Mathematical Proof of Four-Color Theorem

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Abstract

The method and basic theory are far from traditional graph theory. Maybe they are the key factor of success. *K4 regions* (every region is adjacent to other 3 regions) are the max adjacent relationship, four-color theorem is true because more than 4 regions, there must be a non-adjacent region existing. Non-adjacent regions chain can be color by the same color and decrease color consumption.

1. Introduce

How many different colors are sufficient to color the regions on a map in such a way that no two adjacent regions have the same color? After examining a wide variety of different planar graphs, one discovers the apparent fact that every graph, regardless of size or complexity, can be colored with just four distinct colors.

The famous four color theorem, sometimes known as the four-color map theorem or Guthrie's problem. In mathematical history, there had been numerous attempts to prove the supposition, but these so-called proofs turned out to be flawed. There had been accepted proofs that a map could be colored in using more colors than four, such as six or five, but proving that only four colors were required was not done successfully until 1976 by mathematicians Appel and Haken, although some mathematicians do not accept it since parts of the proof consisted of an analysis of discrete cases by a computer. But, at the present time, the proof remains viable. It is possible that an even simpler, more elegant, proof will someday be discovered, but many mathematicians think that a shorter, more elegant and simple proof is impossible.

In the mathematical field of graph theory, a complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete digraph is a directed graph in which every pair of distinct vertices is connected by a pair of unique edges (one in each direction). Kn denotes the complete graph on n vertices. K1 through K4 are all planar graphs. However, every planar drawing of a complete graph with five or more vertices must contain a crossing, and the non-planar complete graph K5 plays a key role in the characterizations of planar graphs.

2. Four color theorem

(2.1) For any subdivision of the spherical surface into non-overlapping regions, it is always possible to mark each of the regions with one of the numbers 1, 2, 3, 4, in such a way that no two adjacent regions receive the same number.

In fact, if the four-color theorem is true on spherical surface, it is also true on plane surface. Because the map is originate from sphere, and plane surface is part of spherical surface.

3. Strategy

K4 regions (every region is adjacent to other 3 regions) are the max adjacent relationship, four-color theorem is true because more than 4 regions, there must be a non-adjacent region existing. Non-adjacent regions can be color by the same color and decrease color consumption.

Another important theorem is that the map regions can divide into one complete graph and several non-adjacent relationship chains. Every non-adjacent relationship chain can be colored by the same color.

4. Basic axiom

(4.1) Coloring the regions on a map has nothing to do with the region shape.

This is the only one axiom in proof. It's obviously true. Color only depends on adjacent relationship.

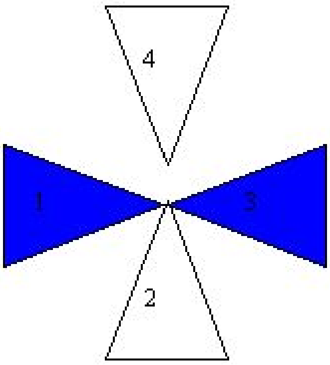
Theorem (4.2)

All color solutions for boundary adjacent regions can apply to point adjacent regions or non-adjacent regions.

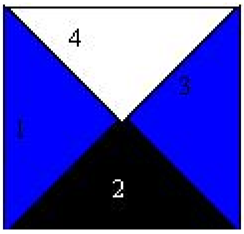
We define adjacent regions as those that share a common boundary of non-zero length. Regions, which meet at a single point or limited points, are not considered to be "adjacent".

Because point adjacent regions are not considered to be "adjacent", any color solution can apply to point adjacent regions, include the color solution of boundary adjacent regions. The free degree of non-adjacent region is limitless. So any color solution of boundary adjacent regions can apply to point adjacent regions and non-adjacent regions.

For example:



Scenario a: non-adjacent and point adjacent



Scenario b: boundary adjacent

All color solutions for Scenario b can apply to Scenario a.

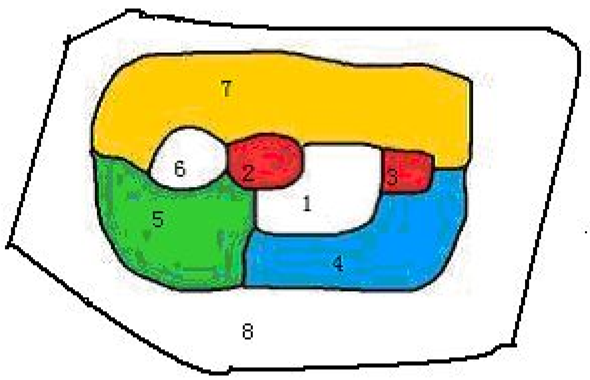
Theorem (4.3)

Any irregular regions map can transform into a circle regions map. The color solution for circle regions map can also apply to the irregular regions map

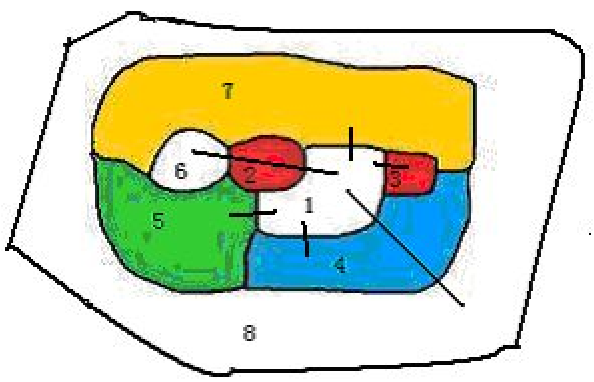
Because basic axiom (4.1) $\Rightarrow$ any irregular regions map can transform into circle-shaped, ring-shaped or fan-shaped.

If circle-shaped, ring-shaped or fan-shaped are point adjacent or non-adjacent, transform into boundary adjacent, finally, to transform into a circle map, ring-shaped and fan-shaped surround circle. Because of Theorem (4.2), the color solution of map transformed can apply to the color solution of map transforming before.

For example:



This an irregular map.



To ensure arbitrary map can be transform into circle map, first select a circle center, second draw a line from center to region, the least region number crossed over is the layer number of ring. From 1 to 6, the least region number is 2, from 1 to 8, the least region number is 2, and so both

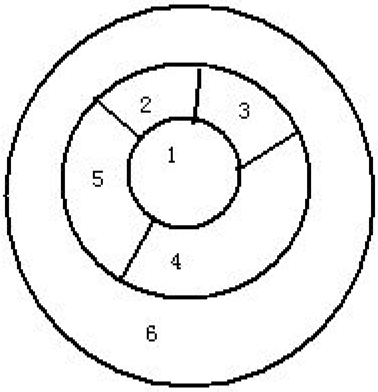
6 and 8 are in layer 3 in circle map. Other regions are in layer 2.

