Mathematical Proof of Four-Color Theorem

By Liu Ran

Abstract

The method and basic theory are far from traditional graph theory. Maybe they are the key factor of success. *K4 countries* (every country is adjacent with other 3 countries) are the max adjacent relationship, four-color theorem is true because more than 4 countries, there must be a non-adjacent country existing. Non-adjacent countries chain can be color by the same color and decrease color consumption.

1. Introduce

How many different colors are sufficient to color the countries on a map in such a way that no two adjacent countries have the same color? After examining a wide variety of different planar graphs, one discovers the apparent fact that every graph, regardless of size or complexity, can be colored with just four distinct colors.

The famous four color theorem, sometimes known as the four-color map theorem or Guthrie's problem. In mathematical history, there had been numerous attempts to prove the supposition, but these so-called proofs turned out to be flawed. There had been accepted proofs that a map could be colored in using more colors than four, such as six or five, but proving that only four colors were required was not done successfully until 1976 by mathematicians Appel and Haken, although some mathematicians do not accept it since parts of the proof consisted of an analysis of discrete cases by a computer. But, at the present time, the proof remains viable. It is possible that an even simpler, more elegant, proof will someday be discovered, but many mathematicians think that a shorter, more elegant and simple proof is impossible.

In the mathematical field of graph theory, a complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete digraph is a directed graph in which every pair of distinct vertices is connected by a pair of unique edges (one in each direction). Kn denotes the complete graph on n vertices. K1 through K4 are all planar graphs. However, every planar drawing of a complete graph with five or more vertices must contain a crossing, and the non-planar complete graph K5 plays a key role in the characterizations of planar graphs.

2. Four color theorem

(2.1) For any subdivision of the spherical surface into non-overlapping regions, it is always possible to mark each of the regions with one of the numbers 1, 2, 3, 4, in such a way that no two adjacent regions receive the same number.

In fact, if the four-color theorem is true on spherical surface, it is also true on plane surface. Because the map is originate from sphere, and plane surface is part of spherical surface.

3. Strategy

K4 countries (every country is adjacent with other 3 countries) are the max adjacent relationship, four-color theorem is true because more than 4 countries, there must be a non-adjacent country existing. Non-adjacent countries can be color by the same color and decrease color consumption.

Another important theorem is that the map countries can divide into one complete graph and several non-adjacent relationship chains. Every non-adjacent relationship chain can be colored by the same color.

4. Basic axiom

(4.1) Coloring the countries on a map has nothing to do with the country shape.

This is the only one axiom in proof. It's obviously true. Color only depends on adjacent relationship.

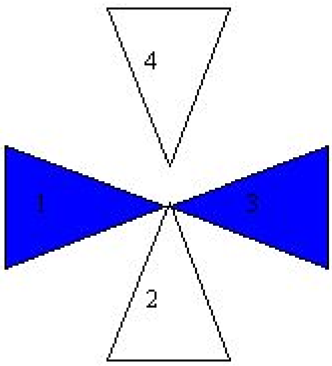
Theorem (4.2)

All color solutions for boundary adjacent countries can apply to point adjacent countries or non-adjacent countries.

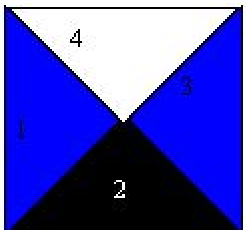
We define adjacent regions as those that share a common boundary of non-zero length. Regions, which meet at a single point or limited points, are not considered to be "adjacent".

Because point adjacent countries are not considered to be "adjacent", any color solution can apply to point adjacent countries, include the color solution of boundary adjacent countries. The free degree of non-adjacent countries is limitless. So any color solution of boundary adjacent countries can apply to point adjacent countries and non-adjacent countries.

For example:



Scenario a: non-adjacent and point adjacent



Scenario b: boundary adjacent

All color solutions for Scenario b can apply to Scenario a.

