullet}\subset\{\mathfrak{q},\mathfrak{p}\}^{\bullet}. \tag{3}$$

That $\mathfrak{q}$ is in ${\color{red}\equiv}_{\|\mathfrak{p}^{\square}\|}$ implies that

$$\mathfrak{q}^{\square} = \|\mathfrak{q}^{\square}\| \wedge \underline{\underline{\overline{\overline{\mathfrak{q}^{\square}}}}} = \|\mathfrak{p}^{\square}\| \wedge \underline{\underline{\overline{\overline{\mathfrak{q}^{\square}}}}}.$$

Since $\mathbf{q}$ is distinct from $\mathbf{p}$, it follows that $\mathbf{q}^{\square}$ and $\mathbf{p}^{\square}$ are distinct elements of the partition $\|\mathbf{p}^{\square}\|$. Consequently

$$\mathbf{p}^{\square} \cap \mathbf{q}^{\square} = \emptyset \tag{4}$$

and

$$\mathbf{p}^{\boxminus} \cap \mathbf{q}^{\boxminus} \xlongequal{\text{by (10.5.8)}} \{\mathbf{y} \in \mathbf{Q} : \mathbf{y}^{\square} \in \left\|\overline{\overline{\underline{\mathbf{p}}}}\right\|\} \cap \{\mathbf{y} \in \mathbf{Q} : \mathbf{y}^{\square} \in \left\|\overline{\overline{\underline{\mathbf{q}}}}\right\|\} = \{\mathbf{y} \in \mathbf{Q} : \mathbf{y}^{\square} \in \|\mathbf{p}^{\square}\|\} = \equiv_{\|\mathbf{p}^{\square}\|}. \tag{5}$$

So we have

$$\{\mathbf{q}, \mathbf{p}\}^{\bullet} = \mathbf{q}^{\blacksquare} \cap \mathbf{p}^{\blacksquare} \xlongequal{\text{by (10.5.10)}} (\mathbf{q}^{\square} \cup \mathbf{q}^{\boxminus}) \cap (\mathbf{p}^{\square} \cup \mathbf{p}^{\boxminus}) = (\mathbf{q}^{\square} \cap \mathbf{p}^{\square}) \cup (\mathbf{q}^{\boxminus} \cap \mathbf{p}^{\boxminus}) \xlongequal{\text{by (4) and (5)}} \equiv_{\|\mathbf{p}^{\square}\|} \tag{6}$$

which, with (3), implies

$$(\equiv_{\|\mathbf{p}^{\square}\|})^{\bullet} \subset \equiv_{\|\mathbf{p}^{\square}\|}. \tag{7}$$

On the other hand, (6) implies

$$\equiv_{\|\mathbf{p}^{\square}\|} = \{\mathbf{q}, \mathbf{p}\}^{\bullet} \subset \mathbf{q}^{\blacksquare} \Longrightarrow \mathbf{q} \in (\equiv_{\|\mathbf{p}^{\square}\|})^{\bullet} \xRightarrow{\text{since } \mathbf{q} \text{ is generic}} \equiv_{\|\mathbf{p}^{\square}\|} \subset (\equiv_{\|\mathbf{p}^{\square}\|})^{\bullet} \tag{8}$$

as well as

$$\{\mathbf{q}, \mathbf{p}\}^{\bullet\bullet} = (\equiv_{\|\mathbf{p}^{\square}\|})^{\bullet}. \tag{9}$$

That (1) holds now follows from (6), (8) and (9).

An analogous argument shows that (2) holds as well. QED

(10.9) Theorem Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. Let $\mathbf{N}$ be a hyperbolic line trace[43] in $\mathbf{L}$. Then

$$\mathbf{N}^{\bullet} = \mathbf{N}^{\circ} \quad \text{and} \quad \mathbf{N}^{\bullet\bullet} = \mathbf{N}. \tag{1}$$

Proof. If $\mathbf{N}^{\bullet} \neq \mathbf{N}^{\circ}$, there would be $\mathbf{p}$ in $\mathbf{Q} \cap \mathbf{N}^{\bullet}$, which would imply that $\mathbf{N} \subset \mathbf{p}^{\square}$. But $\mathbf{p}^{\square}$ is in $\mathcal{T}$, which contradicts (8.23.iii). Consequently $\mathbf{N}^{\bullet} = \mathbf{N}^{\circ}$.

From (8.23.ii) it follows that $\mathbf{N}^{\bullet}$ is a hyperbolic line trace. Thus, by what we have just shown,

$$\mathbf{N}^{\bullet\bullet} = (\mathbf{N}^{\bullet})^{\circ} = (\mathbf{N}^{\circ})^{\circ} \xlongequal{\text{by (8.13.ii)}} \mathbf{N}.$$

QED

(10.10) Theorem Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. Let $\mathbf{N}$ be a parabolic line trace in $\mathbf{L}$, and let $\mathbf{T}$ be the element of $\mathcal{T}$ such that $\mathbf{N} \subset \mathbf{T}$. Then

$$\mathbf{N}^{\bullet} = \mathbf{N}^{\circ} \cup \{[\|\mathbf{T}\|, \overline{\overline{\underline{\mathbf{T}}}}]\} \tag{1}$$

and

$$\mathbf{N}^{\bullet\bullet} = \mathbf{N} \cup \{[\|\mathbf{T}\|, \overline{\overline{\underline{\mathbf{T}}}}]\}. \tag{2}$$

Proof. For any line trace $\mathbf{K}$ we have

$$\mathbf{K}^{\bullet} \xlongequal{\text{by (10.5.10)}} \bigcap_{\mathbf{s} \in \mathbf{K}} \mathbf{s}^{\blacksquare} \xlongequal{\text{by (10.5.9)}} \bigcap_{\mathbf{s} \in \mathbf{K}} (\mathbf{s}^{\square} \cup \mathbf{s}^{\boxminus}) = (\bigcap_{\mathbf{s} \in \mathbf{K}} \mathbf{s}^{\square}) \cup (\bigcap_{\mathbf{s} \in \mathbf{K}} \mathbf{s}^{\boxminus}) \xlongequal{\text{by (10.5.7) and (10.5.8)}}$$

$$\mathbf{K}^{\circ} \cup \{\mathbf{q} \in \mathbf{Q} : \mathbf{K} \subset \mathbf{q}^{\square}\} \xlongequal{\text{by (10.5.4)}} \mathbf{K}^{\circ} \cup \{[\|\mathbf{U}\|, \overline{\overline{\underline{\mathbf{U}}}}] : \mathbf{K} \subset \mathbf{U} \in \mathcal{T}\}. \tag{3}$$

Substituting $\mathbf{N}$ for $\mathbf{K}$ in (3), we see that (1) holds.

Using the fact that the $\mathbf{K}^{\circ}$ is a line trace too[44], that $\mathbf{K}^{\circ\circ} = \mathbf{K}$ [45] and substituting $\mathbf{K}^{\circ}$ for $\mathbf{K}$ in (3), we obtain

$$(\mathbf{K}^{\circ})^{\bullet} = \mathbf{K} \cup \{[\|\mathbf{U}\|, \overline{\overline{\underline{\mathbf{U}}}}] : \mathbf{K}^{\circ} \subset \mathbf{U} \in \mathcal{T}\}. \tag{4}$$

Thus

$$\mathbf{K}^{\bullet\bullet} \xlongequal{\text{by (10.5.10)}} (\mathbf{K}^{\circ} \cup \mathbf{K}^{\ominus})^{\bullet} = (\mathbf{K}^{\circ})^{\bullet} \cap (\mathbf{K}^{\ominus})^{\bullet} \xlongequal{\text{by (4)}}$$

$$(\mathbf{K} \cup \{[\|\mathbf{U}\|, \overline{\overline{\underline{\mathbf{U}}}}] : \mathbf{K}^{\circ}\}) \cap ((\mathbf{K}^{\ominus})^{\circ} \cup (\mathbf{K}^{\ominus})^{\ominus}) = (\mathbf{K} \cap (\mathbf{K}^{\ominus})^{\circ}) \cup (\{[\|\mathbf{U}\|, \overline{\overline{\underline{\mathbf{U}}}}] : \mathbf{K}^{\circ} \subset \mathbf{U} \in \mathcal{T}\} \cap (\mathbf{K}^{\ominus})^{\ominus}) \tag{5}$$

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[43] *Cf.* (8.12) and (8.20).
[44] *Cf.* (8.13.i).
[45] *Cf.* (8.13.ii).