Layer	**Region**
Layer 1	1
Layer 2	2,3,4,5,7
Layer 3	6,8

To ensure to preserve the adjacent relationship in transforming,

Region	**Adjacent region**
1	2,3,4,5,7
2	1,5,6,7
3	1,4,7
4	1,3,7,8,5
5	1,2,4,6,7,8
6	2,5,7
7	1,2,3,4,5,6,8
8	4,5,7

In circle map, the necessary condition of adjacent relationship is between 2 adjacent layers, or between 2 adjacent regions (left, right) in the same layer. Such as below:



Region 1 (layer 1) is adjacent to 2,3(layer 2). Region 3 (layer 2) is adjacent to 2,4(layer 2).

If regions are not in the adjacent layer or more than 2 regions in the same layer, they are sure to be not adjacent. Such as, 6(layer 3) and 1(layer 1) are not adjacent; 2 and 4 are not adjacent in layer 2, because there are 3,4,5 in layer 2, they can't all adjacent to 1.

With the 2 necessary condition of adjacent relationship, check the table one item by one item.

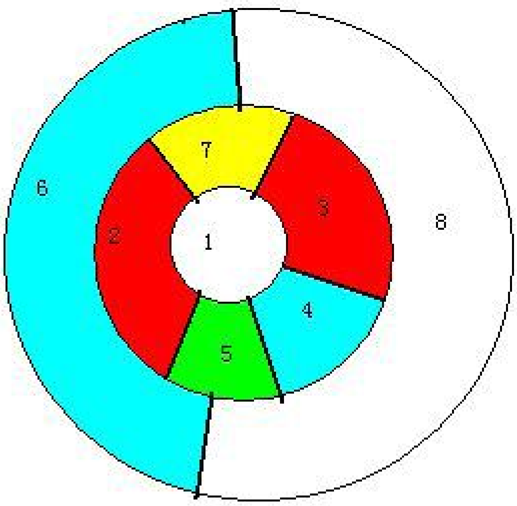
1	2,3,4,5,7
2	1,5,6,7
3	1,4,7
4	1,3,7,8,5
5	1,2,4,6,7,8
6	2,5,7
7	1,2,3,4,5,6,8
8	4,5,7

This is the method to check region 7 and others are similar.

First, remove regions in adjacent layers.

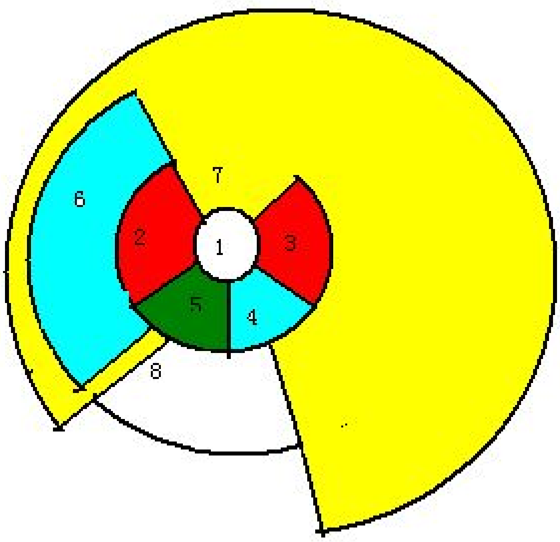
7	2,3,4,5

Because 2,3,4,5 are in the same layer and total 4 > 2, the region 7 can't preserver adjacency.

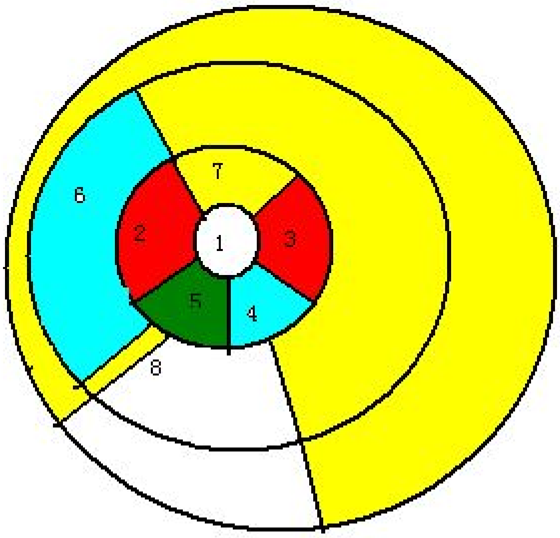


To preserver adjacency, it must to cross layers by region 4, 5 or 7.

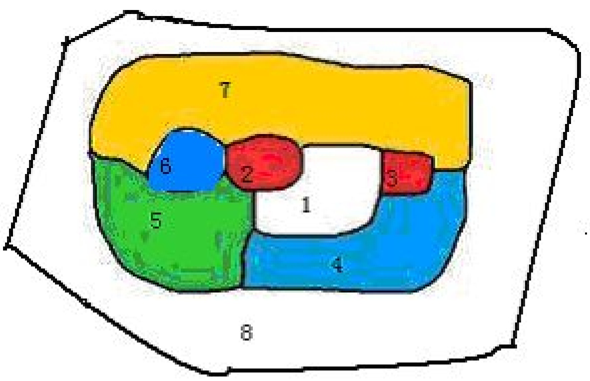
Go back to original map. Region 7 is across over region 6 to adjacent to 5, and is across over region 3 to adjacent to 4. The final map is below.



It equal to the standard circle map below.



Transform irregular map to circle map, 1 is circle center, 6 and 8 are in layer 2, 7 is across layer 2,3,4. Other regions are in layer 1. The boundary adjacent relation is never changed, but some point adjacent or non-adjacent relations are changed to boundary adjacent relation to match the circle map transforming.



The color solution for circle map transformed can apply to irregular map also. Region 6 has changed color, but there is no same color between boundary adjacent regions. It is a color solution qualified.

