The Classical Theory of Light Colors: a Paradigm for Description of Particle Interactions

Nicolae Mazilu & Maricel Agop

Abstract.
The color is an interaction property: of the interaction of light with matter. Classically speaking it is therefore akin to the forces. But while forces engendered the mechanical view of the world, the colors generated the optical view. One of the modern concepts of interaction between the fundamental particles of matter – the quantum chromodynamics – aims to fill the gap between mechanics and optics, in a specific description of strong interactions. We show here that this modern description of the particle interactions has ties with both the classical and quantum theories of light, regardless of the connection between forces and colors. In a word, the light is a universal model in the description of matter. The description involves classical Yang-Mills fields related to color.
Key Words: theory of colors, MacAdam ellipse, quantum chromodynamics, Yang-Mills fields, Hannay angle, Riemannian geometry, trichromacy, dichromacy, blackbody radiation, holographic principle, asymptotic freedom

Introduction

There are two contemporary concepts which, coming from the physics of past, carry, in our opinion, a special meaning for the physics of future: the concept of asymptotic freedom and the holographic principle. The first one helped rounding the quantum chromodynamics (QCD) as a science of strong interactions of what are currently considered fundamental material particles. It advocates, roughly speaking, the light-like behavior of the particles involved in strong interactions. The second one, closely correlated to the idea of asymptotic freedom, but aiming to cover the whole range of fundamental forces, advocates the idea of two degrees of freedom in the description of fundamental interactions. This, again, is a fundamental classical characteristic of light.

These two modern principles stand witness to the fact that the analogy in theoretical physics is still the main tool that works at any level, in any physical theory. Taken as such they would then point out that the light should be part of a universal model in the theory of fundamental interactions of matter, and thus it should become a standard in the modern analogy leading to fundamental forces. We aim to show here that both these principles hold indeed a manifest classical character, whose roots are to be found, with equal chances, both in the classical Newtonian theory of light and in the quantum theory of light as it was constructed by Planck at the beginning of the last century. In other words, the two modern principles carry the burden of continuity with respect to the two physical theories considered nowadays the quintessence of the classical theoretical physics.

Along the analysis to be presented here, it will become clear that the light is indeed a model of any interaction in the universe, inasmuch as it can be revealed by an obvious interaction: the one giving the colors. This interaction has an exquisite classical description, involving only two degrees of freedom, and that description can be done by continuous Lie groups from the SL(2,R) family. There are thus Yang-Mills type fields describing the colors, in the spirit of the classical theory of light established by Hooke and Newton. This turns out to be also a characteristic of light from the quantum mechanical point of view.

The Classical Theory of Light: an Abridged History

First, let us review the classical theory of light, up to the point where it was brought by Hooke and Newton. Following the line of thought given by the idea of analogy, we primarily find the standard of the classical analogy which led to the theory of light itself: this is given by the waves on the surface of quiet water, induced by a stone falling into water. One can say that the making of the theoretical model of light is nothing more than elaborations, at any of its significant moments, on one or another of the parts of that experience. The phenomenology which was the basis of these elaborations for the first theoretical model of light comprises two basic observable facts: reflection and refraction of light. Let's review the original significant moments of the making of concept of light.

First, it was the explanation of the propagation of light. To the extent where it can be explained as a motion, that classical standard of analogy shows that it is a 'motion of a motion', so to speak, but not only that. The concentric water waves show indeed a motion perpendicular to the water surface, which nevertheless 'multiplies', in order to allow for the growth of the circles representing waves. The propagation is therefore a 'motion of the periodic motion' but, being accompanied by a continual growth of the waves, it indicates also a continual generation of that periodic motion, along the very circles representing the crests of waves.

In the case of light, though, the analogous of water surface is missing, to say nothing to the effect that, as a matter of fact, even the analogous of the water is missing. To make up for the want of an equivalent to water, the ether was invented: the light waves are waves in ether. However, the water surface was hard to replace, for the propagation of light takes place in all directions but in space not in a plane, and apparently there is no physically distinguished surface to play the part of an equivalent of water surface, like a space 'section' as it were. The Huygens theory of the wave surface of light simply supresses the need of that equivalent of water surface, going directly over to space matters, with the help of the concept of light ray.

In space the analogous of circles depicting the waves on the surface of quiet water, was first the so-called 'orb', defined by Thomas Hobbes as the material portion between two concentric spheres of close radii, which extends continuously by propagation (**Hobbes, 1644**). What really matters in this definition is only the uniform continuous growth of the sphere, equivalent to the growth of circles on the surface of water, thus giving what seems to be the essence of the propagation phenomenon of light in space. One therefore loses, by simple suppression from the model – thus foreshadowing the later principle of Huygens – the local oscilatory motion characterizing the waves on the face of quiet water. As such, nothing assures us here that the motion of light "within orb" would be a vibratory motion. On the contrary, what seems to be obvious on logical grounds is that such a motion is rather uniform, matching the uniform extension of the orb in space. Further on, Hobbes also defines an extremely important concept that played a fundamental part in the development of the whole theory of light. Disregarding even the evidence of the geometrical divergence of the light rays, he defined the so-called light line, which would represent what happens within the surface of water – therefore along the very circles depicting the water waves. This led implicitly to the concept of physical ray of light, which is a plane figure formed by two parallel straight lines – ideal geometric rays delimiting the physical ray – joined perpendicularly, in each one of their corresponding points, by the light line. The propagation of light could now be defined as a transportation of the light line, and thus it gained some other, more subtle nuances, than the standard given by water waves.

First, the lack of a detailed explanation of the differential properties of this light line started to be noticed. We have reasons to say that to Hobbes the propagation of light is simply the parallel transport of this segment in the Euclidean sense, therefore a transportation which maintains the property of orthogonality with respect to limiting rays all along the physical ray. As long as the light goes straight there are no problems. We have to face them only in the case of change of direction of light, like in reflection and refraction phenomena. Here the transport of the light line must be defined accordingly, and indeed Hobbes does it by methods prefiguring those which established the parallel transfort of the modern differential geometry (**Levi-Civita, 1927**, pp. 100–102). As Hobbes conceives the transport in propagation of light only within classical synthetic geometry, the problem of extension of the light segment within the orb – which is a problem of deformation proper in modern views – remained in suspension. It obviously popped up later on, requiring specific solutions related to the idea of gauge.

At this moment the important intervention of Robert Hooke led to the first rational idea of color, which appears in the case of interaction of light with matter, as represented by refraction and reflection phenomena. Apparently Hooke defines the color by noticing first of all that the physical ray of Hobbes is actually an ideal case. Indeed, in reality the geometrical rays delimiting a physical ray of light should certainly be divergent. They can only be approximately parallel, the approximation being closer to the ideal situation as the distance from the light source grows (**Hooke, 1665, 1705**). Therefore the light line of Hobbes is in reality not perpendicular to the rays delimiting a physical ray of light. However, at great distances from the source of light they can be considered very nearly perpendicular, which in practice is quite sufficient. To Hooke the light line acquires even a special name – orbicular pulse – which suggests the idea of motion characterizing the light within the orb. The idea of pulse would have been suggested to Hooke by a study of Hobbes, who describes the creation of light as a pulsating phenomenon of the kind of systole and diastole, observed in the case of heart (see **Shapiro, 1973, 1975** and the works cited there for an elaboration on the manner in which Hooke could have been influenced by Hobbes). However, this 'orbicular pulse' means a great deal more in the hands of Hooke, along the idea of representation of the way in which the light motion takes place within the orb.