We have

$$\mathsf{K}^{\ominus} \overset{\text{by (10.5.10)}}{=\!=\!=} \mathsf{K}^{\bullet} \cap \mathbf{Q} \overset{\text{by (10.1.i)}}{=\!=\!=} \bigcap_{\mathsf{x}\in\mathsf{K}} \mathsf{x}^{\boxminus} \overset{\text{by (10.5.7)}}{=\!=\!=}$$

$$\bigcap_{\mathsf{x}\in\mathsf{K}} \{[\|\mathsf{U}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{U}}}}}}}\,] : \mathsf{x}\in\mathsf{U}\in\mathcal{T}\} = \{[\|\mathsf{U}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{U}}}}}}}\,] : \mathsf{K}\in\mathsf{U}\in\mathcal{T}\}. \tag{6}$$

From (6) follows that $\mathsf{K} \subset (\mathsf{K}^{\ominus})^{\circ}$ which with (4) and (5), yield

$$\mathsf{K}^{\bullet\bullet} = \mathsf{K} \cup \Big(\{[\|\mathsf{U}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{U}}}}}}}\,] : \mathsf{K}^{\circ}\in\mathsf{U}\in\mathcal{T}\} \cap (\mathsf{K}^{\ominus})^{\ominus}\Big). \tag{7}$$

We have

$$\mathsf{N}^{\ominus} \overset{\text{by (6)}}{=\!=\!=} \{[\|\mathsf{T}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}}}}}}}\,]\} \Longrightarrow (\mathsf{N}^{\ominus})^{\ominus} \overset{\text{by (10.5.8)}}{=\!=\!=} \bigcap_{\mathsf{U}\in\left\|\overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}}}}}}}\right\|} [\|\mathsf{U}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{U}}}}}}}\,]$$

and

$$\{[\|\mathsf{U}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{U}}}}}}}\,] : \mathsf{N}^{\circ}\in\mathsf{U}\in\mathcal{T}\} = \{[\|\mathsf{T}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}}}}}}}\,]\}$$

and so substitution of N for T in (7) yields (2). QED

(10.11) Theorem Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. Let N be an elliptic line trace in $\mathbf{L}$, and let T_1 and T_2 be the elements of $\mathcal{T}$ such that $\mathsf{N}^{\circ} = \mathsf{T}_1 \cap \mathsf{T}_2$. Then

$$\mathsf{N}^{\bullet} = \mathsf{N}^{\circ} \cup \{[\|\mathsf{T}_1\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_2}}}}}}\,], [\|\mathsf{T}_2\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_1}}}}}}\,]\} \tag{1}$$

and

$$\mathsf{N}^{\bullet\bullet} = \mathsf{N} \cup \{[\|\mathsf{T}_1\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_1}}}}}}\,], [\|\mathsf{T}_2\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_2}}}}}}\,]\}. \tag{2}$$

Proof. We have seen that, for any line trace K in $\mathbf{L}$, (10.10.3), (10.10.6) and (10.10.7) hold. Substituting N for K in (10.10.3), we obtain (1).

Substituting N for K in (10.10.6), we obtain

$$\mathsf{N}^{\ominus} \overset{\text{by (6)}}{=\!=\!=} \{[\|\mathsf{T}_1\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_2}}}}}}\,], [\|\mathsf{T}_2\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_1}}}}}}\,]\} \Longrightarrow (\mathsf{N}^{\ominus})^{\ominus} \overset{\text{by (10.5.8)}}{=\!=\!=}$$

$$\bigcap_{\mathsf{U}\in\left(\left\|\overline{\overline{\underline{\underline{\|\mathsf{T}_1\| \wedge \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_2}}}}}}}}}}\right\| \cap \left\|\overline{\overline{\underline{\underline{\|\mathsf{T}_1\| \wedge \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_2}}}}}}}}}}\right\|\right)} [\|\mathsf{U}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{U}}}}}}}\,] = \{[\|\mathsf{T}_1\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_1}}}}}}\,], [\|\mathsf{T}_2\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_2}}}}}}\,]\}$$

and

$$\{[\|\mathsf{U}\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{U}}}}}}}\,] : \mathsf{N}^{\circ}\in\mathsf{U}\in\mathcal{T}\} = \{[\|\mathsf{T}_1\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_1}}}}}}\,], [\|\mathsf{T}_2\|, \overline{\overline{\overline{\underline{\underline{\underline{\mathsf{T}_2}}}}}}\,]\}$$

and so substitution of N for K in (10.10.7) yields (2). QED

(10.12) Recapitulation and Definitions In (10.5) we defined a pre-polar $^{\blacksquare}$ and polar $^{\bullet}$ and in (10.7) we showed that the first axiom (10.1.i) of a space polar operator was satisfied.

In (10.8), (10.9), (10.10) and (10.11) we described the lines of the polar operator. To recapitulate, we consider distinct points $\mathbf{p}$ and $\mathbf{q}$ of $\mathbf{S}$. If both of them are in $\mathbf{Q}$ and $\mathbf{q}^{\square}$ is either a left or a right coset of $\mathbf{p}^{\square}$, then (10.8) implies that all elements in the corresponding column or row comprise the set

$$\{\mathbf{p}, \mathbf{q}\}^{\bullet\bullet} = \{\mathbf{p}, \mathbf{q}\}^{\bullet}. \tag{1}$$

If $\mathbf{p}$ and $\mathbf{q}$ are in $\mathbf{Q}$ but not in a common row or column, then (8.13.ii) and (10.11.2) imply that

$$\{\mathbf{p}, \mathbf{q}\}^{\bullet\bullet} \cap \mathbf{L} \text{ is an elliptic line trace.} \tag{2}$$

If $\mathbf{p}$ is in $\mathbf{L}$ and $\mathbf{q}$ is in $\mathbf{Q}$, then either (2) holds or

$$\{\mathbf{p}, \mathbf{q}\}^{\bullet\bullet} \cap \mathbf{L} \text{ is a parabolic line trace.} \tag{3}$$

If $\mathbf{p}$ and $\mathbf{q}$ are both in $\mathbf{L}$, then either (2) holds, (3) holds or

$$\{\mathbf{p}, \mathbf{q}\}^{\bullet\bullet} \cap \mathbf{L} \text{ is a hyperbolic line trace.} \tag{4}$$

In the case of (1) we shall speak of a **quadric line**, in the case of (2) an **elliptic line**, in the case of (3) a **parabolic line** and in the case of (4) a **hyperbolic line**.

We proceed to show that $^\bullet$ is a space polar operator.

(10.13) Theorem Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. Two lines of $\mathbf{S}$ cross if and only if their polars are lines which cross.

Proof. We already know that polars of lines are lines. Suppose that $\mathbf{K}$ and $\mathbf{N}$ are lines such that $\mathbf{K}^\bullet \cap \mathbf{N}^\bullet$ is a singleton. It will suffice to show that $\mathbf{N}\cap\mathbf{K}$ is non-void. We consider the various cases *seriatim*.

$\xrightarrow{\mathbf{K}\text{ and }\mathbf{N}\text{ both quadric}}$ It is without loss of generality that we may presume that $\mathbf{K}$ is of the form $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}}$.