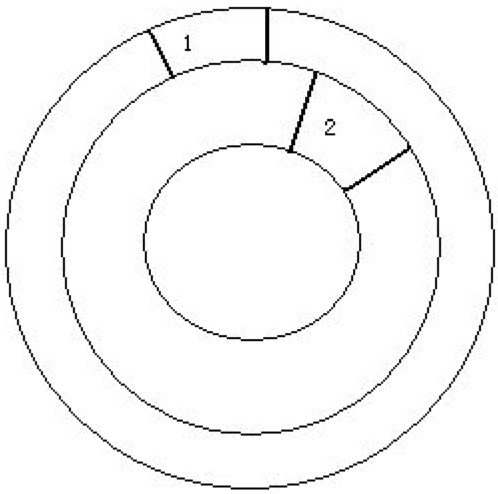
5. Terminology

To describe conveniently, I have defined some terms in circle map.

Solution(n, color1,color2,…,colork) is a color solution qualified to color all of n regions by color in *{color1,color2,…,colork}.*

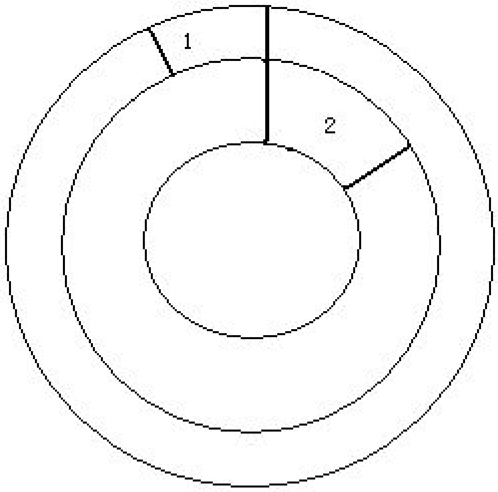
Non-adjacent regions as those no point met.

For example: 1 is non-adjacent to 2 in below circle map.



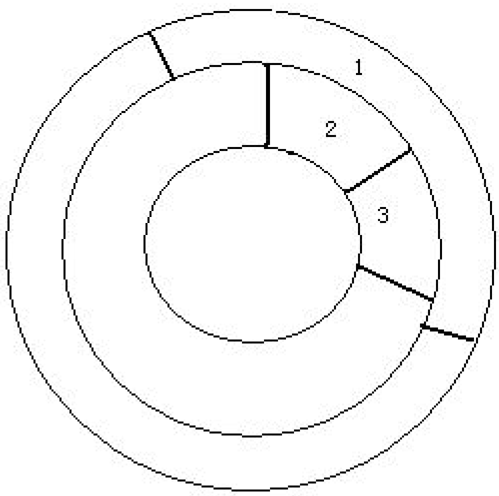
Point adjacent regions as those that meet at a single point or limited points

For example: 1 is point adjacent to 2 in below circle map.



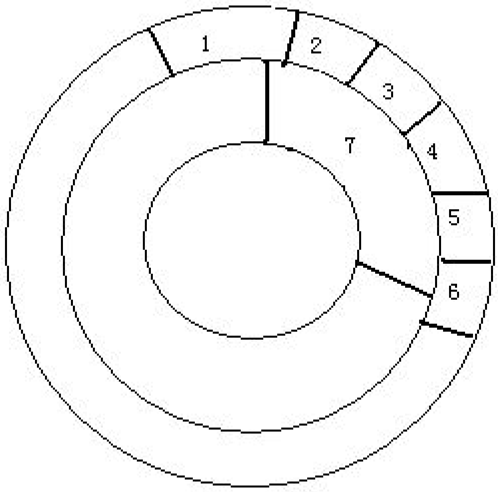
Boundary adjacent regions as those that share a common boundary of non-zero length.

For example: 1, 2, 3 are all boundary adjacent to other 2 regions in below circle map.



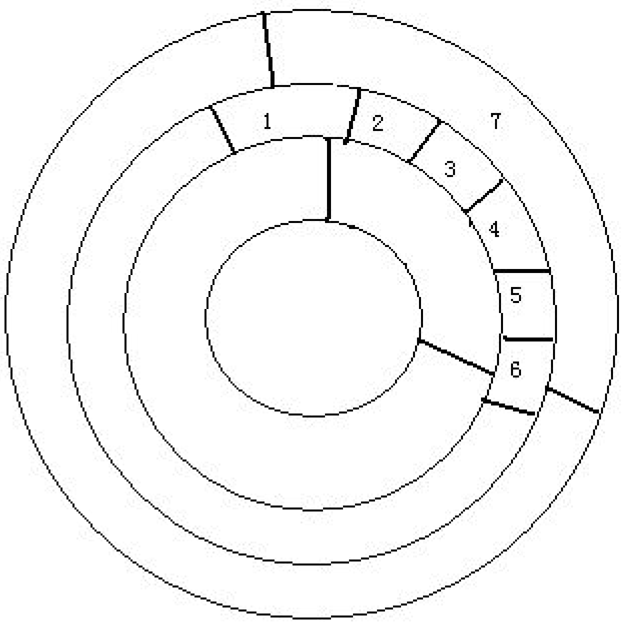
Covered is that the least regions in upper ring have covered one nation in lower ring. Especially, the least N regions in upper ring covering 1 nation in lower ring calls *N Covered,* all the regions in upper ring covering 1 nation in lower ring calls *full Covered.*

For example: 1, 2, 3, 4, 5, 6 are covering 7 in below circle map, that is 6 covering.



Supported is that the least regions in lower ring have covered one nation in upper ring. Especially, the least N regions in lower ring covering 1 nation in upper ring calls *N Supported*, all the regions in lower ring covering 1 nation in upper ring calls *full Supported.*

For example: 1, 2, 3, 4, 5, 6 are supporting 7 in below circle map, that is 6 supporting.

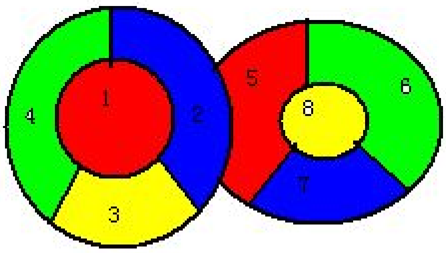


Color is to color region by one or more than one colors. It is recoded as *Color(region) = { color}.* If a region can be colored by more than 1 colors, it can be recoded as *Color(region) = {color1/color2…/colork}.* Such as *Color(3) = {yellow/green/gray}.* Region 3 is colored by *{yellow}* now, but Region 3 has the freedom to color by *{ gree}* or *{gray}.*

Main color is *{color1}* in *Color(region) = {color1/color2…/colork}*, which color the region in fact.