For, as we have noticed, the idea of propagation in space has suppressed the oscillatory motion, which, regardless of its direction, is essential in the clasical standard of analogy that led to the concept of light: the circular waves on the face of quiet water. Even the idea that some motion, other than that directly representing the propagation per se, might be associated with the light was momentarily lost: the light line of Hobbes is a purely geometric concept after all! Hooke calls back into question the problem, and solves it once and for all: within the orb we have definitely motion; the problem is to decide what kind of motion this is. And he reaches, speculatively, the epoch-making conclusion, that the motion characterizing the light within orb is vibratory, therefore an oscillatory motion, like that of the waves on water, perpendicular to its surface. Hooke's reasoning is quite simple, speculating on the fundamental idea that the light is material. In broad lines, one could say that he just notices that if the light would be characterized by a continuous motion within a body – like the Hobbes' orb – this would lead to rupture, as it surely happens in the processes of continual deformation of materials. As, however, no transparent body is apparently destroyed by the light passing through it, one should conclude that the motion characterizing the light is vibratory, with a very small amplitude ("short" to Hooke), reestablishing periodically the matter in every one of its points. This motion is localized along the light line of Hobbes so that this one is actually an 'orbicular pulse', approximately perpendicular to the geometrical rays defining the physical ray.

Here Hooke introduces the first rational concept of color ever, and thus begins the modern history of theoretical chromodynamics, as understood *mot à mot* according to its Greek name: the dynamics of color. First, the color is not revealed but only by light, when it touches the material bodies, i.e. by interaction. According to Hooke, the color is explained by the fact that through refraction the orbicular pulse acquires an inclination on the limiting rays, becoming slightly nonorthogonal to the geometrical rays, to a specific extent though, depending on the color carried by them. Due to this, in a natural ray, the orbicular pulse is "broken" into segments, each one representing a homogeneous ray of specific color. So, for the natural light, the orbicular pulse appears as a broken, even folded line, joining the defining geometrical rays, which carry two fundamental colors – red and blue – from which all the other colors are constructed by the action of matter upon light.

This idea – the first true physical theory of colors – was killed in the cradle, so to speak, by Newton's observations to the effect that the orbicular pulse is actually a cross-sectional surface phenomenon. It is indeed performed transversally with respect to the direction of light ray, but not in a definite plane containing that direction. In other words, the color is indeed a parameter of homogeneity of the physical ray of light, but spatial not planar. A physical ray of light of a given color is symmetrical around the direction of propagation – in modern terms it has axial symmetry (**Newton, 1952**). So if one speaks about Hooke's fragmentation of the orbicular pulse, with each fragment representing a color, this process does not preserve the axial plane of the original pulse in the transversal section of the physical ray. The physical ray itself is therefore not a plane figure, as Hobbes and Hooke presented it, but has a volume as the experience on light shows. If the light carries many colors, for instance if it is white, then it is transversally inhomogeneous, i.e. it is no more axially symmetric, although not as geometrical shape, but with respect to color. As such, it is indeed composed from axially symmetric color-homogeneous rays, that can be exhibited as an elongated spectrum, when passing the light ray through a prism. This is, in broad strokes, the newtonian conclusion of the celebrated prism experiments, and the ground for Newton's discussion on colors, which later on, during 19th century, generated the physical theory of light spectrum, and implicitly led to the theory of quanta.

This moment of our knowledge has a particular significance when judged through the formulation of the modern holographic principle (**'t Hooft, 1993**; **Susskind, 1994**). Indeed by the physical ray of Hoobes and Hooke, the idea of planeness, as contained in the waves on the water, is certainly preserved. Newton's intervention sets things in order according to the observations on the light itself. Formulated in modern terms, the Newtonian conclusion is: the physics of light itself shows that the planeness is not preserved in the geometrical form of the physical ray of light, *but in a general property of symmetry*, abstract we might say, that can be expressed in two variables. In other words two degrees of freedom suffice to describe the light! Let's elaborate a little more on this statement.

Quite obviously, Newton realized that, from the point of view of the experience on light itself, something is missing in the concept of physical ray of Hooke. That concept cannot explain the fact that the experimental white light ray is spatially homogeneous and isotropic in the cross section, when the ray is constructed by appropriate circular holes. According to Hooke's idea, the cross section should appear on a screen differently colored on a certain direction of that screen. It appears this way indeed, but only when one complicates the construction of the ray by passing the light through a prism. Only this procedure isolates homogeneous, axially symmetric, "sub-rays" of the same color, in terms of which one can explain the elongated spectrum in the manner of Hooke. This is, however, a general idea of symmetry, and it does not refer by any means to the geometrical shape of the ray. In other words: it is the variation of light color that has two-dimensional extension, not the ray itself. Thus, one can say that the study of light per se adds this important conclusion to the very idea of wave, above and beyond the classical standard of analogy that helped constructing the concept thus far.

Another important point in Newton's observations is the one usually connected with the particle theory of light. True, sometimes – and quite often at that, one might say – Newton slips into the direct association of light with material particles, but his definition of the light ray, which starts the celebrated *Opticks*, is extremely cautious and does not reduce by any means to the idea of particle in the classical connotation. So much the less reduces it to the geometrical concept of straight line. It certainly pays to reproduce here that definition (**Newton, 1952**, p.1; see also the French edition from 1787), for it comprises a whole philosophy, which later, with the occasion of quanta of light, was labelled as 'revolution':

> By the Rays of Light I understand its least Parts, and those as well Successive in the same Lines, as Contemporary in several Lines.

A consideration of the subsequent text of Newton, explaining this definition, shows two further essential points of the Newtonian natural philosophy. First, Newton refers the definition of ‘parts’ to the experimental possibility to build them: the “least Parts” should not, by any means, be understood as ‘particles’ in the classical connotation. These last ones can exist ‘under our eyes’ as it were, being stable at the time scale of the common experience. They do not require experimental action in order to be defined. On the other hand the ‘least Parts” of light must be defined accordingly, therefore experimentally. And their definition depends on the experimental capability to exhibit them. On one hand one has to have the physical possibility of discerning a direction of propagation of light, by holes in screens for instance, and on the other hand one should have the physical possibility to stop the light by a screen. It is the interplay of these two experimental procedures, obviously in an ideal theoretical form, that defines the least parts of light, the way Newton conceives them.

Therefore, coming again to the modern theoretical environment, Newton defines the rays of light in the modern manner in which the elementary particles are defined: only by experimental capability. Indeed the modern elementary particles are closer to the least parts of light as defined by Newton, than to the matter as intuitively understood. It is in this sense that one can talk indeed of a particle theory of light to Newton. But then one can demonstrate – and we will make this obvious in the present work – that as such the light can be theoretically considered the paradigm of any theory of fundamental interactions. The discovery of the asymptotic freedom just shows it. Of course, in order to do this, one has to use an explicit holographic principle.

A second point made explicit by Newton himself is that the idea of geometrical ray as a straight line is directly connected with the infinite speed of light. As, however, the experimental evidence of the epoch showed that the light has a finite speed, he declares that he was forced to concede to his definition of the ray reproduced above. It is here an essential point of the natural philosophy of light, that certainly needs elaboration and, we have to confess, we were unable to find it properly documented to Newton. The statement, of course, can be proved within the modern differential geometry, and constitutes one of the most subtle points of the wave-particle transition, seen, again, as an ‘asymptotic freedom’.

Classical Light in Terms of Two Degrees of Freedom

In this section we present the Hooke’s and Newton’s results as geometrical theorems. Obviously, the classical theory of light is naturally correlated with the classical differential geometry of surfaces, by the very concept of wave surface. Thus a light ray in vacuum, for instance, can be imagined as a trajectory, in some relation with the local normal of the wave surface. The propagation of light, taken in the initial connotation as the process that should include naturally the reflection and refraction phenomena, can be imagined, first and foremost, as a variation of the normal direction to the evolving wave surface. Of course, with this concept we took a leap over time, coming closer to modern ones, where the Hobbes’ idea of orb metamorphosed into that of wave surface, by bringing in the difraction among the experimental facts characterizing the light per se (**Fresnel, 1827**). However, the presentation of the development of concept in its historical continuity would not be helpful here, inasmuch as we follow a logical continuity, which can only be exhibited by a theory of colors. In order to reach that logical continuity Fresnel’s theory had to be rounded, so to speak, by the idea of gauge, which, as we will show here, can be naturally supported only by such an idea of color, leading directly to a Yang-Mills type theory. And, of course, this idea can in turn be naturally extended to characterize the modern concept of particle. Only the classical theory of light, as briefly reviewed in the previous section, would naturally allow for such an extension, an the present section shows the way to it.