By (10.8)

$$\mathbf{K}^\bullet = \mathbf{K}. \tag{1}$$

If $\mathbf{N}^\bullet$ were of the form $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{q}^\square}}}}}$, then it would either be all of $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}}$, or not intersect $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}}$ at all. Consequently $\mathbf{N}^\bullet$ must be of the form $\equiv_{\|\mathbf{q}^\square\|}$ for some $\mathbf{q}\in\mathbf{Q}$. By (10.8.1) we know that $\mathbf{N} = \mathbf{N}^\bullet$. Thus

$$\mathbf{N}\cap\mathbf{K} = \equiv_{\|\mathbf{q}^\square\|} \cap \mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}} = \{[\|\mathbf{q}^\square\|, \overline{\overline{\underline{\underline{\mathbf{p}^\square}}}}]\}.$$

$\xrightarrow{\mathbf{K}\text{ quadric and }\mathbf{N}\text{ elliptic}}$ As above, we presume that $\mathbf{K}$ is of the form $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}}$ and that (1) holds. Since $\mathbf{N}$ is elliptic, by (10.11) there exist $\mathbf{A},\mathbf{B}\in\mathcal{T}$ distinct such that

$$\mathbf{N} = (\mathbf{A}\cap\mathbf{B})\cup\{[\|\mathbf{A}\|, \overline{\overline{\underline{\underline{\mathbf{B}}}}}], [\|\mathbf{B}\|, \overline{\overline{\underline{\underline{\mathbf{A}}}}}]\} \text{ and } \mathbf{N}^\blacksquare = \big((\|\mathbf{A}\|\wedge\overline{\overline{\underline{\underline{\mathbf{B}}}}})\cap(\|\mathbf{B}\|\wedge\overline{\overline{\underline{\underline{\mathbf{A}}}}})\big)\cup\{[\|\mathbf{A}\|, \overline{\overline{\underline{\underline{\mathbf{A}}}}}], [\|\mathbf{B}\|, \overline{\overline{\underline{\underline{\mathbf{B}}}}}]\}.$$

Since $\mathbf{N}^\blacksquare$ and $\mathbf{K}^\blacksquare$ intersect, either $[\|\mathbf{A}\|, \overline{\overline{\underline{\underline{\mathbf{A}}}}}]$ or $[\|\mathbf{B}\|, \overline{\overline{\underline{\underline{\mathbf{B}}}}}]$ is in $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}}$. Without loss of generality we shall presume that it is $[\|\mathbf{A}\|, \overline{\overline{\underline{\underline{\mathbf{A}}}}}]$: that $\overline{\overline{\underline{\underline{\mathbf{A}}}}} = \overline{\overline{\underline{\underline{\mathbf{p}^\square}}}}$. It follows that $[\|\mathbf{B}\|, \overline{\overline{\underline{\underline{\mathbf{A}}}}}]$ is in $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}}$, and so

$$\mathbf{N}\cap\mathbf{K} = \{[\|\mathbf{B}\|, \overline{\overline{\underline{\underline{\mathbf{A}}}}}]\}.$$

$\xrightarrow{\mathbf{K}\text{ quadric and }\mathbf{N}\text{ parabolic}}$ As before, we presume that $\mathbf{K}$ is of the form $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}}$ and that (1) holds. Since $\mathbf{N}$ is parabolic, by (10.10) there exists $\mathbf{T}\in\mathcal{T}$ such that

$$\mathbf{N} = (\mathbf{T}\cap\mathbf{N})\cup\{[\|\mathbf{T}\|, \overline{\overline{\underline{\underline{\mathbf{T}}}}}]\} \text{ and } \mathbf{N}^\blacksquare = (\mathbf{T}\cap\mathbf{N}^\bullet)\cup\{[\|\mathbf{T}\|, \overline{\overline{\underline{\underline{\mathbf{T}}}}}]\}.$$

Since $\mathbf{N}^\blacksquare$ and $\mathbf{K}^\blacksquare$ intersect, it follows that $[\|\mathbf{T}\|, \overline{\overline{\underline{\underline{\mathbf{T}}}}}]$ is in $\mathrm{IIII}_{\underline{\underline{\overline{\overline{\mathbf{p}^\square}}}}}$. Consequently

$$\mathbf{N}\cap\mathbf{K} = \{[\|\mathbf{T}\|, \overline{\overline{\underline{\underline{\mathbf{T}}}}}]\}.$$

$\xrightarrow{\mathbf{K}\text{ quadric and }\mathbf{N}\text{ hyperbolic}}$ In view of (10.9), this case cannot occur.

$\xrightarrow{\text{polars not in }\mathbf{Q}\text{ but intersect in }\mathbf{Q}}$ Let $\mathbf{p}$ be in $\mathbf{K}^\bullet\cap\mathbf{N}^\bullet\cap\mathbf{Q}$. From (10.5.10) and (10.5.4) we know that

$$\big((\mathbf{K}\cap\mathbf{L})\cup(\mathbf{N}\cap\mathbf{L})\big)\subset\mathbf{p}^\square \xrightarrow{\text{by (10.5.5)}} \mathbf{p}\in\mathbf{N}\cap\mathbf{K}.$$

$\xrightarrow{\text{polars do not intersect in }\mathbf{Q}}$ Then there is an element $\mathbf{p}$ of $\mathbf{L}$ such that $\mathbf{p}\in\mathbf{K}^\bullet\cap\mathbf{N}^\bullet$. From (8.30) follows that either $\mathbf{K}\cap\mathbf{N}\cap\mathbf{L}$ is non-void or there exists $X\in\mathcal{T}$ such that $\mathbf{K}^\circ\cup\mathbf{N}^\circ\subset X$. In the latter case we have

$$[\|X\|, \overline{\overline{\underline{\underline{X}}}}]\in(\mathbf{N}\cap\mathbf{K}).$$

QED

(10.14) Theorem Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. Let $(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ be a triangular triple, in the sense of (10.1.3). Then there exists $\mathbf{p}\in\mathbf{S}$ such that $\{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}^{\bullet\bullet} = \mathbf{p}^\blacksquare$.

Proof. Let $\mathbf{K}$ be the line $\{\mathbf{a},\mathbf{b}\}^{\bullet\bullet}$, and let $\mathbf{N}$ be the line $\{\mathbf{a},\mathbf{c}\}^{\bullet\bullet}$. We prove the theorem case by case:

$\xRightarrow{(\mathbf{K}\cup\{\mathbf{c}\})\subset\mathbf{Q}}$ In view of (10.5.5) we may, without loss of generality, presume that

$$\overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}} = \overline{\overline{\underline{\underline{\mathbf{b}^{\square}}}}}. \tag{1}$$

By (10.5.8) we have

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\} \subset [\,\|\mathbf{c}\|, \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}\,]^{\boxminus} \subset [\,\|\mathbf{c}\|, \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}\,]^{\blacksquare} \xRightarrow{\text{by (10.2.vi)}} \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet\bullet} \subset [\,\|\mathbf{c}\|, \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}\,]^{\blacksquare}. \tag{2}$$

Since $\|\mathbf{a}^{\square}\|$ either equals $\|\mathbf{b}^{\square}\|$ or is disjoint from it, and since $\mathbf{a}$ and $\mathbf{b}$ are distinct, it follows from (1) that $\|\mathbf{a}^{\square}\| \cap \|\mathbf{b}^{\square}\| = \emptyset$. Suppose that $\mathbf{r}\in\mathbf{Q}$ is in $\mathbf{a}^{\blacksquare}\cap\mathbf{b}^{\blacksquare}$. Then (10.5.8) implies that

$$\mathbf{r}^{\blacksquare} \in \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}. \tag{3}$$

Since $\mathbf{c}$ is not in $\mathbf{K}$, we know that $\overline{\overline{\underline{\underline{\mathbf{c}}}}}$ cannot equal $\overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}$ — thus $\overline{\overline{\underline{\underline{\mathbf{c}^{\square}}}}} \cap \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}} = \emptyset$.. Suppose that $\mathbf{r}\in\mathbf{Q}$ is in also in $\mathbf{c}^{\blacksquare}$. Then (10.5.8) implies that $\mathbf{r}^{\blacksquare}\in\|\mathbf{c}^{\square}\|$. Consequently (3) implies that

$$\mathbf{r} = [\,\|\mathbf{c}^{\square}\|, \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}\,] \Longrightarrow (\mathbf{a}^{\blacksquare}\cap\mathbf{b}^{\blacksquare}\cap\mathbf{c}^{\blacksquare}) \subset [\,\|\mathbf{c}^{\square}\|, \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}\,] \xRightarrow{\text{by (10.2.i)}}$$

$$[\,\|\mathbf{c}^{\square}\|, \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}\,]^{\blacksquare} \subset (\mathbf{a}^{\blacksquare}\cap\mathbf{b}^{\blacksquare}\cap\mathbf{c}^{\blacksquare})^{\bullet} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet\bullet}.$$

This with (3) implies that

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet\bullet} = [\,\|\mathbf{c}^{\square}\|, \overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}\,]^{\bullet}. \tag{4}$$

$\xRightarrow{\mathbf{b}\in\mathbf{Q}\text{ and }\mathbf{N}\subset\mathbf{Q}}$ If we interchange the roles of $\mathbf{b}$ and $\mathbf{c}$ in the above, we obtain (4) again.