Backup color is *{color2…/colork}* in *Color(region) = {color1/color2…/colork}*, which doesn't color the region in fact, but which has the freedom to color by *{color2…/colork}.*

Dependent color is a region can be colored by more than 1 color, and depends on another multiple color region. It is recoded as *Color(region) = {color1/(dependent region)}.* For example:



Region 5 can be colored by *{red/green/yellow}.* Region 6,7,8 depend on the color of region 5. It is record as *Color(5)= {red/green/yellow}, Color(6)= {green/(5)} , Color(7)= {blue/(5)} , Color(8)= {yellow/(5)} .*

Major color is to color regions in a ring with 2 alternate colors. But if there is odd number of regions in a ring, the head and tail are the same color.

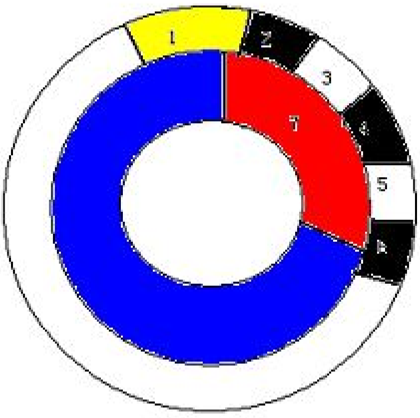
Isolating color is to isolate the same major color with 1 another color in a ring, which is of odd number of regions.

For example: We can see the example in below circle map.

Major color of ring 1 is white color and no isolate color, record as *Major(1) = {white}, Isolating(1) = {};*

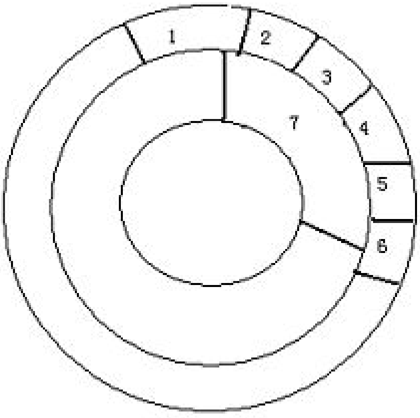
Major color of ring 2 is red and blue color and no isolate color, record as *Major(1) = {read, blue}, Isolating(1) = {};*

Major color of ring 3 is black and white color and isolate color is yellow color, record as *Major(3) = {white, black}, Isolating(3) = {yellow}.*



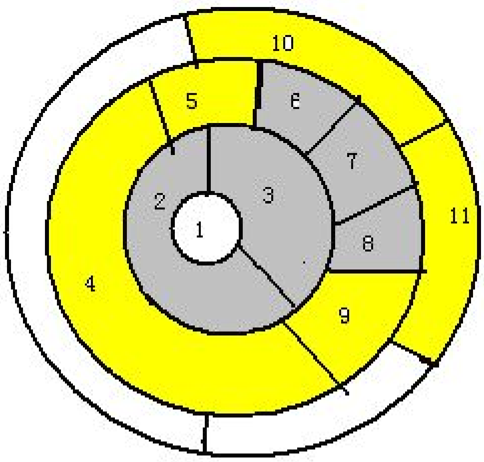
Region number is the total region number of a ring.

For example: region number of ring 3 is 7 in below circle map. Record as *Region(3) = 7.*



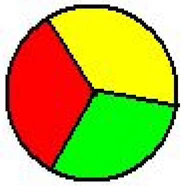
Border regions are all the boundary regions between colored and uncolored. It's the frontier of regions colored.

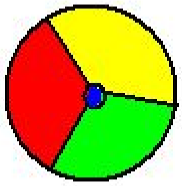
Home regions are the entire boundary regions closed by *Border regions*. It's the home of regions colored. For example:



Regions 1 to 11 are colored, the *Border regions* are marked as yellow color, which are close border to seal all regions colored. *Home regions* are marked as gray color.

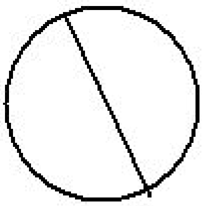
Empty region is a point, which is not a real region, only a proving tool. For example, 3 regions is equivalent to 3 regions and a *Empty region.*



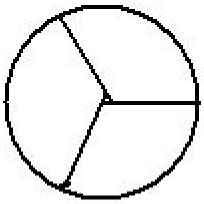


Kn regions are k regions are all adjacent. Anyone of k region is adjacent to other k-1 regions.

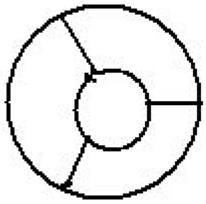
For example: *k2 regions* are in below circle map.



k3 regions are in below circle map. Anyone region is adjacent to other two regions.



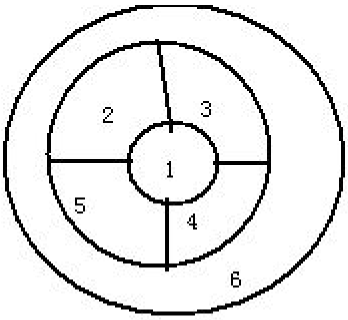
K4 regions are in below circle map. Anyone region is adjacent to other three regions.



Graph theory has proven *K4 regions* are the max adjacent relationship in planar graph.

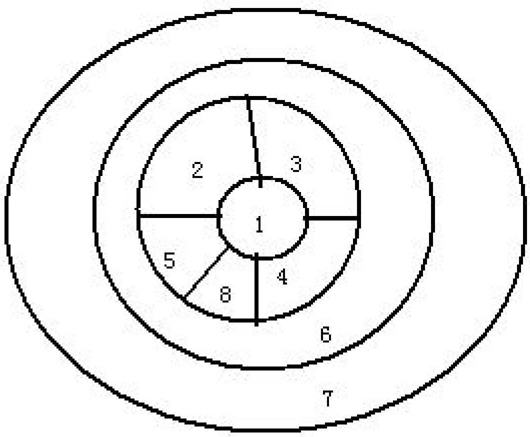
NKn regions are k regions are all non-adjacent. Anyone of k region is non-adjacent to other k-1 regions.

For example: *Nk2 regions* are in below.



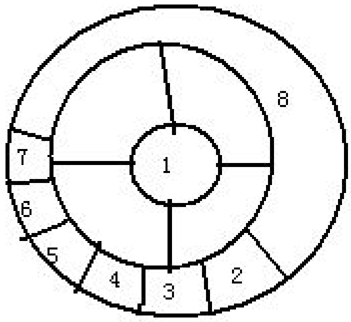
Region 1 and 6, 2 and 4, 3 and 5 are all *Nk2 regions*

Nk3 regions are in below.