The classical differential geometry of the surfaces is entirely constructed on a general manner of conception of the notion of neighborhood (**Фиников, 1952**). Any point of a surface belongs to a neighborhood *of a certain order* of another point. Between the neighborhoods of different orders, in the same surface, there is a connection, uniquely characterizing the surface. In order to see how this philosophy works, assume a formal Taylor expansion of the position vector on the surface, something of the form:

$$\vec{r} = \vec{x} + d\vec{x} + d^2\vec{x} + d^3\vec{x} + \ldots$$

This formula suggests the orders of neighborhoods through the natural orders of the differentials of the positions $\vec{r}$ and $\vec{x}$. For instance, the second differential is the variation of the first differential, therefore the differential geometry on a first order neighborhood is given by vectors that are second order differentials. The third differential is the variation of the second differential, therefore it reflects a differential geometry for which the second order neighborhoods are the basic 'showground', etc. However, it turns out that there is really only one essential 'showground', namely the first order neighborhood, inasmuch as the whole geometry in a point of the wave surface, and the attached physics of course, can be described in terms of the first order differentials.

First, one can recognize the departure of $\vec{r}$ from the surface at the location $\vec{x}$ by the projection $(\vec{r} - \vec{x})$ along the normal to surface at $\vec{x}$, whose unit vector we denote $\hat{n}$ as usual. We have

$$\hat{n} \cdot (\vec{r} - \vec{x}) = \hat{n} \cdot d\vec{x} + \hat{n} \cdot d^2\vec{x} + \hat{n} \cdot d^3\vec{x} + \ldots \tag{1}$$

This means that the vector $(\vec{r} - \vec{x})$ does not belong to the surface in all of its local neighborhoods, but has departures of different orders from it, and these become apparent as we get closer to the surface in order to be able to distinguish its details. As by the very definition we have to admit that the vector $d\vec{x}$ *is within the surface*, the first term in equation (1) is null, so that

$$\hat{n} \cdot (\vec{r} - \vec{x}) = \hat{n} \cdot d^2\vec{x} + \hat{n} \cdot d^3\vec{x} + \ldots \tag{2}$$

Therefore, the first-order neighborhood of the surface is characterized by this differential relation. The length of the first-order differential of the position vector is the first fundamental form of the surface, or the metric. The right hand side of the equation (2) represents details that become obvious as we gradually approach the point $\vec{x}$ of the surface. The first term in equation (2) is the second fundamental form of the surface. In the intrinsic geometry of the surface the invariants related to the second fundamental form are different measures of the local curvature of the surface.

We don't think that we can escape, in any physical problem, and so much the less in the case of light, from the bounds of analogy altogether. Here it comes with the idea of the coarsening of the wave surface due to matter. This is a phenomenon of 'fragmentation' of surface, making it have a certain degree of 'roughness'. The 'roughness' is reflected in the local variation of normal unit vector of the surface, and this is exactly what happens in the cases of reflection and refraction of light analyzed by Hooke and Newton. And, as well known, the variation of that unit normal is a vector *within the surface*, whose components are two differential 1-forms, designated here as ω_1^3 and ω_2^3, which we call the *curvature differential forms* (see **Flanders, 1989**). Either this vector, or one related in a certain way to it, should be connected with the 'orbicular pulse' of Hooke. Thus Hooke's idea of representation of the colors by an angle with respect to the position vector has, in the classical theory of surfaces, quite a natural representation.

Now, in ordinary physical situations, the idea of roughness, is naturally connected with that of a friction force: the roughness is variable with the 'friction' mechanically representing the interaction between matter formations. The 'friction' is essentially a surface phenomenon. The 'friction' force is usually zero whenever the surface is smooth, a condition characterized, in a first instance, by the lack of variation of the unit normal to surface, therefore by the fact that the curvature differential forms are null. According to one of the Cartan lemmas (**Фиников, 1948**), this 'friction' force, considered as a surface force, should be a differential 2-form, which can be written as

$$f = \omega_1^3 \wedge \phi^1 + \omega_2^3 \wedge \phi^2 \tag{3}$$

where ϕ^1 and ϕ^2 are two *conveniently chosen* differential 1-form, and '$\wedge$' means exterior multiplication of the differential forms. The equation (3) incorporates the previous logic, according to which the force is zero whenever there is not geometrical roughness, i.e. there is no variation of the normal to surface. However, this is only a necessary condition.

If the conveniently chosen auxiliary forms are the components of the *first fundamental form*, i.e. the components of the position vector $d\vec{x}$ in the tangent plane, s^1 and s^2 say, then equation (3) simply offers the definition of the curvature matrix as a limiting case where the 'friction' forces are zero. Indeed, in the case of null 'friction' force in equation (3), another one of the Cartan lemmas shows that we must have (see **Guggenheimer, 1977**)

$$\begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = -\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \cdot \begin{pmatrix} s^1 \\ s^2 \end{pmatrix} \tag{4}$$

with α, β, γ some parameters – *the curvature parameters*. This defines the classic symmetrical curvature matrix. The curvature parameters are thereby *external parameters*, a feature bestowed upon them by the Cartan lemma. They are always submitted to variation due to some physical causes contained in the matter interacting with light. Their geometrical evaluation can be made just by measurements on the surface as usual (**Lowe, 1980**).

If, however, the *'friction' force is permanent*, like in the case of light through ether, i.e. it always exists and is nonzero by some physical reasons rather than geometrical, we can express it as a differential 2-form:

$$f = \Phi_{\alpha\beta} s^\alpha \wedge s^\beta; \quad \alpha, \beta = 1,2$$

Then the equation (3) can be read in the form

$$(\omega_1^3 - \Phi_{12} s^2) \wedge s^1 + (\omega_2^3 - \Phi_{21} s^1) \wedge s^2 = 0$$

Applying again the Cartan lemmas' considerations, we have, in a compact 'Dirac' writing

$$\langle \omega^3 | - \langle s | \cdot \mathbf{\Phi} = \langle s | \cdot \mathbf{b} \quad \therefore \quad \langle \omega^3 | = \langle s | \cdot (\mathbf{b} + \mathbf{\Phi})$$

where $\mathbf{\Phi}$ is the 2×2 skew-symmetric matrix having the unique element $\Phi_{12} \equiv \phi$, $\mathbf{b}$ the usual curvature matrix from equation (4) and $|s\rangle$ is the column vector from that equation. We do recognize here a curvature matrix which is no more symmetrical, but contains also a 'twisting' naturally accompanying the 'roughness' of surface. From among the usual measures of the curvature, the *mean curvature* of the surface remains unchanged by such physical forces, however its *Gaussian curvature* is changed:

$$2H \equiv \mathrm{tr}(\mathbf{b} + \mathbf{\Phi}) = \alpha + \gamma; \quad K \equiv \det(\mathbf{b} + \mathbf{\Phi}) = \alpha\gamma - \beta^2 + \phi^2$$

Such forces do not change the second fundamental form of the surface per se. In the case of light, they cannot appear therefore when the propagation of light is made without refraction or reflection, but should nevertheless be obvious in like processes. This fact validates, to a large extent, the electromagnetic image of light, even within the limits of the classical geometrical optics. It has, however, some other, more fundamental connotations related to the wave surface, which become apparent if we take into consideration the classical considerations of Hooke and Newton.