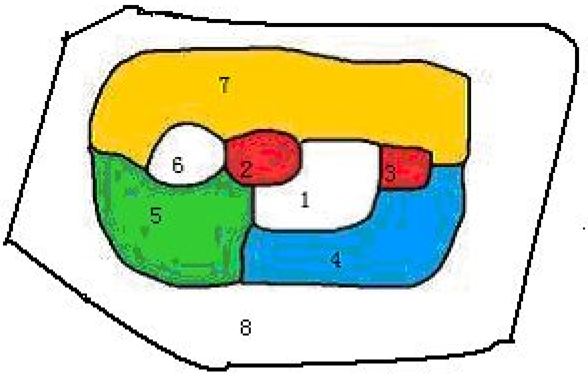
Theorem (4.3)

Any irregular countries map can transform into a circle countries map. The color solution for circle countries map can also apply to the irregular countries map

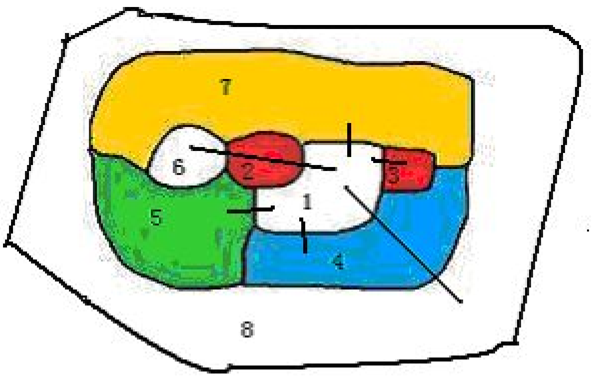
Because basic axiom (4.1) $\Rightarrow$ any irregular countries map can transform into circle-shaped, ring-shaped or fan-shaped.

If circle-shaped, ring-shaped or fan-shaped are point adjacent or non-adjacent, transform into boundary adjacent, finally, to transform into a circle map, ring-shaped and fan-shaped surround circle. Because of Theorem (4.2), the color solution of map transformed can apply to the color solution of map transforming before.

For example:



This an irregular map.



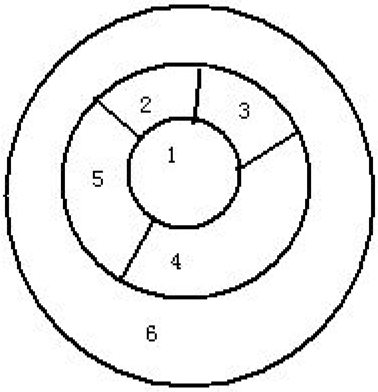
To ensure arbitrary map can be transform into circle map, first select a circle center, second draw a line from center to country, the least country number crossed over is the layer number of ring. From 1 to 6, the least country number is 2, from 1 to 8, the least country number is 2, and so both 6 and 8 are in layer 3 in circle map. Other countries are in layer 2.

Layer	**Country**
Layer 1	1
Layer 2	2,3,4,5,7
Layer 3	6,8

To ensure to preserve the adjacent relationship in transforming,

Country	**Adjacent country**
1	2,3,4,5,7
2	1,5,6,7
3	1,4,7
4	1,3,7,8,5
5	1,2,4,6,7,8
6	2,5,7
7	1,2,3,4,5,6,8
8	4,5,7

In circle map, the necessary condition of adjacent relationship is between 2 adjacent layers, or between 2 adjacent countries (left, right) in the same layer. Such as below:



Country 1 (layer 1) is adjacent with 2,3(layer 2).   Country 3 (layer 2) is adjacent with 2,4(layer 2).

If countries are not in the adjacent layer or more than 2 countries in the same layer, they are sure to be not adjacent. Such as, 6(layer 3) and 1(layer 1) are not adjacent; 2 and 4 are not adjacent in layer 2, because there are 3,4,5 in layer 2, they can't all adjacent with 1.