Region (2,4 ,7) or (3,8,7) are all *Nk3 regions*

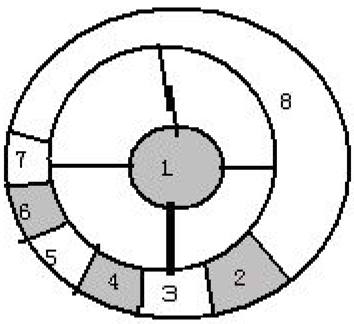
Nk3 regions are in below.



Region (1,2,4,6) are *Nk4 regions*

Any *NKx regions* can divide into (x-1) *NK2 regions chains*.

For example: *Nk4 regions* (1,2,4,6) = (1,2) + (2,4) + (4,6) = 3 *NK2 regions*



6. Preliminary theorem

(6.1) *K4 regions* have only 3 scenarios in circle map.

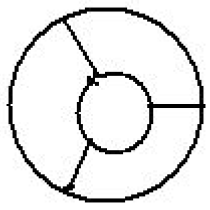
Because in a ring, one region can only adjacent to 2 regions at most => there are at most 3 regions in a ring, but *K4 regions* have 4 regions > 3 => *K4 regions* are at least in 2 rings.

If total of rings >=3, there must be one ring insulating another ring. =>there must be 2 regions being non-adjacent.=> total of rings <=2,

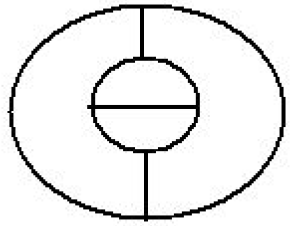
Because total of rings >=2 and total of rings < 2 => total of rings =2.

Total of rings =2 and *K4 regions* have 4 regions => *K4 regions* have only 3 scenarios in circle map. I.e.

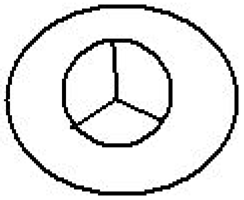
(6.1.1) *region(1) = 1, region (2) = 3*.



(6.1.2) *region(1) = 2, region (2) = 2*.



(6.1.3) *region(1) = 3, region (2) = 1*.

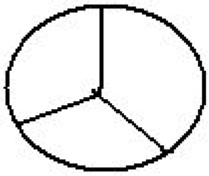


Is there any region across rings? No. Because there is only 2 adjacent rings, regions can keep adjacent relationship in adjacent rings, don't' need to cross rings.

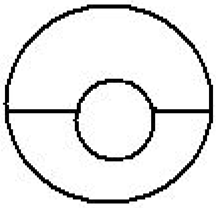
(6.2) *K3 regions* have only 3 scenarios in circle map.

Similarly, we can get 3 scenarios.

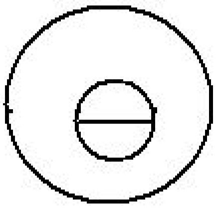
(6.2.1) *region (1) = 3*.



(6.2.2) *region(1) = 1, region (2) = 2.*



(6.2.3) *region(1) = 2, region (2) = 1.*



To prove conveniently, we can unify (6.1) and (6.2). For *K3 regions*, we regard there is a *empty region* in *home regions*. Then *K3 regions* become *K4 regions*

(6.2.1) *region (1) = 1 empty region , region (2) = 3.*

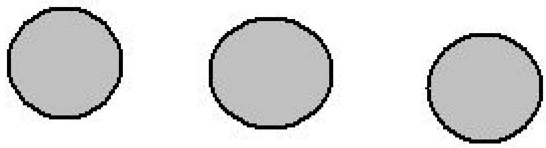
(6.2.2) *region (1) = 1+ 1 empty region, region (2) = 2.*

(6.2.3) *region (1) = 2+ 1 empty region, region (2) = 1.*

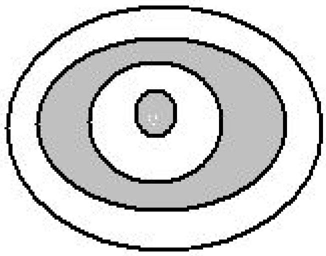
7. Four color theorem

There are N regions. Firstly, search max adjacent relationship.

If no *K2 regions*, all regions are *non-adjacent*, one color is sufficient.



Max adjacent relationship is *K2 regions,* all regions are at most adjacent to one region, and two colors are sufficient.



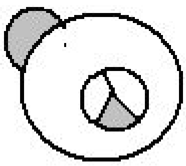
Max adjacent relationship is *K3 regions,* we can add a *empty region* in ring 1, it becomes *K4 regions. border regions* are formed.

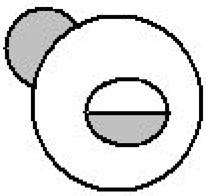
Max adjacent relationship is *K4 regions*, because *K4 regions* have two rings, *border regions* are formed now.

Next we select the 5th adjacent region to color. There are 3 scenarios:

Scenario 1. There is only 1 region in *border regions*

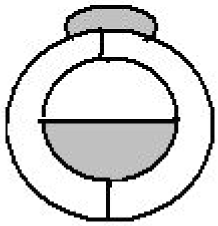
We can always find one non-adjacent regions in *border regions,* because there are 3 regions in home of *K4 regions,* and there are 2 regions in home of *K3 regions* .

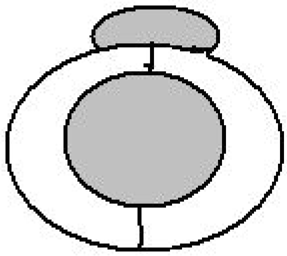




Scenario 2. There are 2 regions in *border regions*

We can always find one non-adjacent region in *border regions* or *home regions*, because there are 2 regions in home of *K4 regions* ,and there is 1 region in home of *K3 regions.*

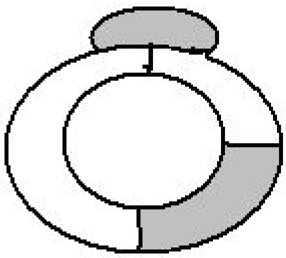


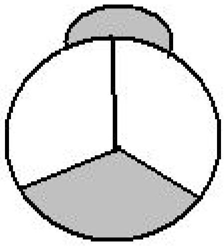


Scenario 3. There are 3 regions in *border regions.*

Scenario 3.1. the 5th region is adjacent to less than 3 regions in *border regions.*

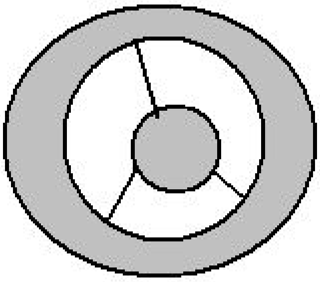
We can always find one non-adjacent region in *home regions.*