Indeed, let's just consider the second fundamental form. It is represented by the first term in equation (2) which, in view of the orthogonality between the normal to surface and the elementary displacements within surface, can be written as:

$$\hat{n}\cdot d^2\vec{x} \equiv -d\hat{n}\cdot d\vec{x} = \langle s\,|\,\mathbf{b}\,|\,s\rangle \tag{5}$$

In this expression we used the previous equation (4). This tells us that the second fundamental form of the wave surface, being a measure of the projection of the variation of normal unit vector to the displacements within the wave surface, can be taken as a measure of the color of light in the sense of Hooke. The orbicular pulse should therefore be taken in relation with the wave surface, according to Newton's conception of light. This will justify the modern idea of three-dimensionality of the manifold of colors: if the second fundamental form of the wave surface represents the color of light, then the color space is certainly a linear threedimensional space, as claimed in the modern theory of colors (**Schrödinger, 1920**). Indeed, one can prove that the binary quadratics do form a linear threedimensional space.

The variation of second fundamental form can generally occur by both the variation of the curvature parameters and by the variation of the components of the first fundamental form in the tangent plane of the wave surface. In a compact 'Dirac notation' this variation can be written as

$$\delta\mathrm{II} = \langle ds|\mathbf{b}|s\rangle + \langle s|d\mathbf{b}|s\rangle + \langle s|\mathbf{b}|ds\rangle \tag{7}$$

Here II denotes the second fundamental form. We assumed that, in expressing this variation, the rules of usual differentiation apply. There are, therefore, two main contributions to the variation of the second fundamental form in this approach. One of them is due to the variation of the components of the first fundamental form in the tangent plane the other is due to the variation of the curvature matrix itself. This last variation is the one we are after, for it represents the variation of the second fundamental form strictly due to some physical reasons, for instance the interaction of light with the material environment. The question arises therefore: when is the variation of the second fundamental form strictly due to the interaction with the medium hosting the light?

At least formally, the answer is quite obvious from equation (7): in those cases in which its variation is due strictly to the variation of the curvature matrix, i.e. when in equation (7) only the middle term remains. This can happen in cases where the sum of the two extreme terms from the right hand side of (7) is zero:

$$\langle ds|\mathbf{b}|s\rangle + \langle s|\mathbf{b}|ds\rangle = 0 \tag{8}$$

Assuming now the existence of an 'evolution' of the vector $|s\rangle$, such that

$$|ds\rangle = \mathbf{a}|s\rangle \tag{9}$$

where **a** is a real matrix, the condition (9) becomes:

$$\langle s|(\mathbf{a}^t\mathbf{b} + \mathbf{ba})|s\rangle = 0$$

Here the superscript 't' stands for 'transposed'. In other words, when the position in the first order 'showground' of a point of the wave surface evolves according to equation (9), then the second fundamental form varies strictly due to the curvature parameters if

$$\mathbf{a}^t\mathbf{b} + \mathbf{ba} = \mathbf{0} \tag{10}$$

A solution of this equation presents itself immediately in the form **a** = **I**·**b**, where **I** is the fundamental skewsymmetric 2×2 matrix. The condition (10) is satisfied in view of the symmetry of **b** and the antisymmetry of **I**. The projection of the vector $|ds\rangle$ from equation (9) along vector $|s\rangle$ is, in that particular case, the geodesic torsion of the surface. Incidentally, the condition (10) is satisfied also for the inverse of the matrix **a**. Then, if the curvature parameters are constants, the evolution (9) preserves the second fundamental form of the surface. In that particular case one can properly call the second fundamental form a wavelength. This is the classical way to description of color, leading, through the dispersion law, to the modern theory of coherence.

Nonconstant Curvature: the Case of Light Line Deformation

What about the cases when the curvature parameters are not constants? The physics of light is then dictated by both the current values of the curvature parameters and their variations. In the first order neighborhood of a certain point of the wave surface, the variations of the curvature parameters can be described by the deformations of the surface. This concept was left behind by Hooke – his orbicular pulse is strictly a periodic motion – and has not been even considered by Newton. Let us consider it here within the classical differential geometry of a wave surface.

In the case of a symmetric curvature matrix, the equation (9) can be integrated even without considering a time parameter, thus leading to ellipses on the wave surface as dictated by constant distance from the tangent plane. This is in fact the classical case of interpretation of the second fundamental form of a surface (**Struik, 1988**). One can say that, in case we are able to discover a time, the motion dictated by equation (9) is a harmonic two-dimensional motion: just as Hooke claims for his orbicular pulse. However, the orbicular pulse should not be only this displacement. Indeed, as we already noticed, Hooke did not take into consideration the inherent deformation accompanying the expansion of wave, and one might even add that he could not do it. Yet, within the differential theory of surfaces the deformation of the orbicular pulse comes as quite a natural concept. We will illustrate this statement with the help of the classical idea of infinitesimal deformation of wave surface (**Guggenheimer, 1977**, pp. 245ff).

A deformation of the wave surface is infinitesimal if the first fundamental form in a point coincides with the first fundamental form in the corresponding point of the deformed surface. This definition mimicks what Hooke assumes happening inside transparent bodies penetrated by light. If the deformation can be expressed by a small parameter, say ε, in the form

$$\vec{r} = \vec{x} + \varepsilon\vec{z}$$

then the first fundamental forms at $\vec{r}$ and $\vec{x}$ coincide when ε is infinitely small. This sets important restrictions upon vector $\vec{z}$, amounting to the fact that its differential should be always perpendicular to $d\vec{x}$ in the first order 'showground'. If this condition is expressed by a relation of the form

$$d\vec{z} = \vec{y} \times d\vec{x} \tag{11}$$

then the vector $\vec{y}$ cannot be quite arbitrary: its component along the normal to surface needs to be constant, while the in-surface components have differentials that should be expressed linearly in the components of the fundamental 1-form:

$$\begin{pmatrix} -v^2 \\ v^1 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \cdot \begin{pmatrix} s^1 \\ s^2 \end{pmatrix}; \quad \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \equiv d\vec{y} \tag{12}$$

Then, as a consequence of the fact that $d\vec{y}$ is an exact differential vector, the parameters A, B, C must satisfy, besides other restricting differential relations, an algebraic apolarity with respect to the second fundamental form of the surface. It is this condition then, which is expected to prevail whenever the vector $\vec{y}$ is dictated by external circumstances of a physical nature imposed during the process of deformation of surface, because these circumstances are external and local. One should thus have:

$$v^1 \wedge \omega_1^3 + v^2 \wedge \omega_2^3 = 0 \quad \therefore \quad \alpha C + \gamma A - 2\beta B = 0 \tag{13}$$

For an algebraic – and thus, in fact, physical – interpretation of this result, let's notice that, because $d\vec{y}$ is an 'intrinsic' vector with respect to surface (its component along the normal to surface is null), the cross product of this vector with the elementary displacement on surface is oriented along the normal to surface. This vector is

$$d\vec{y} \times d\vec{r} = \{A(s^1)^2 + 2Bs^1s^2 + C(s^2)^2\}\hat{n} \tag{14}$$

Consequently its magnitude is a quadratic form, algebraically apolar to the second fundamental form of the surface. One can therefore say that it is added to the second fundamental form, thus changing the local curvature of the surface. According to Hooke's idea it changes the color of light. As the parameters A, B, C, just like α, β, γ, are introduced by external reasons they can be taken as representing the matter acting upon light. Is this image reasonable?

Within classical phenomenology, matter interacts with light only because of its space extension. Incidentally, one could say that, in the modern times, only the idea of material point made possible the electromagnetic theory of light. But, continuing the reasoning within the classical phenomenology, based on reflection and refraction experiments, an extended part of matter has naturally a surface separating it from the environment. The coefficients A, B, C can then be taken as representing the surface of matter from the very same point of view which refers to the wave surface, as described above, i.e. they are the coefficients of the second fundamental form of the surface of the matter interacting with light. In this situation the equation (14) carries an important dynamical connotation, occuring in the form of a theorem which appears as a working hypothesis to Newton.