$\xRightarrow{\mathbf{K}\subset\mathbf{Q}\text{ and }\mathbf{c}\in\mathbf{L}}$ Again we may presume that (1) holds. Since $\overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}$ is a partition of $\mathbf{L}$, one of its elements $\mathbf{P}$ contains $\mathbf{c}$. From (10.5.4) and (10.5.8) we obtain

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\}\in[\,\|\mathbf{P}\|, \overline{\overline{\underline{\underline{\mathbf{P}}}}}\,]^{\blacksquare}. \tag{5}$$

Suppose that $\mathbf{r}$ is in $\mathbf{a}^{\blacksquare}\cap\mathbf{b}^{\blacksquare}\cap\mathbf{c}^{\blacksquare}$. As above we have (3), and we also have from (10.5.7) that $\mathbf{c}\in r^{\blacksquare}$. Since $\overline{\overline{\underline{\underline{\mathbf{a}^{\square}}}}}$ is a partition of $\mathbf{L}$, it thus follows from (3) that

$$\mathbf{r}^{\blacksquare} = \mathbf{P} \Longrightarrow \mathbf{r} = [\,\|\mathbf{P}\|, \overline{\overline{\underline{\underline{\mathbf{P}}}}}\,].$$

This means that

$$(\mathbf{a}^{\blacksquare}\cap\mathbf{b}^{\blacksquare}\cap\mathbf{c}^{\blacksquare}) \subset \{[\,\|\mathbf{P}\|, \overline{\overline{\underline{\underline{\mathbf{P}}}}}\,]\} \xRightarrow{\text{by (10.2.i)}} [\,\|\mathbf{P}\|, \overline{\overline{\underline{\underline{\mathbf{P}}}}}\,]^{\blacksquare} \subset (\mathbf{a}^{\blacksquare}\cap\mathbf{b}^{\blacksquare}\cap\mathbf{c}^{\blacksquare})^{\bullet} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet\bullet}$$

which, with (5), implies that

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet\bullet} = [\,\|\mathbf{P}\|, \overline{\overline{\underline{\underline{\mathbf{P}}}}}\,]^{\bullet}. \tag{6}$$

$\xRightarrow{\mathbf{N}\subset\mathbf{Q}\text{ and }\mathbf{b}\in\mathbf{L}}$ Interchanging the roles of $\mathbf{b}$ and $\mathbf{c}$ in the above paragraph, we obtain (6) again.

$\xRightarrow{\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbf{L}}$ The sets $\{\mathbf{a},\mathbf{b}\}^{\circ}$ and $\{\mathbf{a},\mathbf{c}\}^{\circ}$ are distinct line traces in $\mathbf{a}^{\square}$. Either we have that

$$\{\mathbf{a},\mathbf{b}\}^{\circ}\cap\{\mathbf{a},\mathbf{c}\}^{\circ} = \emptyset \tag{7}$$

or

$$(\exists\,\mathbf{p}\in\mathbf{L})\quad \mathbf{p}\in(\{\mathbf{a},\mathbf{b}\}^{\circ}\cap\{\mathbf{a},\mathbf{c}\}^{\circ}). \tag{8}$$

Suppose that (7) holds. Then

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\circ} = \{\mathbf{a},\mathbf{b}\}^{\circ}\cap\{\mathbf{a},\mathbf{c}\}^{\circ} = \emptyset \tag{9}$$

and from (8.30) follows that there exists $\mathbf{T}\in\mathcal{T}$ such that $(\{\mathbf{a},\mathbf{b}\}^{\circ\circ}\cup\{\mathbf{a},\mathbf{c}\}^{\circ\circ})\subset\mathbf{T}$. Consequently

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\} \subset [\,\|\mathbf{T}\|, \overline{\overline{\underline{\underline{\mathbf{T}}}}}\,]^{\blacksquare}. \tag{10}$$

If $\mathbf{r}$ is in $\mathbf{a},\mathbf{b},\mathbf{c}^{\ominus}$, then $\mathbf{r}$ is in $\mathbf{a}^{\boxminus}\cap\mathbf{b}^{\boxminus}\cap\mathbf{c}^{\boxminus}$ and so $\{\mathbf{a},\mathbf{b},\mathbf{c}\}\subset\mathbf{r}^{\square}$. By (8.14.i) either $\mathbf{r}^{\square}=\mathbf{T}$ or $\mathbf{r}^{\square}\cap\mathbf{T}$ is a line trace. Since $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ cannot be a subset of a line trace, it follows that $\mathbf{r}^{\square}=\mathbf{T}$: we have

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\ominus} = [\,\|\mathbf{T}\|, \overline{\overline{\underline{\underline{\mathbf{T}}}}}\,] \xRightarrow{\text{by (9)}} \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet} = [\,\|\mathbf{T}\|, \overline{\overline{\underline{\underline{\mathbf{T}}}}}\,].$$

Consequently

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet\bullet} = [\,\|\mathbf{T}\|,\, \underline{\underline{\overline{\overline{\mathbf{T}}}}}\,]^{\bullet}. \tag{11}$$

Now suppose that (8) holds. Then

$$\mathbf{p} \in \{\mathbf{a},\mathbf{b}\}^{\circ} \cap \{\mathbf{a},\mathbf{c}\}^{\circ} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\circ} \subset \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet}. \tag{12}$$

Let $\mathbf{r}$ be any element of $\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\circ}$. Then

$$\mathbf{r} \in ((\{\mathbf{a},\mathbf{b}\}^{\circ\circ})^{\circ} \cap (\{\mathbf{a},\mathbf{c}\}^{\circ\circ})^{\circ}) = \{\mathbf{a},\mathbf{b}\}^{\circ} \cap \{\mathbf{a},\mathbf{c}\}^{\circ} \xrightarrow{\text{by (8.11.iv)}} \mathbf{r} = \mathbf{p} \tag{13}$$

since $\{\mathbf{a},\mathbf{b}\}^{\circ} \neq \{\mathbf{a},\mathbf{c}\}^{\circ}$. Assume that $\mathbf{s}$ were any element of $\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\text{◓}}$. Then

$$\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbf{s}^{\square} \xRightarrow{\text{by (8.15)}} (\{\mathbf{a},\mathbf{b}\}^{\circ\circ} \cup \{\mathbf{a},\mathbf{c}\}^{\circ\circ}) \subset \mathbf{s}^{\square}. \tag{14}$$

Furthermore, in view of (12), we have

$$(\{\mathbf{a},\mathbf{b}\}^{\circ\circ} \cup \{\mathbf{a},\mathbf{c}\}^{\circ\circ}) \subset \mathbf{p}^{\square} \xRightarrow{\text{by (14)}} (\{\mathbf{a},\mathbf{b}\}^{\circ\circ} \cup \{\mathbf{a},\mathbf{c}\}^{\circ\circ}) \subset \mathbf{p}^{\square} \cap \mathbf{s}^{\square}.$$

But (8.14.i) implies that $\mathbf{p}^{\square} \cap \mathbf{s}^{\square}$ is a line trace: an absurdity. This with (13) implies that

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\circ} \cup \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\text{◓}} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\circ} = \{\mathbf{p}\} \Longrightarrow \{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\bullet\bullet} = \mathbf{p}^{\blacksquare}. \tag{15}$$

$\xRightarrow{\mathbf{K} \not\subset \mathbf{Q},\ \mathbf{N} \not\subset \mathbf{Q} \text{ and } \mathbf{a},\mathbf{b},\mathbf{c} \in \mathbf{Q}}$ Here $\mathbf{K}$ and $\mathbf{N}$ are elliptic lines in $\mathbf{S}$. Thus we can replace $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ by elements of $\mathbf{L}$ to obtain (15).

$\xRightarrow{\mathbf{K} \not\subset \mathbf{Q},\ \mathbf{a},\mathbf{b} \in \mathbf{Q} \text{ and } \mathbf{c} \in \mathbf{L}}$ Here $\mathbf{K}$ is an elliptic line and $\mathbf{N}$ is either an elliptic or a parabolic line. Thus we can replace $\mathbf{a}$, $\mathbf{b}$ by elements of $\mathbf{L}$ to obtain (15).

$\xRightarrow{\mathbf{N} \not\subset \mathbf{Q},\ \mathbf{b},\mathbf{c} \in \mathbf{Q} \text{ and } \mathbf{a} \in \mathbf{L}}$. We can interchange a and c in the above to achieve the same result.