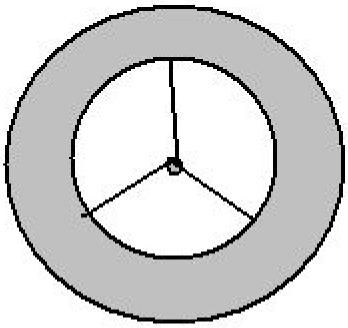




Scenario 3.2. the 5th region is adjacent to 3 regions in *border regions.*

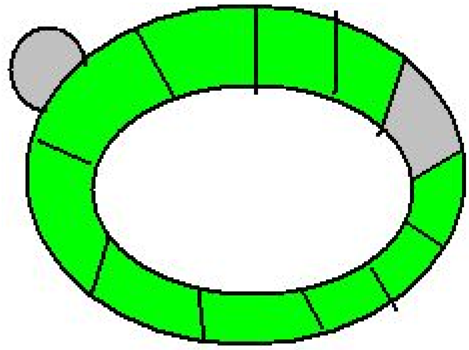
We can always find one non-adjacent region in *home regions.* For *K3 regions,* we can find the *empty regions.*



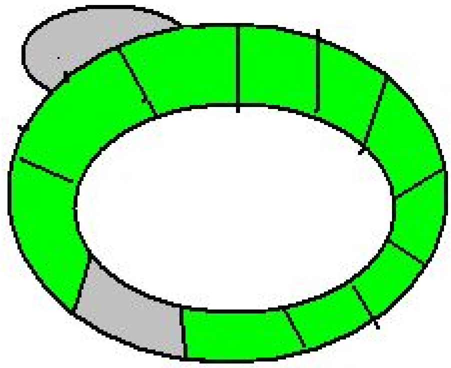


The *border regions* become longer and longer. When *border regions* are more than 3 regions, there are 3 scenarios also. Assume to color the kth region and *border regions* are above three.

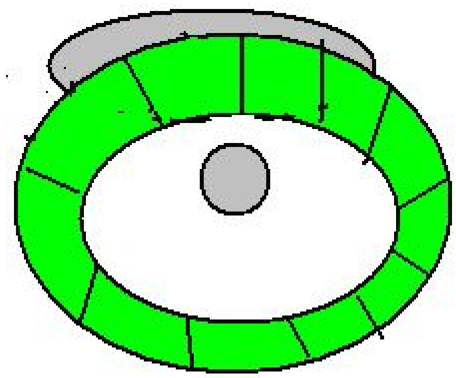
(7.1) When the kth region is adjacent to one region in *border regions.* We can always find at least two non-adjacent regions in *border regions.*



(7.2) When the k^{th} region is adjacent to two regions in *border regions*. We can always find at least one non-adjacent region in *border regions*.



(7.3) When the k^{th} region is adjacent to 3 or more than 3 regions in *border regions*. We can always find at least one non-adjacent region in *home regions*.

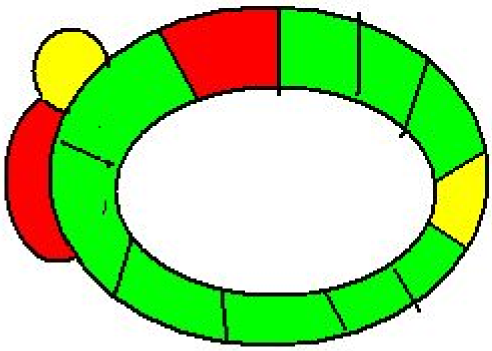


Above proof indicates every region can find *non-adjacent region*, every region is in *NK2 regions* except regions in ring 1 and ring 2.

Next to prove all the *NK2 regions* are not overlapped.

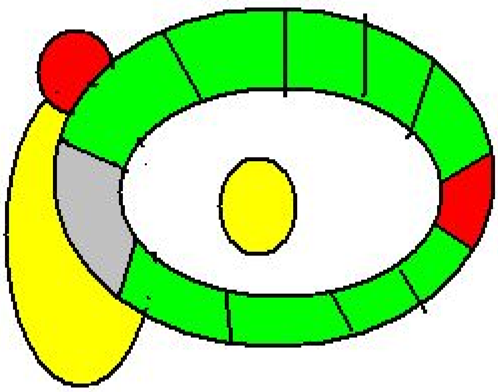
Assume to color the (k+1)th region

In scenario (7.1) (7.2), the (k+1)th region is adjacent to less than 3 regions of *border regions and* not adjacent to all *border regions.* We can always find another non-adjacent region in *border regions,* because *border regions* are above three. The 2 *NK2 regions* are not overlapped.



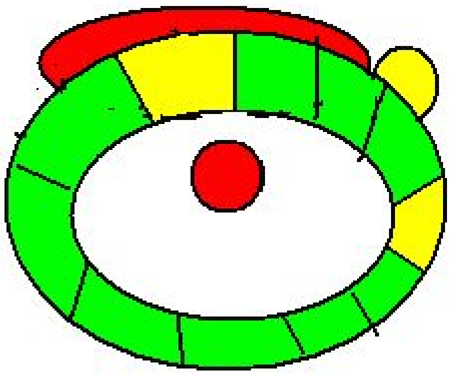
In scenario (7.1) (7.2), the (k+1)th region is adjacent to 3 or more than 3 of *border regions* or is adjacent to all *border regions*

We can always find a non-adjacent region in *home regions.*



How to ensure the *home regions* are sufficient? Because the (k+1)th region is adjacent to 3 or more than 3 of *border regions* or is adjacent to all *border regions*, it must *full cover* at least one region(gray color region). It means that one region is from *border regions* to *home regions.* Though we have consumed one home region, we add at least one home region also. The *home regions* do not decrease.

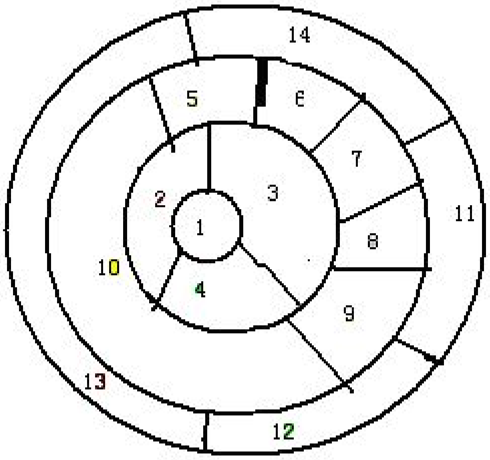
In scenario (7.3), the $(k+1)^{th}$ region is always non-adjacent to the regions which are *full covered* by the k^{th} region or in *border regions*.