The corollary of this whole classical theory would be that the matter itself behaves like light, inasmuch as it interacts with light through its surface, as it is actually the case. The quantitative expression of this interaction is then the bilinear form from equation (13) which is zero only in the case of infinitesimal deformations. The point is that the matter extension can be geometrically described just like the extension of light, by a second fundamental form, representing a color. This, again, corresponds to the natural fact that the color only appears at the interaction of the matter with the light. But then, a normal acceleration occurs at the interaction point, for the second fundamental form is a measure of an acceleration normal to surface (see equation (5) above). Or else, as in the case of matter acceleration means inertia, therefore, according to the classical principles, it means a force. Whence, the idea that the matter acts upon light with a force along the normal direction to its surface. This is a fundamental hypothesis to Newton, used by him to prove the laws of refraction (**Newton, 1952**, p. 79ff). As it turns out, it is actually a geometrical theorem.

Quantum Theory of Light

The classical theory of light, as extracted from the phenomenology comprising the experiments of reflection and refraction, is not the only one pointing explicitly to a holographic principle. The same happens with the quantum theory of light, whereby the two degrees of freedom appear to a more abstract level, while having also a more involved physical meaning. Indeed, from a purely statistical theoretical point of view, the Planck moment in the physics of light (**Planck, 1900**) reveals two distinct theoretical sides. The first one is the heuristic side, closely related to the Gaussian aspect of the statistics of light fluctuations. According to Max Born, this was the source of inspiration in establishing the famous connection between the fluctuations of the spectral density and the equilibrium temperature of the blackbody radiation, leading to the idea of quantum. The second side of the Planck moment of the physics of light is the proper quantum side, whereby the probability distributions characterizing the blackbody radiation are of the quadratic variance function type. The quantum is required here by the condition that the blackbody radiation spectrum should satisfy the Wien displacement law. The contemporary theory of light colors, and of colors in general, seems mainly related to the first side of this moment of physics. Let's show a way to it

Now, the two degrees of freedom involved in the holographic principle are more intricate, yet closer to the proper holographic description of light. Indeed, the Planck's original Gaussian, represents two processes of fluctuation, at low and high temperatures, and is uncorrelated (**Mazilu, 2010**). When considered, however, in the general, correlated form, the probability density of this Gaussian would be something of the form:

$$p_{XY}(x,y)=\frac{\sqrt{ac-b^2}}{2\pi}\exp\left\{-\frac{1}{2}\left(ax^2+2bxy+cy^2\right)\right\} \tag{15}$$

where X and Y are, as we said, the two characteristic fluctuation processes, playing the part of the two degrees of freedom, originally constituting the thermal light at low and high temperature, respectively. The classical theory of color can be constructed on this statistics, as it has an interesting twist on it.

Indeed, in the classical theory of color, we don't specify these two random processes by temperature regimes, because in general we cannot associate a physical temperature with the color. The problem of associating a temperature to the color was not solved yet (**MacAdam, 1977**), and we don't think will ever be solved. For once, the thermodynamically defined absolute temperature is not physically supported for light as classically defined. This issue led to Planck theory in the first place. On the other hand, from a statistical point of view, the temperature goes into a parameter characterizing the distribution of colors in a more elaborate way than it does in the Planck's statistics. Thus, let's just say, for the sake of the present argument, that in the case of light measurements in general we have to do with two stochastic processes X and Y, participating in the composition of a color. If ever in need of a statistical evaluation of the parameters a, b, c of the density from equation (15) above, we have at our disposal the maximum information entropy principle, for instance, giving their values by

$$a=\frac{\mathrm{var}(y)}{D},\quad c=\frac{\mathrm{var}(x)}{D},\quad b=-\frac{\mathrm{cov}(xy)}{D},\quad D\equiv\mathrm{var}(x)\,\mathrm{var}(y)-[\mathrm{cov}(xy)]^2 \tag{16}$$

Here 'var' and 'cov' denote the variance and the covariance of the experimental data on X and Y.

This characterization of the color measurements – the so-called dichromatic characterization – is closely related to a plane centric affine geometry. This is to say that if one insists in characterizing the measurements of light in a plane, which is obviously the natural way to consider these measurements ever since the first ideas on light came out (**Hoffman, 1966**), the geometry of this plane is the centric affine geometry. The group of this plane geometry is given by the infinitesimal generators

$$X_1=y\frac{\partial}{\partial x};\quad X_2=\frac{1}{2}\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right);\quad X_3=-x\frac{\partial}{\partial y} \tag{17}$$

while the group of the space of values a, b, c is given by infinitesimal generators

$$X_1=-a\frac{\partial}{\partial b}-2b\frac{\partial}{\partial c};\quad X_2=-a\frac{\partial}{\partial a}+c\frac{\partial}{\partial c};\quad X_3=2b\frac{\partial}{\partial a}+c\frac{\partial}{\partial b} \tag{18}$$

These are two realizations of the same $\mathfrak{sl}(2,R)$ algebraical structure. The second one has intransitive action, which allows transitivity only along specific manifolds, given by constant discriminant of the quadratic form from the exponent of equation (15).

The probability density (15) itself can be presented as a *joint invariant* of the two actions (17) and (18), with the help of Stoka theorem (**Stoka, 1968**). According to this theorem, any joint invariant of the two actions is an arbitrary continuous function of the two algebraic formations

$$ax^2+2bxy+cy^2,\quad ac-b^2 \tag{19}$$

Obviously (15) is only a special case of this theorem. By the same token, the straight lines through origin x = y = 0 can be presented as joint invariants of two actions (17), while the joint invariants of two actions (18), one in the variables a, b, c, the other in the variables α, β, γ, say, are arbitrary functions of the following three algebraic formations (**Mazilu, 2004**):

$$\alpha\gamma-\beta^2,\quad ac-b^2,\quad a\gamma+c\alpha-2b\beta \tag{20}$$

These facts can give good reasons for a few further observations related to the classical theory of colors.

The argument along these lines allows us to put forward, both in a classical Hooke-type theory, and in the modern theory of fluctuations of light, the Newtonian idea of a general two-dimensional symmetry involved in the description of light and matter. This leads further to a representation of color in connection with MacAdam discovery of the meaning of quadratic forms for which the discriminant from equation (19) is positive (**MacAdam, 1942**). First of all, we have to be a little more specific about the probability densities like that from (15). Thus, for instance, consider that the background light color in a measurement process on a certain plane has the normal density

$$p_{XY}\left(x, y|\alpha,\beta,\gamma\right) \equiv \frac{\sqrt{\alpha\gamma-\beta^2}}{2\pi}\exp\left\{-\frac{1}{2}\left(\alpha x^2+2\beta xy+\gamma y^2\right)\right\} \tag{21}$$

in two variables X and Y, of which we don't know too much for now, other than that they are characterized by the statistics α, β and γ, as suggested before. All we know for sure is that, in practice, X and Y are some kind of projections on unspecified planes, that happen to be experimentally realizable, and that they represent two colors (the so-called property of dichromacy). At this moment, the theory is therefore dichromatic. Now, let us say that the two processes are jointly participating to give a third process, and all we know of this *participation* is that it is some kind of addition of them. More specifically, we will suppose that this third process is a kind of weighted sum of the two processes, having the general form

$$Z = \mu X + \nu Y \tag{22}$$

This is, for instance, the case of *initial conditions* in the case of the harmonic oscillator, under the condition of a proper gauging of light. The participations μ and ν are, in this particular case, given by the two solutions of a second order differential equation. The problem now is to find the probability density of the stochastic process Z. This can be done by following a known statistical routine, and the final result is

$$P_Z(z) = \sqrt{\frac{\alpha\gamma-\beta^2}{2\pi(\alpha\nu^2-2\beta\mu\nu+\gamma\mu^2)}}\exp\left\{-\frac{1}{2}\frac{\alpha\gamma-\beta^2}{\alpha\nu^2-2\beta\mu\nu+\gamma\mu^2}z^2\right\} \tag{23}$$

This is a Gaussian type probability density, having a zero mean and the variance

$$\sigma_Z^2 = \frac{\alpha\nu^2-2\beta\mu\nu+\gamma\mu^2}{\alpha\gamma-\beta^2} \tag{24}$$

Such a probability density is particularly attractive in constructing the one related to characterizing the differentials of the three statistics, given their values.