$\xRightarrow{\mathbf{a} \in \mathbf{Q} \text{ and } \mathbf{b},\mathbf{c} \in \mathbf{L}}$ Here $\mathbf{K}$ is either an elliptic or a parabolic line and so we can replace $\mathbf{a}$ with an element of $\mathbf{L}$ to obtain (15).

$\xRightarrow{\mathbf{c} \in \mathbf{Q} \text{ and } \mathbf{a},\mathbf{c} \in \mathbf{L}}$ We may interchange $\mathbf{a}$ with $\mathbf{c}$ in the above to obtain the same result.

We have covered all the essentially differing cases, and so from (4), (6), (11) and (15) follows Theorem (10.14). QED

(10.15) Theorem Let $\mathcal{T}$ be a meridian aggregate for a meridian libra $\mathbf{L}$. Then the operator $^{\blacksquare}$ on $\mathbf{S}$ is a space polar operator. Thus $\mathbf{S}$ and $\mathbf{L}$ constitute a three dimensional projective space.

Proof. It follows from (10.5.10) that $^{\bullet}$ is the polar operator induced by the pre-polar operator $^{\blacksquare}$ on $\mathbf{S}$. It follows from (10.7), (10.13) and (10.14) respectively, that axioms (10.1.1), (10.1.2) and (10.1.3), respectively, are fulfilled.

By definition[46], a meridian aggregate of balanced sets has dimension[47] at least 4. This dimension is equal to the cardinality of each element of ▥ and of each element of ▤. In particular it follows that each quadric line has at least 4 elements.

Suppose that $\mathbf{a}$ and $\mathbf{b}$ are generic but distinct elements of $\mathbf{L}$. It follows from Lemma (8.10) that there exists a basis[48] for the representation $\overset{\boxed{a}}{\rho}$ and elements $q, r \in F$ such that

$$\overset{\boxed{a}}{\rho}_{\mathbf{a}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \{\mathbf{a},\mathbf{b}\}^{\circ\circ} = \{\begin{pmatrix} e & rd \\ qd & e \end{pmatrix} : e^2 \neq qrd^2\},\ \text{and}$$

$$\overset{\boxed{a}}{\rho}_{\mathbf{b}} = \begin{cases} \begin{pmatrix} 1 & r \\ q & 1 \end{pmatrix}, & \text{if } \mathbf{b} \notin \mathbf{a}^{\square}; \\ \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}, & \text{if } \mathbf{b} \in \mathbf{a}^{\square}. \end{cases} \tag{1}$$

[46] *Cf.* (8.4).
[47] *Cf.* (6.17).
[48] *Cf.* (8.8).

Let e be any element of the representation space such that $e^2 \neq qr$ and $e \neq 0$. Define

$$\boldsymbol{W} \equiv \begin{cases} \{\mathbf{a}, \mathbf{b}, (\overset{\boxed{a}}{\rho})^{-1}(\begin{pmatrix} 1 & -r \\ -q & 1 \end{pmatrix})\}, & \text{if } \mathbf{b} \notin \mathbf{a}^{\square}; \\ \{\mathbf{a}, \mathbf{b}, (\overset{\boxed{a}}{\rho})^{-1}(\begin{pmatrix} e & -r \\ -q & e \end{pmatrix})\}, & \text{if } \mathbf{b} \notin \mathbf{a}^{\square}. \end{cases}$$

Thus the cardinality of $\boldsymbol{W}$ is 3 in either case, and $\boldsymbol{W} \subset \{\mathbf{a}, \mathbf{b}\}^{\circ\circ}$. It follows that elliptic, parabolic, and hyperbolic lines in $\mathbf{S}$ have cardinality at least 3. Consequently axiom (10.1.4) is fulfilled for $^{\bullet}$. QED

(10.16) Remarks The reader may wish to refer to Figure (11) of Section (1) in conjunction with Theorem (10.17) *infra*, Figure (22) of Section (9) in conjunction with Theorem (10.19) *infra* and Figure (23) of Section (9) with Theorem (10.20) *infra*.

(10.17) Theorem Let $\mathbf{S}$ be any three dimensional projective space wherein lines have at least four elements. Let $\mathbf{Q}$ be a ruled quadric surface in $\mathbf{S}$. Let $\mathbf{L}$ be the complement of $\mathbf{Q}$ in $\mathbf{S}$. Let $\mathfrak{C}$ and $\mathfrak{R}$ be the reguli associated with $\mathbf{Q}$ and let, for each $\mathbf{a} \in \mathbf{L}$, $\hat{\mathbf{a}}$ be the function described in (10.4). Then there exists a unique meridian libra operator $\lfloor , , \rfloor$ on $\mathbf{L}$ such that

$$(\forall\, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{L}) \quad \lfloor \mathbf{a}, \mathbf{b}, \mathbf{c} \rfloor\hat{} = \hat{\mathbf{a}} \circ (\hat{\mathbf{b}})^{-1} \circ \hat{\mathbf{c}}. \tag{1}$$

Proof. The projective space $\mathbf{S}$ has a commutative coordinate field which is associated to a meridian by (4.13). This meridian is associated to a meridian libra $\mathbf{L}$ with meridian aggregate $\mathcal{T}$ by (8.7). The libra set $\mathbf{L} \cup (\boxed{|||} \times \boxed{\equiv})$ is given the structure of a three dimensional projective space by (10.15). Since three dimensional projective spaces with the same underlying field are isomorphic, and since all ruled reguli are projectively equivalent, we may identify $\mathbf{S}$ with $\mathbf{L} \cup (\boxed{|||} \times \boxed{\equiv})$ and $\mathbf{Q}$ with $\boxed{|||} \times \boxed{\equiv}$. The operation carried over from $\mathbf{L}$ is evidently as in (1). QED

(10.18) Notation Let $\mathfrak{C}$ and $\mathfrak{R}$ be the reguli of a ruled quadric surface $\mathbf{Q}$. For $\mathbf{s} \in \mathbf{Q}$ we will write $\boldsymbol{/}\mathbf{s}\boldsymbol{/}$ for the element of $\mathfrak{C}$ containing $\mathbf{s}$ and $\boldsymbol{\backslash}\mathbf{s}\boldsymbol{\backslash}$ for the element of $\boxed{\equiv}$ containing $\mathbf{s}$. For $\mathbf{C} \in \mathfrak{C}$ and $\mathbf{R} \in \mathfrak{R}$ we write $\mathbf{C} \wedge \mathbf{R}$ for the element of which the singleton is $\mathbf{C} \cap \mathbf{R}$. For $\mathbf{a}, \mathbf{b} \in \mathbf{L}$ we define

$$\mathbf{a} \bowtie \mathbf{b} | \mathbf{Q} \ni \mathbf{s} \hookrightarrow (\hat{\mathbf{b}}^{-1}(\boldsymbol{\backslash}\mathbf{s}\boldsymbol{\backslash})) \wedge (\hat{\mathbf{a}}(\boldsymbol{/}\mathbf{s}\boldsymbol{/})) \in \mathbf{Q}. \tag{1}$$

Evidently each function $\mathbf{a} \bowtie \mathbf{b}$ sends elements of $\mathfrak{C}$ to $\mathfrak{R}$ and *vice versa*.

(10.19) Theorem Let $\mathbf{S}$ be any three dimensional projective space wherein lines have at least four elements. Let $\mathbf{Q}$ be a ruled quadric surface in $\mathbf{S}$. Let ϕ be any bijection of $\mathbf{Q}$ such that its restriction to any element of $\mathfrak{C}$ is a projective mapping onto an element of $\mathfrak{R}$. Then there exist elements $a, \mathbf{b} \in \mathbf{L}$ such that

$$\phi = \mathbf{a} \bowtie \mathbf{b}. \tag{1}$$

Proof. We saw in the *proof* of (10.17) that the complement $\mathbf{L}$ of $\mathbf{Q}$ in $\mathbf{S}$ can be identified with a libra $\mathbf{L}$, and $\mathbf{Q}$ with the product $\boxed{|||} \times \boxed{\equiv}$. We do this and apply (9.10) for the case in which ρ is the $\mathcal{T}$-inner representation. Thus there exist $\mathbf{a}, \mathbf{b} \in \mathbf{L}$ such that $\overset{\leftrightarrow}{\rho}_{[\mathbf{a}, \mathbf{b}]} = \phi$. From (5.15) and (10.18.1) follows that (1) holds. QED