All regions are in *NK2 regions(*except ring 1 and ring 2 regions*)*, these *NK2 regions* are connected with header and tail, and become *NK2 regions chains*. The header of every *NK2 regions chain* is in the first *K4 regions.* Ring 1 and ring 2 regions are perhaps in *NK2 regions chain*, perhaps not. There are at most 4 *NK2 regions chains.* Every *NK2 regions* chain is colored by one color. Every *NK2 regions chain* is at most adjacent to 3 others *NK2 regions chains.*

(7.4) Any map can divide into one complete graph *Kn regions*(n<=4) and several (<=4) *NK2 regions chains.*

For example:



First to search *K4 regions,* (1,2,3,4).

Next to find *NKn regions (NK2 regions chains)* .

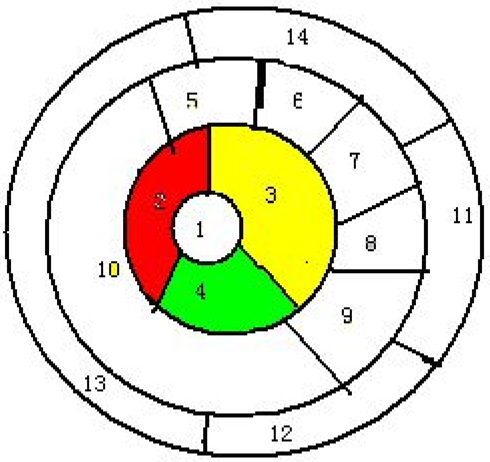
NK4 regions (4,5,7,12) = (4,5)+(5,7)+(7,12);

NK3 regions (1,9,14) = (1,9)+(9,14);

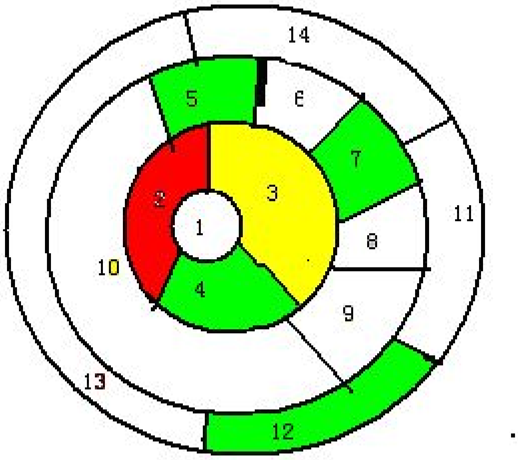
NK4 regions (2,6,8,13) = (2,6)+(6,8)+(8,13);

NK3 regions (3,10,11) = (3,10)+(10,11);

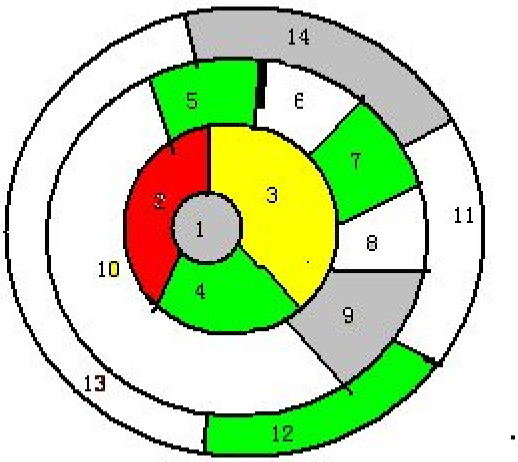
Next to color *K4 regions* by 4 colors.



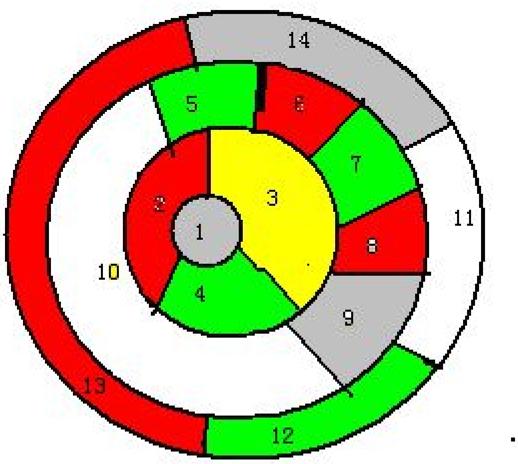
Color *NK4 regions* (4,5,7,12) = (4,5)+(5,7)+(7,12) by 1 color;



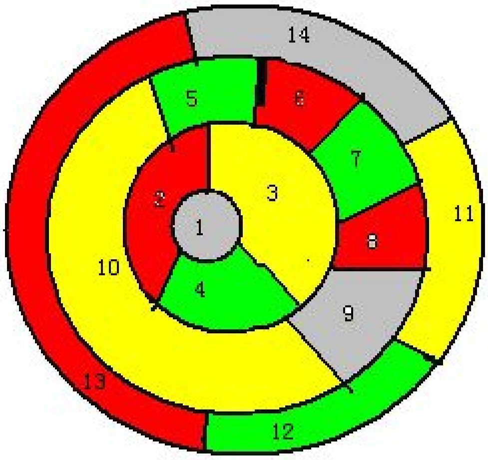
Color *NK3 regions* (1,9,14) = (1,9)+(9,14) by 1 color;



Color *NK4 regions* (2,6,8,13) = (2,6)+(6,8)+(8,13) by 1 color;



Color *NK3 regions* (3,10,11) = (3,10)+(10,11) by 1 color;

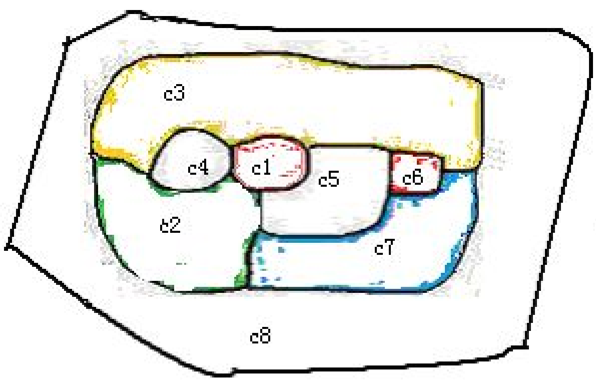


Finally, four-color theorem is proven now!