Indeed, the equation (24) is indication of the nature of an 'intensity variable' so to speak. It obviously satisfies the Stoka theorem, and indicates that the quadratics are essential in the statistics related to the 'trichromacy' theory of colors. One can see directly that the trichromacy is due to the fact that there is a 'dichromatic' moment in the theory of color space, related to the experimental procedures. Indeed, as we mentioned before, from algebraical point of view, the set of binary quadratics like those occuring in the exponent of a bivariate Gaussian, is a linear three-dimensional space. Whence the basic theoretical support for the idea that the color space should be three-dimensional, even though not necessarily Euclidean. This, of course, gives even more reasons for considering the quadratic as fundamental in the theory of light colors.

There should be, therefore, a way to the color of light, giving consistency to the ideas regarding the trichromacy of light colors directly through a general quadratic statistical variable Z(X, Y), obtained, by dichromacy, in the measurement process of its values:

$$z(x, y) \equiv \frac{1}{2}(ax^2+2bxy+cy^2) \tag{25}$$

This statistical variable then characterizes a specific plane of illumination, *no matter of the orientation of that plane*, because the quadratic is form-invariant by any central projection. We have thus to find the probability density of this variable, under condition that the plane of light is characterized by the a priori probability density as given, for instance, in equation (21). That probability density satisfies, of course, the Stoka theorem, and the probability density of Z should also satisfy that theorem, in the precise sense that it must be a function of the algebraical formations from equation (20). This leaves us with a functionally undetermined probability density though, even if we impose some natural constraints in order to construct it.

Proceeding nevertheless directly, in the usual manner of the statistical practice, we are able to solve the problem, at least in this particular case. Thus, we have to find first the characteristic function of the variable (25). As known, this is the expectation of the imaginary exponential of Z, using (21) as probability density. Performing this operation directly, we get:

$$\left\langle e^{i\zeta Z}\right\rangle = \frac{1}{2\pi\sqrt{1+(i\zeta)\dfrac{a\gamma+c\alpha-2b\beta}{ac-b^2}+(i\zeta)^2\dfrac{\alpha\gamma-\beta^2}{ac-b^2}}} \tag{26}$$

In view of (20), this characteristic function certainly satisfies the Stoka theorem, which thus reveals its right place in the physical theory. Like the Wien displacement law in the case of selection of the physically correct spectrum for blackbody radiation, the Stoka theorem should also serve for the selection of the right probability density in the case of light colors in general. Anyway, the sought for probability density can then be found by a routine Fourier inversion of (26), based on tabulated formulas (**Gradshteyn, Ryzhik, 1994; 2007**, the examples 3.384(21); 6.611 (18); 9.215(16)&(17)):

$$p_Z(z\,|\,a,b,c) = \sqrt{AB}\exp\left(-\frac{A+B}{2}z\right)\cdot I_0\left(\frac{A-B}{2}z\right) \tag{27}$$

Here I_0 is the modified Bessel function of order zero, and A, B are two constants to be calculated from the formulas

$$A+B=\frac{2b\beta-a\gamma-c\alpha}{ac-b^2};\quad AB=\frac{\alpha\gamma-\beta^2}{ac-b^2};\quad A>B \tag{28}$$

Again, this probability density obviously satisfies the Stoka theorem, as it is a function of the joint invariants from equation (20). And so do the mean and the standard deviation of the variable Z, for they can be calculated as

$$\langle Z\rangle \equiv \frac{1}{2}\frac{A+B}{AB}=\frac{1}{2}\frac{2b\beta-a\gamma-c\alpha}{ac-b^2};\quad \mathrm{var}(Z)\equiv\frac{1}{2}\frac{A^2+B^2}{A^2B^2}=\frac{1}{2}\left(\frac{2b\beta-a\gamma-c\alpha}{ac-b^2}\right)^2-\frac{\alpha\gamma-\beta^2}{ac-b^2} \tag{29}$$

We thus have the interesting conclusion that the essential statistics related to variable Z do not depend but on the coefficients of the background color distribution, and the values of the parameters entering the expression of the color Z. On one hand, this means that the geometry of the color space is dictated by the statistical characteristics of the plane of projection and by the physics describing the color, naturally incorporated in the variable Z. For instance Z can represent the energy of a harmonic oscillator, or even the wavelength of light when described by the wave surface. On the other hand, our result shows that the color space is actually characterized by a Riemannian metric of negative curvature, which is the current tenet in the theory of color. Let’s show this.

Light as a Stochastic Process

One usually insists, and with good reasons at that, upon the fact that the geometry of the color space is not an Euclidean one, but a general Riemannian geometry (see **Schrödinger, 1920**; English translations of these works in **MacAdam, 1970**; see also **Wyszecki, Stiles, 1982** for a pertinent comprehensive review of the theories of colors in all their aspects). In such circumstances, the Riemannian metric carries a special statistical significance whereby the components of the metric tensor are covariances of the three color coordinates (**Silberstein, 1938, 1943**). However, this meaning of the metric does not seem to be theoretically secured. Yet one works this way, and the results confirm the manner of approach everywhere in the classical theory of color. There should be therefore some fundamental truth there, whose formal expression is not yet obvious. And there is, of course, a fundamental truth here, giving deeper physical grounds to the holographic principle in an unexpected form related to the classical theory of colors.

First, the previous statistical theory can help us secure, from a theoretical point of view, a purely statistical connotation in the color space. Assume indeed, that a, b and c are some variations of the 'background' parameters α, β and γ, respectively. It thus turns out that this variation, dZ say, of the color measure Z, is dictated only by the variations of its coefficients, and it is a process having, according to equation (29), the following expectation and variance:

$$\overline{dZ} \equiv \frac{1}{2}\frac{A+B}{AB} = \frac{1}{2}\frac{2\beta d\beta - \gamma d\alpha - \alpha d\gamma}{\alpha\gamma - \beta^2};$$
$$\overline{[\Delta(dZ)]^2} \equiv \frac{1}{2}\frac{A^2+B^2}{A^2B^2} = \frac{1}{2}\left(\frac{2\beta d\beta - \gamma d\alpha - \alpha d\gamma}{\alpha\gamma - \beta^2}\right)^2 - \frac{d\alpha d\gamma - (d\beta)^2}{\alpha\gamma - \beta^2} \tag{30}$$

Here a bar over the symbol means average using the probability density given by equation (27). From these formulas we get a statistic having a special geometrical meaning:

$$\overline{[\Delta(dZ)]^2} - \overline{dZ}^2 \equiv \overline{dZ^2} - 2\overline{dZ}^2 = \frac{1}{4}\left(\frac{2\beta d\beta - \gamma d\alpha - \alpha d\gamma}{\alpha\gamma - \beta^2}\right)^2 - \frac{d\alpha d\gamma - (d\beta)^2}{\alpha\gamma - \beta^2} \tag{31}$$

The right hand side of this formula carries indeed a special meaning: it is the Riemannian metric which can be built by the methods of absolute geometry for the space of the 2×2 matrices, having the singular matrices as points of the absolute quadric (**Mazilu, Agop, 2012**). In fact, one can prove, and we will show this immediately, that the quadratic form (31) is just the Cartan-Killing metric of the certain action of the 2×2 real matrices. This is indeed of the quadratic form

$$\frac{1}{4}({\omega_2}^2 - 4\omega_1\omega_3) \tag{32}$$

where $\omega_{1,2,3}$ are three 1-forms representing three conservation laws of the SL(2,R), and has the exquisite interpretation already mentioned. Meanwhile, let's notice that, from a stochastic point of view, the process of physical variation of the parameters of the quadratic form is 'almost' a Lévy-type process with three parameters (**Lévy, 1965**), in the sense that the elementary distance is decided by the variance function. This validates indeed the statistical interpretation of the metric of the space of colors, but raises instead another problem related to the coordinates representing the colors. This problem indicates, in turn, the feasibility of another, more special, approach of the geometry of colors, leading to the idea of Yang-Mills fields even in the classical case.