(10.20) Theorem Let $\mathbf{S}$ be any three dimensional projective space wherein lines have at least four elements. Let $\mathbf{Q}$ be a ruled quadric surface in $\mathbf{S}$. Let ϕ be any bijection of $\mathbf{Q}$ such that its restriction to any element of $\mathfrak{C}$ is a projective mapping onto another element of $\mathfrak{C}$. Let $\mathbf{a}$ be any point in $\mathbf{S}$ not in $\mathbf{Q}$. Then there exist $\mathbf{p}, \mathbf{q} \in P$ but not in $\mathbf{Q}$ such that

$$\phi = (\mathbf{p} \bowtie \mathbf{q}) \circ (\mathbf{a} \bowtie \mathbf{a}). \tag{1}$$

Proof. Proceeding as in the proof of (10.19), but applying to (9.14) instead of (9.10), we obtain $\mathbf{b}, \mathbf{c}, \mathbf{d}, \boldsymbol{e} \in \mathbf{a}^{\circ}$ such that

$$\overset{\leftrightarrow}{\rho}_{[\lfloor \boldsymbol{e}, \mathbf{a}, \mathbf{c} \rfloor, \lfloor \mathbf{d}, \mathbf{a}, \mathbf{b} \rfloor]} \circ \overset{\leftrightarrow}{\rho}_{[\mathbf{a}, \mathbf{a}]} = \phi.$$

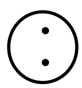
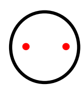
We let $\mathbf{p} \equiv \lfloor \boldsymbol{e}, \mathbf{a}, \mathbf{c} \rfloor$ and $\mathbf{q} \equiv \lfloor \mathbf{d}, \mathbf{a}, \mathbf{b} \rfloor$. QED

11. Libras as Scales

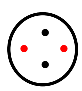
(11.1) Introduction The old Latin word for a set of scales or balances was *libra*. These often were depicted as two round plates suspended from a balance beam. In making a mathematical model of this idea we posit that the objects we weigh come from a certain set L and that we distinguish between different elements of L only in so far that they have different effect on the scales. In particular, if we place one object from L in the center of each plate, then the plates will be in equilibrium just when the objects are the same. Thus any non-trivial use of the libra will require at least two objects on each of the two scales.

Fig. 24: Libra: a Set of Scales

For precision we shall assume that each of the two scales has two opposing marks to indicate where weights from L are to be placed: little black disks on the left-hand scale, and little red disks on the right-hand scale:

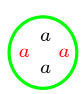
 and  .

The weights on these two scales pull against one another, and so one may alternatively visualize them as being super-imposed with the weights on the black marks pressing down, and the weights on the red marks pulling up:

 .

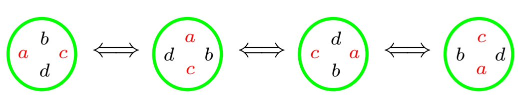
We shall assume that rotating the these superimposed scales will make no difference in determining equilibrium.

When we place objects on the scales we shall color them black to indicate they are on the left-hand scale and red to indicate that they are on the right-hand scale. We shall color the boundary of the scales green to indicate that the scales are in equilibrium. For example, our first postulate for our mathematical model of a libra will be that when a common element a of L is placed on each disk, then the scales are in equilibrium:



. (1)

Besides preservation of equilibrium under rotation of the scales,



we shall also postulate that equilibrium is preserved when permuting the disks via the "Klein-4" permutations: for all $a, b, c, d \in L$,

$$(\forall\, a, b, c, d \in L) \quad \left(\begin{smallmatrix} & b & \\ a & & c \\ & d & \end{smallmatrix}\right) \iff \left(\begin{smallmatrix} & a & \\ b & & d \\ & c & \end{smallmatrix}\right) \iff \left(\begin{smallmatrix} & d & \\ c & & a \\ & b & \end{smallmatrix}\right) \iff \left(\begin{smallmatrix} & c & \\ d & & b \\ & a & \end{smallmatrix}\right). \tag{2}$$

Postulate 3 formalizes the assertion that elements of L differ only in so far as they influence equilibrium of the scales:

$$(\forall\, a, b, c \in L)(\exists!\, d \in L) \quad \left(\begin{smallmatrix} & b & \\ a & & c \\ & d & \end{smallmatrix}\right). \tag{3}$$

We have yet to decide what it means for scales with with more that two objects apiece to be in equilibrium — we shall do so in terms of the scales with two objects apiece. Suppose that we have three objects on each scale, and so six on the superimposed scales:

$$\left(\begin{smallmatrix} v & u & z \\ w & x & y \end{smallmatrix}\right) .$$

The upper half elements determine an element r of L such that $\left(\begin{smallmatrix} & u & \\ v & & z \\ & r & \end{smallmatrix}\right)$ and the lower half determine another element s such that $\left(\begin{smallmatrix} & s & \\ w & & y \\ & x & \end{smallmatrix}\right)$. We shall define equilibrium for $\left(\begin{smallmatrix} v & u & z \\ w & x & y \end{smallmatrix}\right)$ to mean that $r = s$. In view of the equivalences

$$\left(\begin{smallmatrix} & s & \\ w & & y \\ & x & \end{smallmatrix}\right) \iff \left(\begin{smallmatrix} & w & \\ x & & s \\ & y & \end{smallmatrix}\right) \iff \left(\begin{smallmatrix} & x & \\ w & & y \\ & s & \end{smallmatrix}\right)$$

the definition becomes

$$\left(\begin{smallmatrix} v & u & z \\ w & x & y \end{smallmatrix}\right) \iff (\exists\, r \in L) \quad \left(\begin{smallmatrix} & u & \\ v & & z \\ & r & \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} & x & \\ w & & y \\ & r & \end{smallmatrix}\right).$$

In devising this definition of equilibrium of six objects we separated the scales by an imaginary horizontal line down the center. This was somewhat arbitrary however as this is just one of three ways to effect such a separation:

$$\left(\begin{smallmatrix} v & u & z \\ w & x & y \end{smallmatrix}\right) \qquad \left(\begin{smallmatrix} v & u & z \\ w & x & y \end{smallmatrix}\right) \qquad \left(\begin{smallmatrix} v & u & z \\ w & x & y \end{smallmatrix}\right).$$

If we had chosen the middle separation above, the definition would have been

$$\left(\begin{smallmatrix} v & u & z \\ w & x & y \end{smallmatrix}\right) \iff (\exists\, s \in L) \quad \left(\begin{smallmatrix} & y & \\ z & & x \\ & s & \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} & v & \\ u & & w \\ & s & \end{smallmatrix}\right)$$

while if we had chosen the right-hand separation, the definition would have been

$$\left(\begin{smallmatrix} v & u & z \\ w & x & y \end{smallmatrix}\right) \iff (\exists\, t \in L) \quad \left(\begin{smallmatrix} & w & \\ x & & v \\ & t & \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} & z & \\ y & & u \\ & t & \end{smallmatrix}\right).$$

In order to make these three *prima facie* different definitions equivalent, our fourth and last postulate for a libra will be that, for all $u, v, w, x, y, z \in L$

$$(\exists\, r \in L)\ \begin{smallmatrix} & u & \\ v & & z \\ & r & \end{smallmatrix} \text{ and } \begin{smallmatrix} & x & \\ w & & y \\ & r & \end{smallmatrix} \iff (\exists\, s \in L)\ \begin{smallmatrix} & y & \\ z & & x \\ & s & \end{smallmatrix} \text{ and } \begin{smallmatrix} & v & \\ u & & w \\ & s & \end{smallmatrix} \iff (\exists\, t \in L)\ \begin{smallmatrix} & w & \\ x & & v \\ & t & \end{smallmatrix} \text{ and } \begin{smallmatrix} & z & \\ y & & u \\ & t & \end{smallmatrix}. \qquad (4)$$

This **libra** which we have just defined has a rather elegant characterization within the context of algebra. Postulate 3 begs introduction of a notation for d in terms of a, b, and c: we denominate d with $\lfloor a, b, c \rfloor$:

$$(\forall\, a, b, c \in L) \quad \begin{smallmatrix} & b & \\ a & & c \\ & \lfloor a,b,c \rfloor & \end{smallmatrix}.$$