With the 2 necessary condition of adjacent relationship, check the table one item by one item.

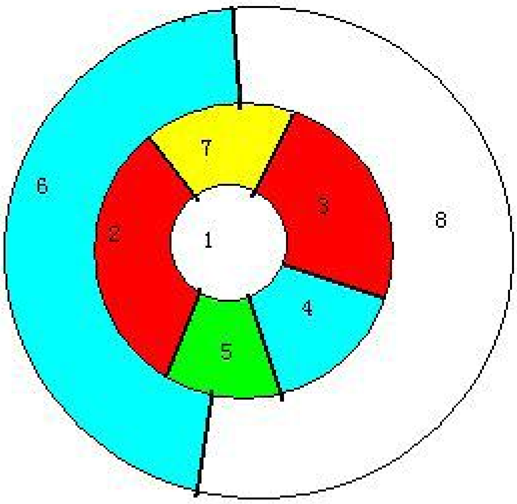
1	2,3,4,5,7
2	1,5,6,7
3	1,4,7
4	1,3,7,8,5
5	1,2,4,6,7,8
6	2,5,7
7	1,2,3,4,5,6,8
8	4,5,7

This is the method to check country 7 and others are similar.

First, remove countries in adjacent layers.

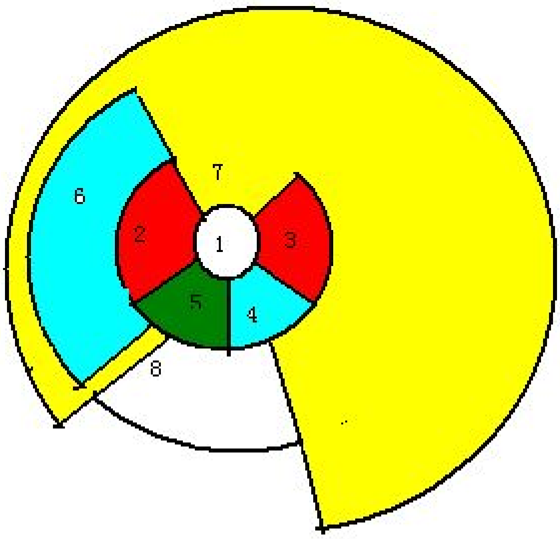
7	2,3,4,5

Because 2,3,4,5 are in the same layer and total 4 > 2, the country 7 can't preserver adjacency.

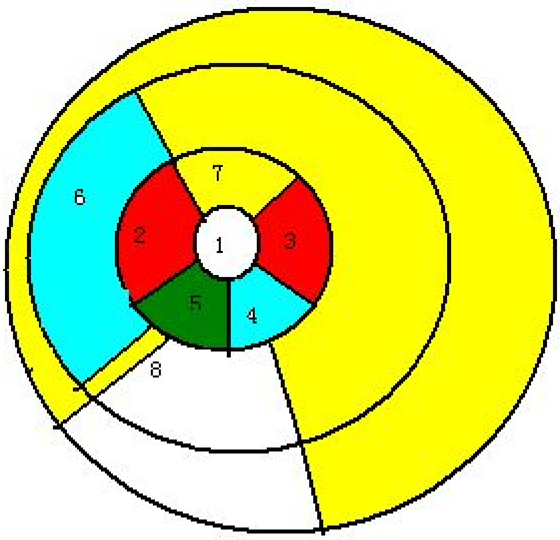


To preserver adjacency, it must to cross layers by country 4, 5 or 7.