Resnikoff's Special Theory of Colors

Notice indeed that, actually, it is not the variable dZ we are after, but the parameters dα, dβ and dγ, and they can be assumed to have zero averages, without any problem. Equations (30) and (31) are then just constraining *control* equations, related to a space coherence of light for instance. Indeed, we usually measure the wavelength in order to get the characteristics of light and, when referred to the wave surface, the wavelength is a quadratic form in the parameters of the plane of dichromatic measurements. Howard

Resnikoff introduced as representative for what he calls the ‘perceptual lights’ a set of 2×2 symmetric matrices (**Resnikoff, 1972**):

$$\boldsymbol{\alpha} \equiv \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{33}$$

involving directly the second-order statistics. The determinant of this symmetric matrix is taken as the *brightness variable* of the light, to be constructed from the three basic color perceptions. Resnikoff suggests that the entries of the matrix (33) are to be taken as color coordinates. In that case the coordinate β can be chosen to be the regular ‘B’ – the quantifier for the ‘blue’ color in an RGB color scheme – of course, in the cases where the brightness of light thus calculated is positive. For a certain situation β has therefore to play the part of a correlation when statistically considered in the case of dichromatic basic variables. The choice is not unique, for there are three manners of calculating this brightness on a certain range of the color parameters RGB, in order to satisfy the positivity requirement, but let us go with it just for the sake of illustration. Thus, if we take, in the manner of Resnikoff:

$$\xi = \sqrt{\alpha\gamma - \beta^2}; \quad u = \frac{\beta}{\alpha}; \quad v = \frac{\sqrt{\alpha\gamma - \beta^2}}{\alpha} \tag{34}$$

the matrix (33) becomes

$$\boldsymbol{\alpha} = (\xi/v)\begin{pmatrix} 1 & u \\ u & u^2 + v^2 \end{pmatrix} \tag{35}$$

In this case we have by direct calculation:

$$\boldsymbol{\alpha}^{-1}d\boldsymbol{\alpha} = d\ln(\xi/v)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1/v^2)\begin{pmatrix} -udu & -(u^2 - v^2)du - 2uvdv \\ du & udu + 2vdv \end{pmatrix} \tag{36}$$

and the Resnikoff metric is just the Cartan-Killing metric of this group of matrices, given by:

$$\mathrm{tr}[(\boldsymbol{\alpha}^{-1}d\boldsymbol{\alpha})\cdot(\boldsymbol{\alpha}^{-1}d\boldsymbol{\alpha})] = 2\left\{\left(\frac{d\xi}{\xi}\right)^2 + \frac{du^2 + dv^2}{v^2}\right\} \tag{37}$$

Now, the matrix (36) has the general form:

$$\boldsymbol{\alpha}^{-1}d\boldsymbol{\alpha} = d(\ln\xi)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1/v^2)\begin{pmatrix} -udu - vdv & -(u^2 - v^2)du - 2uvdv \\ du & udu + vdv \end{pmatrix} \tag{38}$$

and carries a special meaning in the geometrical theory of color. In order to reveal this meaning let’s consider the quadratic forms in their utmost generality, from the general standpoint that their coefficients do represent lights or color coordinates, as suggested by Resnikoff.

Differential Dichromacy: the MacAdam Ellipses

The general equation of a conic section is a quadratic equation of the form

$$f(x,y) \equiv \alpha x^2 + 2\beta xy + \gamma y^2 + 2ax + 2by + c = 0 \tag{39}$$

This time in the quadratic form we have included the possibility of an arbitrary center – not just the origin – whose coordinates are related to the coefficients a, b through a linear homogeneous relation determined by α, β and γ. There is a merit, given by handling simplicity among others, in using again the ‘notation of Dirac’. This also allows for a suggestive interpretation of the final geometrical results. In such notation the equation (39) can be written as

$$f(x,y) \equiv \langle x|\boldsymbol{\alpha}|x\rangle + 2\langle a|x\rangle + c = 0 \tag{40}$$

where we used the following identification:

$$|a\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} \quad \therefore \quad \langle a| \equiv \begin{pmatrix} a \\ b \end{pmatrix}^t = \begin{pmatrix} a & b \end{pmatrix}$$

This vector represents the relative position of the center of the conic in the known geometrical sense:

$$\boldsymbol{\alpha}|x_c\rangle + |a\rangle = |0\rangle \quad \therefore \quad |x_c\rangle = -\boldsymbol{\alpha}^{-1}|a\rangle \tag{41}$$

If we refer the conic to this center, by means of the translation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \leftrightarrow \quad |x\rangle = |x_c\rangle + |\xi\rangle$$

the equation (40) becomes purely quadratic in coordinates, although otherwise inhomogeneous:

$$\langle \xi|\boldsymbol{\alpha}|\xi\rangle - \langle x_c|\boldsymbol{\alpha}|x_c\rangle + c = 0 \tag{42}$$

This algebra is now used in constructing an argument for the matrix representation of colors.

Within the framework of Resnikoff representation presented above, the problem of identification of a center of color in a plane of measurement – what we would like to call the MacAdam's problem (**MacAdam, 1942**) – has an explicit algebraical expression. Indeed, we can simply represent the repeated targeting of "the same geometrical color center" by the differential equations $dx_c = dy_c = 0$. Then the condition (41) comes formally down to the following matrix differential equation:

$$|0\rangle \equiv d\left(\boldsymbol{\alpha}^{-1}|a\rangle\right) = (d\boldsymbol{\alpha}^{-1})|a\rangle + \boldsymbol{\alpha}^{-1}|da\rangle \tag{43}$$

Obviously this equation limits the set of possible conics having the same geometric center. Using the definition of the inverse of a matrix, to the effect that $\boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\alpha}$ is the identity matrix, one can easily prove by direct differentiation the matrix differential relation $d\boldsymbol{\alpha}^{-1} = -\boldsymbol{\alpha}^{-1}\cdot d\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^{-1}$, so that from equation (41) we must have

$$|da\rangle = (d\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^{-1})|a\rangle \tag{44}$$

Thus the condition of fixed center comes actually down to a certain evolution of the vector $|a\rangle$, dictated by the matrix of the quadratic form from the equation of the conic section and its variation. In detail, the equation (44) can be written as

$$\begin{pmatrix} da \\ db \end{pmatrix} = \frac{1}{\alpha\gamma - \beta^2} \begin{pmatrix} \gamma d\alpha - \beta d\beta & \alpha d\beta - \beta d\alpha \\ \gamma d\beta - \beta d\gamma & \alpha d\gamma - \beta d\beta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{45}$$

The matrix governing the evolution in the right hand side of this equation can be further adjusted to a special form:

$$\boldsymbol{\Omega} \equiv d(\ln\sqrt{\Delta}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\omega_2/2 & \omega_1 \\ -\omega_3 & \omega_2/2 \end{pmatrix}; \quad \Delta \equiv \alpha\gamma - \beta^2 \tag{46}$$

Δ is therefore the determinant of $\boldsymbol{\alpha}$, i.e. Resnikoff's brightness squared, and we denoted

$$\omega_1 = \frac{\alpha d\beta - \beta d\alpha}{\Delta}; \quad \omega_2 = \frac{\alpha d\gamma - \gamma d\alpha}{\Delta}; \quad \omega_3 = \frac{\beta d\gamma - \gamma d\beta}{\Delta} \tag{47}$$

three differential forms generated by the elements of the matrix of quadratic form representing the family of local colors, and their differentials. When calculated in the coordinates from equation (34) these differential forms are

$$\omega_1 = \frac{du}{v^2}; \quad \omega_2 = 2\frac{udu + vdv}{v^2}; \quad \omega_3 = \frac{(u^2 - v^2)du + 2uvdv}{v^2} \tag{48}$$

showing explicitly that the matrix from equation (46) is the transposed of that from equation (38)

Thus, the proposed representation of Resnikoff's has actually a firm physical basis, in relation to MacAdam's ellipses. Indeed, assume that we are to identify a certain center, as in MacAdam experiments. The center is the one position satisfying equation (41), and therefore asks for the differential correlation (44), which turns out to be an equation of motion for the vector $|a\rangle$. When the Resnikoff's matrix is taken as shown, i.e. representing an ellipse, then the motion of the center $|a\rangle$ itself is along an ellipse, which is the real case with MacAdam results. Therefore, the MacAdam's ellipse gives indeed a statistical interpretation to differentials of the elements of color in Resnikoff's representation.