We collect some properties of this trinary operator $\lfloor , , \rfloor$. First consider $a, b \in L$ and set $c \equiv \lfloor a, a, b \rfloor$. By (1) we have $\begin{smallmatrix} & a & \\ a & & a \\ & a & \end{smallmatrix}$ and by (3) we have $\begin{smallmatrix} & a & \\ a & & b \\ & c & \end{smallmatrix}$, which by (2) implies $\begin{smallmatrix} & b & \\ c & & a \\ & a & \end{smallmatrix}$. Thus $\begin{smallmatrix} a & a & a \\ c & b & a \end{smallmatrix}$, which in turn implies that there exists $s \in L$ such that $\begin{smallmatrix} & a & \\ a & & b \\ & s & \end{smallmatrix}$ and $\begin{smallmatrix} & a & \\ a & & c \\ & s & \end{smallmatrix}$. By (2) we have $\begin{smallmatrix} & a & \\ a & & s \\ & b & \end{smallmatrix}$ and $\begin{smallmatrix} & a & \\ a & & s \\ & c & \end{smallmatrix}$. By (3) we have $b = c$: that is,

$$\lfloor a, a, b \rfloor = b.$$

Setting $d \equiv \lfloor b, a, a \rfloor$ we have $\begin{smallmatrix} & a & \\ b & & a \\ & d & \end{smallmatrix}$ and so $\begin{smallmatrix} & b & \\ a & & d \\ & a & \end{smallmatrix}$ which, with $\begin{smallmatrix} & a & \\ a & & a \\ & a & \end{smallmatrix}$, implies $\begin{smallmatrix} a & a & a \\ b & d & a \end{smallmatrix}$. Thus there exists $s \in L$ such that $\begin{smallmatrix} & a & \\ a & & b \\ & s & \end{smallmatrix}$ and $\begin{smallmatrix} & a & \\ a & & d \\ & s & \end{smallmatrix}$. It follows that $\begin{smallmatrix} & a & \\ a & & s \\ & b & \end{smallmatrix}$ and $\begin{smallmatrix} & a & \\ a & & s \\ & d & \end{smallmatrix}$, which implies that $b = d$: that is

$$\lfloor b, a, a \rfloor = b.$$

Now we investigate $n \equiv \lfloor \lfloor a, b, c \rfloor, d, e \rfloor$ for $a, b, c, d, e \in L$. Letting $m \equiv \lfloor a, b, c \rfloor$ we have

$$\begin{smallmatrix} & b & \\ a & & c \\ & m & \end{smallmatrix} \text{ and } \begin{smallmatrix} & d & \\ m & & e \\ & n & \end{smallmatrix} \Longrightarrow \begin{smallmatrix} & e & \\ n & & d \\ & m & \end{smallmatrix} \Longrightarrow \begin{smallmatrix} a & b & c \\ n & e & d \end{smallmatrix} \Longrightarrow (\exists\, s \in L) \begin{smallmatrix} & d & \\ c & & e \\ & s & \end{smallmatrix} \text{ and } \begin{smallmatrix} & a & \\ b & & n \\ & s & \end{smallmatrix} \Longrightarrow \begin{smallmatrix} & b & \\ a & & s \\ & n & \end{smallmatrix}.$$

It follows that $s = \lfloor c, d, e \rfloor$ and $n = \lfloor a, b, s \rfloor$, which means that

$$\lfloor \lfloor a, b, c \rfloor, d, e \rfloor = \lfloor a, b, \lfloor c, d, e \rfloor \rfloor.$$

(11.2) Definitions and Notation The three properties we have just derived for the trinary operator $\lfloor , , \rfloor$ on L motivate the definition of (3.2): that a libra operator is a trinary operator on a set L satisfying the following

$$(\forall\, a, b \in L) \quad \lfloor a, a, b \rfloor = \lfloor b, a, a \rfloor = b; \qquad (1)$$

$$(\forall\, a, b, c, d, e \in L) \quad \lfloor \lfloor a, b, c \rfloor, d, e \rfloor = \lfloor a, b, \lfloor c, d, e \rfloor \rfloor. \qquad (2)$$

In this case, if $a, b, c, d \in L$ and $d = \lfloor a, b, c \rfloor$, we shall write

$$\begin{smallmatrix} & b & \\ a & & c \\ & d & \end{smallmatrix} \text{ or } \begin{smallmatrix} & a & \\ d & & b \\ & c & \end{smallmatrix}.$$

(11.3) Theorem Let $\lfloor , , \rfloor$ be a libra operator for a set L. Then the four postulates enunciated in (11.1) hold.

Proof. That (11.1.1) holds follows directly from (11.2.1). To prove (11.1.2) we have the following series of implications:

$$\begin{smallmatrix} & b & \\ a & & c \\ & d & \end{smallmatrix} \Longrightarrow d = \lfloor a,b,c \rfloor \Longrightarrow \lfloor b,a,d \rfloor = \lfloor b,a,\lfloor a,b,c \rfloor \rfloor = \lfloor \lfloor b,a,a \rfloor, b, c \rfloor = \lfloor b,b,c \rfloor = c \Longrightarrow \begin{smallmatrix} & a & \\ b & & d \\ & c & \end{smallmatrix};$$

$$\begin{smallmatrix} & a & \\ b & & d \\ & c & \end{smallmatrix} \Longrightarrow c = \lfloor b,a,d \rfloor \Longrightarrow \lfloor c,d,a \rfloor = \lfloor \lfloor b,a,d \rfloor, d, a \rfloor = \lfloor b,a,\lfloor d,d,a \rfloor \rfloor = \lfloor b,a,a \rfloor = b \Longrightarrow \begin{smallmatrix} & d & \\ c & & a \\ & b & \end{smallmatrix};$$

$$\begin{smallmatrix} & d & \\ c & & a \\ & b & \end{smallmatrix} \Longrightarrow b = \lfloor c,d,a \rfloor \Longrightarrow \lfloor d,c,b \rfloor = \lfloor d,c,\lfloor c,d,a \rfloor \rfloor = \lfloor \lfloor d,c,c \rfloor, d, a \rfloor = \lfloor d,d,a \rfloor = a \Longrightarrow \begin{smallmatrix} & c & \\ d & & b \\ & a & \end{smallmatrix};$$

$$\begin{smallmatrix} & c & \\ d & & b \\ & a & \end{smallmatrix} \Longrightarrow a = \lfloor d,c,b \rfloor \Longrightarrow \lfloor a,b,c \rfloor = \lfloor \lfloor d,c,b \rfloor, b, c \rfloor = \lfloor d,c,\lfloor b,b,c \rfloor \rfloor = \lfloor d,c,c \rfloor = d \Longrightarrow \begin{smallmatrix} & b & \\ a & & c \\ & d & \end{smallmatrix}.$$

That Postulate (11.1.3) holds follows from the definition of $\begin{smallmatrix} & b & \\ a & & c \\ & d & \end{smallmatrix}$. That Postulate (11.1.4) holds follows from the three following series of implications:

$$(\exists\, r \in L)\ \begin{smallmatrix} & u & \\ v & & z \\ & r & \end{smallmatrix} \text{ and } \begin{smallmatrix} & x & \\ w & & y \\ & r & \end{smallmatrix} \Longrightarrow \lfloor v,u,z \rfloor = r = \lfloor w,x,y \rfloor \Longrightarrow$$

$$z = \lfloor u,u,z \rfloor = \lfloor \lfloor u,v,v \rfloor, u, z \rfloor = \lfloor u,v,\lfloor v,u,z \rfloor \rfloor = \lfloor u,v,\lfloor w,x,y \rfloor \rfloor \Longrightarrow$$

$$\lfloor z,y,x \rfloor = \lfloor \lfloor u,v,\lfloor w,x,y \rfloor \rfloor, y, x \rfloor = \lfloor u,v,\lfloor \lfloor w,x,y \rfloor, y, x \rfloor \rfloor = \lfloor u,v,\lfloor w,x,\lfloor y,y,x \rfloor \rfloor \rfloor =$$

$$\lfloor u,v,\lfloor w,x,x \rfloor \rfloor = \lfloor u,v,w \rfloor \Longrightarrow (\exists\, s \in L)\ \begin{smallmatrix} & y & \\ z & & x \\ & s & \end{smallmatrix} \text{ and } \begin{smallmatrix} & v & \\ u & & w \\ & s & \end{smallmatrix};$$

$$(\exists\, s \in L)\ \begin{smallmatrix} & y & \\ z & & x \\ & s & \end{smallmatrix} \text{ and } \begin{smallmatrix} & v & \\ u & & w \\ & s & \end{smallmatrix} \Longrightarrow \lfloor z,y,x \rfloor = s = \lfloor u,v,w \rfloor \Longrightarrow$$