8. Verification and Demo

One example to verify and explain:

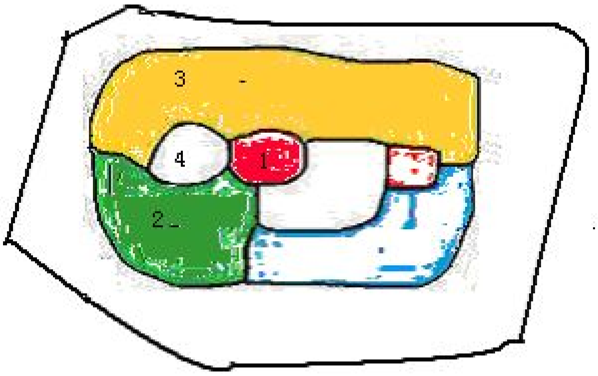
To descript clearly, all regions are marked the number by the color order in advance. The order number (region number) map is below,



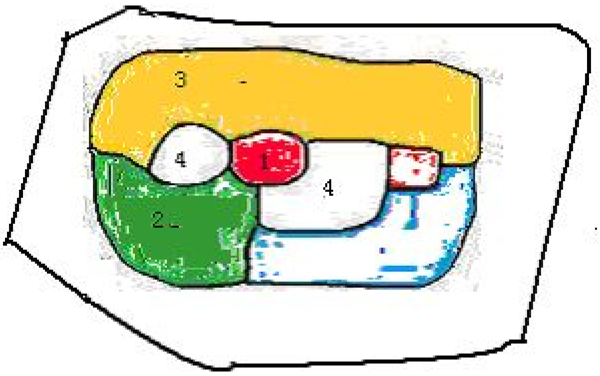
The algorithm bases on (7.4), is simpler than the example in section 7.

1. Search and color max adjacent relationship of complete graph.
2. Find *NK2 regions chains* and colored with more possibilities.

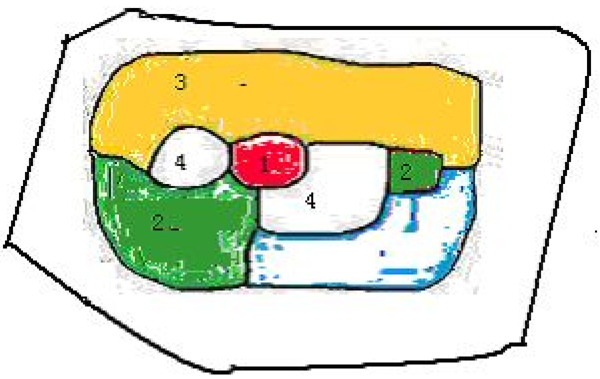
Firstly, search *K4 regions* and colored by {1,2,3,4}



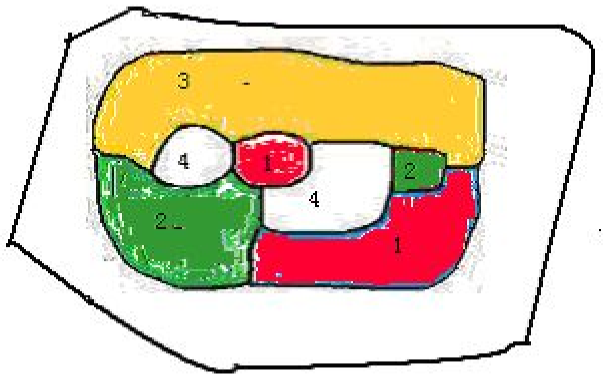
Next, we can select an adjacent region c5, which is adjacent to 3 regions in border. So it is colored by color of home region {4}



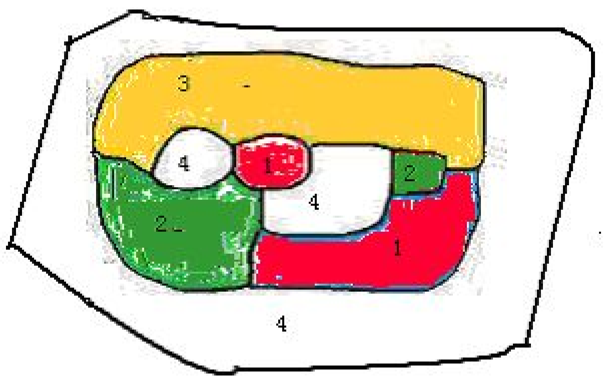
Next, we can select an adjacent region c6, which is adjacent to 2 regions in border. So it’s non-adjacent to a region in border and home, can be colored by {2/4}. But color c4 is adjacent to it, so color is {2}.



Next, we can select an adjacent region c7, which is adjacent to 3 regions in border. It can find a non-adjacent region in *home regions,* and color is {1/4}. Because the red region c1 is *full coved*, which become *home region* from *border region* and color c5 is adjacent to it, so color is {1}.



Next, we can select an adjacent region c8, which is adjacent to all border regions. It can find a non-adjacent region in *home regions*. It can be colored by {1/2/4}, and exclude adjacent region c2, c7. Final color is {4}.



The *NK2 regions chains* are in the below table.

First *K4 regions*	***NK2 regions chains***			
C1	C7			
C2	C6			
C3				
C4	C5	C8		

All regions are in the *NK2 regions chains* except first *K4 regions*, which either are the header of *NK2 regions chains*, or are not in the *NK2 regions chains*, e.g. c3.

Every *NK2 regions chain* is at most adjacent to other 3 *NK2 regions chains*. So four colors are sufficient to color any planar or spherical map.

9. Conclusion

Four-color theorem is an interesting phenomenon, but there is a rule hidden the phenomenon. The max adjacent relationship on a surface decides how many colors are sufficient. More than max adjacent regions, almost every region is in a non-adjacent chain. Every non-adjacent chain can decrease color consumption.

Any planar or spherical map is comprised of one max adjacent relationship of complete graph and several non-adjacent relationship chains whose header is in the complete graph.

REFERENCES

[1] AHLFORS, L.V. Complex Analysis, McGraw-Hill Book Company, 1979.

[2] DHARWADKER, A. & PIRZADA, S. Graph Theory, Orient Longman and Universities Press of India, 2008.

[3] LAM, T.Y. A First Course in Noncommutative Rings, Springer-Verlag, 1991.

[4] ROTMAN, J.J. An Introduction to the Theory of Groups, Springer-Verlag, 1995.

[5] VAN LINT, J.H. & WILSON, R.M. A Course in Combinatorics, Cambridge University Press, 1992.