A General Dynamics of Color

Now, a few algebraical relations among the differential forms (48) are in order. They form a basis of a $\mathfrak{sl}(2,R)$ algebra. The following differential relations can be directly calculated:

$$d\wedge\omega_1 = \frac{\alpha}{\sqrt{\Delta}}\Theta; \quad d\wedge\omega_2 = \frac{2\beta}{\sqrt{\Delta}}\Theta; \quad d\wedge\omega_3 = \frac{\gamma}{\sqrt{\Delta}}\Theta \tag{49}$$

where Θ is the differential 2-form

$$\Theta \equiv \frac{\alpha d\beta\wedge d\gamma + \beta d\gamma\wedge d\alpha + \gamma d\alpha\wedge d\beta}{\Delta^{3/2}} \tag{50}$$

The 2-form Θ is closed because it is the exterior differential of a 1-form:

$$\Theta \equiv d\wedge\psi; \quad \psi \equiv \frac{\alpha+\gamma}{\sqrt{\Delta}} d\left(\tan^{-1}\frac{2\beta}{\alpha-\gamma}\right) \tag{51}$$

representing the *Hannay angle* of this problem. In our context it gives a way to 'objectify', so to speak, the subjective experimental evaluations of colors, and has certainly everything in common with the original angle (**Hannay, 1985**; **Berry, 1985**).

On the other hand, we can verify the following relations:

$$\omega_1\wedge\omega_2 = \frac{\alpha}{\sqrt{\Delta}}\Theta; \quad \omega_2\wedge\omega_3 = \frac{\gamma}{\sqrt{\Delta}}\Theta; \quad \omega_3\wedge\omega_1 = -\frac{\beta}{\sqrt{\Delta}}\Theta \tag{52}$$

Thus, from (49) and (52) we have the characteristic equations of a $\mathfrak{sl}(2,R)$ structure:

$$d\wedge\omega_1 - \omega_1\wedge\omega_2 = 0; \quad d\wedge\omega_3 - \omega_2\wedge\omega_3 = 0; \quad d\wedge\omega_2 + 2(\omega_3\wedge\omega_1) = 0 \tag{53}$$

Using these relations we can draw an important conclusion: the quadratic forms associated with the matrix in Resnikoff representation of light perceptuals are actually fluxes of color in the color space, induced by the 'subjective' uncertainty in determining a color. Indeed, the quadratic form conserved along MacAdam's evolution can be written as $\langle a|\boldsymbol{\omega}|a\rangle$, where $\boldsymbol{\omega}$ is a symmetric matrix of 1-forms in Resnikoff's perceptuals. One can thus construct the 2-form

$$\langle a|d\wedge\boldsymbol{\omega}|a\rangle = \langle a|\boldsymbol{\alpha}|a\rangle\frac{\Theta}{\sqrt{\Delta}}; \quad \boldsymbol{\omega} \equiv \begin{pmatrix} \omega_1 & \omega_2/2 \\ \omega_2/2 & \omega_3 \end{pmatrix} \tag{54}$$

where we have used the equations (49). As the 2-form Θ is a flux, the analogous of the solid angle in the usual Euclidean space, the quadratic form $\langle a|\boldsymbol{\alpha}|a\rangle$ is indeed the intensity of a flux of colors in the color space thus defined. One might say that the human eye is driven, in evaluating the light, by a flux of colors correlated to Hannay's angle.

Conclusions and Outlook

The concepts, especially the physical ones have their internal dynamics. The present work advocates the idea of a continuity of this dynamics: first of the quantum theory with respect to classical theory, then of both theories as regarded through the modern idea of the holographic principle. Resuming this last principle, one could say that the light is a universal model of the physical world. We just tried to make this statement more explicit.

In order to conclude the work nothing would come better than a few words excerpted from the articles that founded the holographic principle in its modern form. First, the words of Gerardus ’t Hooft:

> We would like to advocate here a somewhat extreme point of view. We suspect that there simply are not more degrees of freedom to talk about than the ones one can draw on a surface… *The situation can be compared with a hologram of a three dimensional image on a two-dimensional surface*. The image is somewhat blurred because of limitations of the hologram technique, but the blurring is small compared to the uncertainties produced by the usual quantum mechanical fluctuations. *The details of the hologram on the surface itself are intricate and contain as much information as is allowed by the finiteness of the wavelength of light – read the Planck length*. (**’t Hooft, 1993**; *our Italics*).

Involving the quantum mechanics here has raised problems. Leonard Susskind advocates no existing contradiction, and is even more precise as to the involvement of the quantum limit:

> According to ’t Hooft it must be possible to describe all phenomena within V by a set of degrees of freedom which reside on the surface bounding V. *The number of degrees of freedom should be no larger than that of a two dimensional lattice with approximately one binary degree of freedom per Planck area. In other words the world is in a certain sense a two dimensional lattice of spins* (**Susskind, 1994**; *our Italics*).

We have shown that these statements do have a historical lineage, starting with Newton, and continuing with the quantum theory of light, which occasioned the idea of holography in the first place. The present work showed that Newton’s theory of light means actually that the light supports the idea of an abstract symmetry related to color. The corollary of Newton’s work can be properly understood by referring it to Hooke’s rational theory of colors. In short it states: is not the geometrical form of the light ray that should prevail, but the general two-dimensional symmetry. The Planck’s theory of quanta points out to the very same general symmetry property of physical fields. No wonder then, the holography, as well as its quantal basis, should have roots in the classical theory of light, and the holographic principle thus turns out to be a physically sound universal principle, inasmuch as the light carries the information in the universe. But there is more to it.

First, the light can be taken as a sound physical model of the theory of interactions of material particles, defined in the modern way, i.e. experimentally, which is plainly a Newtonian way of seeing the particles. The particles here are not material points in the classical sense, but have a space extension. Thus they have a surface, and this surface is the one through which the interaction takes place. The present work shows that the interaction is then described by a $\mathfrak{sl}(2,\mathrm{R})$ Lie algebra. This approach offers a rationale to the classical theory of color, seen as a theory of interaction of light with the matter. Moreover, it offers a general view of the theory of fundamental interactions by what we call a Resnikoff-type of representation of interactions.

But the implications of a theory that uses a Resnikoff’s representation of colors, whereby they are quantitatively described by the entries of a 2×2 symmetric real matrix, are far more intricate from physical theoretical point of view. Indeed, such a representation has an outstanding theoretical meaning. A matrix is obviously an element of a noncommutative algebra, which can be simply a Yang-Mills field. It turns out that this classical theory of colors is plainly a Yang-Mills theory. It completes the classical theory of light in a natural way, by including the color in it. The classical electromagnetic theory, even though undoubtedly a gauge theory, is not a Yang-Mills theory yet. The present work shows that it takes considerations of color of light in order to render to the theory of light a plain Yang-Mills character. The modern ‘technicolor’ for instance, should be a genuine classical concept. From this point of view, the light itself actually enters the realm of quantum chromodynamics, as it should naturally do, for the everyday color is related to light. But there is more to it: if the mechanism of color is the one explaining the strong interactions, then this color should be classical too. Thus one might figure out why the noncommutativity is the essential ingredient allowing asymptotic freedom in the case of strong interactions: after all, the light is a model of interaction everywhere in the universe, at any level!

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