$$x = \lfloor y,y,x \rfloor = \lfloor \lfloor y,z,z \rfloor, y, x \rfloor = \lfloor y,z,\lfloor z,y,x \rfloor \rfloor = \lfloor y,z,\lfloor u,v,w \rfloor \rfloor \Longrightarrow$$

$$\lfloor x,w,v \rfloor = \lfloor \lfloor y,z,\lfloor u,v,w \rfloor \rfloor, w, v \rfloor = \lfloor y,z,\lfloor \lfloor u,v,w \rfloor, w, v \rfloor \rfloor = \lfloor y,z,\lfloor u,v,\lfloor w,w,v \rfloor \rfloor \rfloor =$$

$$\lfloor y,z,\lfloor u,v,v \rfloor \rfloor = \lfloor y,z,u \rfloor \Longrightarrow (\exists\, t \in L)\ \begin{smallmatrix} & w & \\ x & & v \\ & t & \end{smallmatrix} \text{ and } \begin{smallmatrix} & z & \\ y & & u \\ & t & \end{smallmatrix};$$

and

$$(\exists\, t \in L)\ \begin{smallmatrix} & w & \\ x & & v \\ & t & \end{smallmatrix} \text{ and } \begin{smallmatrix} & z & \\ y & & u \\ & t & \end{smallmatrix} \Longrightarrow \lfloor x,w,v \rfloor = t = \lfloor y,z,u \rfloor \Longrightarrow$$

$$v = \lfloor w,w,v \rfloor = \lfloor \lfloor w,x,x \rfloor, w, v \rfloor = \lfloor w,x,\lfloor x,w,v \rfloor \rfloor = \lfloor w,x,\lfloor y,z,u \rfloor \rfloor \Longrightarrow$$

$$\lfloor v,u,z \rfloor = \lfloor \lfloor w,x,\lfloor y,z,u \rfloor \rfloor, u, z \rfloor = \lfloor w,x,\lfloor \lfloor y,z,u \rfloor, u, z \rfloor \rfloor = \lfloor w,x,\lfloor y,z,\lfloor u,u,z \rfloor \rfloor \rfloor =$$

$$\lfloor w,x,\lfloor y,z,z \rfloor \rfloor = \lfloor w,x,y \rfloor \Longrightarrow (\exists\, r \in L)\ \begin{smallmatrix} & u & \\ v & & z \\ & r & \end{smallmatrix} \text{ and } \begin{smallmatrix} & x & \\ w & & y \\ & r & \end{smallmatrix}.$$

QED

Bibliography

CERTAINE, J.: The ternary operation (abc)=ab^{-1}c of a group. Bull. Amer. Math. Soc. 49 (1943), 869-877.

GRAUSTEIN, WILLIAM C.: Introduction to Higher Geometry. New York: The MacMillan Company, 1937.

SEIDENBERG, ABRAHAM: Lectures in Projective Geometry. Princeton: D. Van Nostrand Company, Inc. 1962.

SIMPSON, JOHN AND EDMUND WEINER: Oxford English Dictionary, second edition. Oxford: Oxford University Press 1989.

STAUDT, KARL VON: Beiträge der Geometrie der Lage, Heft 2. 1857.

TITS, JACQUES: Généralisation des groupes projectifs basés sur la notion de transitivité. Mem. Acad. Roy. Belg. **27** (1952), 115 p.

VEBLEN, OSWALD AND JOHN WESLEY YOUNG: Projective Geometry, Vol I. Boston: Ginn and Company, 1910.

Index of Symbols

$\sim$	[equivalence relation on $\mathcal{M}^{\Upsilon}_{2+}$]	(2.3)
$\left[\begin{smallmatrix} A & B & C \\ \flat & \sharp & \natural \end{smallmatrix}\right]$	[specific element of $\mathbf{Mor}(\mathfrak{M}, \mathcal{M})$]	(2.8)
$\underline{\mathfrak{t}}$	[pair of elements of Υ where $\mathfrak{t}$ is constant]	(2.9)
$\langle A, B, C, D \rangle$	[specific element of $\mathcal{M}^{\Upsilon}_{2+}$]	(2.11)
$\boxed{\spadesuit\heartsuit \mid \diamondsuit\clubsuit}$	[element of $\mathfrak{M}$ all of whose elements are in $\mathcal{M}^{\Upsilon}_{3}$]	(2.13)
$\left[\begin{smallmatrix} U & V & W \\ A & B & C \end{smallmatrix}\right]$	[specific element of $\Gamma(\mathcal{M}, \sim)$]	(2.16)
$\lfloor \mathfrak{t} \rfloor$	[element of $\mathfrak{M}$ of which $\mathfrak{t}$ is a member]	(2.20)
$[x, y]$	[ordered pair]	(2.1)
$\bar{\mathtt{x}}$	[permutation induced automorphism of $\mathfrak{M}$]	(2.21)
$\boxed{\spadesuit\heartsuit\ddagger\diamondsuit\clubsuit}$	[specific element of $\mathfrak{M}$]	(2.41)
$\lfloor x, y, z \rfloor$	[libra product]	(3.2)
$\llbracket f, g, h \rrbracket$	[libra product of functions]	(3.6)
$[\cdot\!\cdot\!\cdot]$	[meridian operatgor]	(4.1)
${}^{B}\llcorner A, C, E \lrcorner^{D}$	[relative libra operator on a meridian]	(4.1)
$\boxed{\begin{smallmatrix} A \leftrightarrow E \\ B \leftrightarrow D \end{smallmatrix}}$	[specific self-inverse bijection]	(4.4)
$\boxed{\begin{smallmatrix} B \updownarrow E \\ D \nearrow C \end{smallmatrix}}$	[specific self-inverse bijection]	(4.6)
$0, 1, \infty$	[basis for a meridian]	(4.11)
$0, 1, \infty$	[basis for a representation space]	(8.8)
$+, \cdot$	[basis arithmetic operations for a meridian]	(4.11)
$\begin{pmatrix} R & S \\ U & V \end{pmatrix}$	[specific Möbius transformation]	(4.11)
$\left\llbracket \begin{matrix} A & B \\ C & D \end{matrix} \right\rrbracket_{(0,1,\infty)}$	[meridian cross-ratio relative to a basis]	(4.12)
$\lfloor r, s, B \rfloor, \lfloor r, B, s \rfloor, \lfloor B, r, s \rfloor$	[cosets of a balanced set in a libra]	(5.1)
$\Vert B \Vert$	[column of an aggregate]	(5.1)
$\overline{\overline{\underline{\underline{B}}}}$	[row of an aggregate]	(5.1)
$\boxed{\boxed{B}}$	[aggregate of a libra]	(5.1)
$\left\Vert \overline{\overline{\underline{\underline{B}}}} \right\Vert$	[family of linear translates of a balanced set]	(5.1)
x	[element of a representation space]	(5.9)
X	[representation space]	(5.9)
$[x \overset{\phi}{=} y]$	[balanced level set induced by a representation]	(5.9)
$\mathcal{T}_{\phi}$	[aggregate induced by a representation]	(5.9)
$\lfloor r, s, t \rfloor_{\phi}$	[libra operator on a quotient space (by a normal balanced set)]	(5.13)
$\lceil a, b, c \rceil$	[obverse libra operator]	(5.15)
$\vdash, , \dashv$	[symmetrization libra operator on the cartesian product of a libra with itself]	(5.15)
$\boxed{\vert\vert\vert}$	[family of all columns of an aggregate]	(6.1)
$\boxed{\equiv}$	[family of all rows of an aggregate]	(6.1)
$a \textcircled{\tau} b$	[inner projection of an aggregate onto itself]	(6.3)
$\hat{\textcircled{\tau}}, \check{\textcircled{\tau}}$	[inner representations of a libra on $\boxed{\vert\vert\vert} \times \boxed{\equiv}$]	(6.3)
$a^{\textcircled{\tau}}$	[symmetric inner projection of an aggregate]	(6.5)
$a^{\hat{\textcircled{\tau}}}$	[inner representation of an element of a libra]	(6.5)
$R \cap S$	[element of a singleton $R \cap S$]	(6.14)
$\backslash\!\backslash \mathrm{a} \backslash\!\backslash$	[diagonal of an aggregate determined by an element of the libra]	(2.23)

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