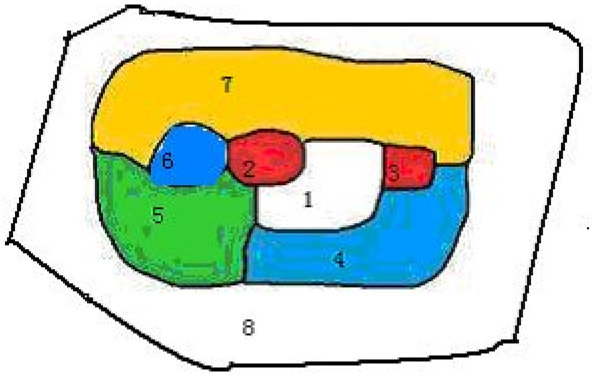
Go back to original map. Country 7 is across over country 6 to adjacent with 5, and is across over country 3 to adjacent with 4. The final map is below.



It equal to the standard circle map below.



Transform irregular map to circle map, 1 is circle center, 6 and 8 are in layer 2, 7 is across layer 2,3,4. Other countries are in layer 1. The boundary adjacent relation is never changed, but some point adjacent or non-adjacent relations are changed to boundary adjacent relation to match the circle map transforming.



The color solution for circle map transformed can apply to irregular map also. Country 6 has changed color, but there is no same color between boundary adjacent countries. It is a color solution qualified.

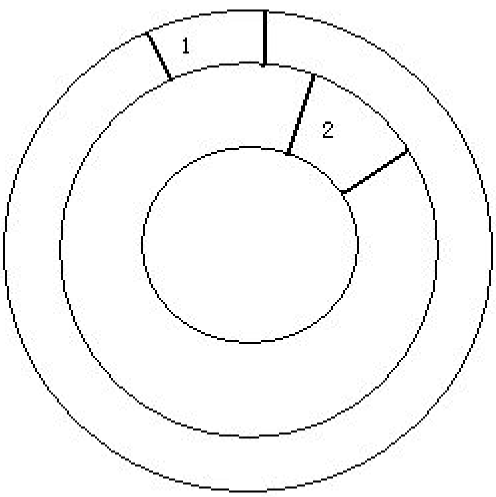
5. Terminology

To describe conveniently, I have defined some terms in circle map.

Solution(n, color1,color2,…,colork) is a color solution qualified to color all of n countries by color in *{color1,color2,…,colork}*.

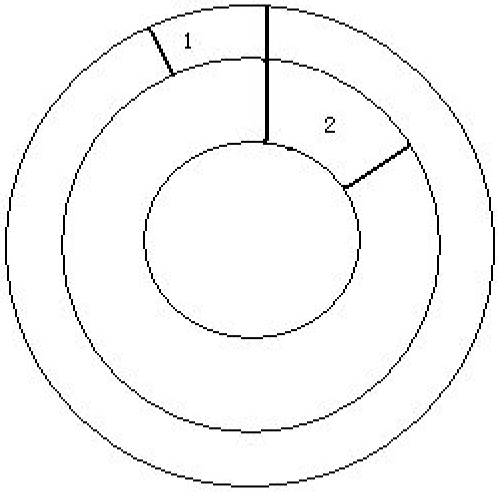
Non-adjacent regions as those no point met.

For example: 1 is non-adjacent with 2 in below circle map.



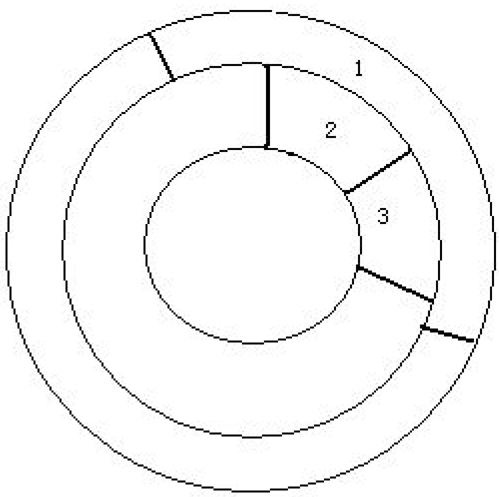
Point adjacent regions as those that meet at a single point or limited points

For example: 1 is point adjacent with 2 in below circle map.



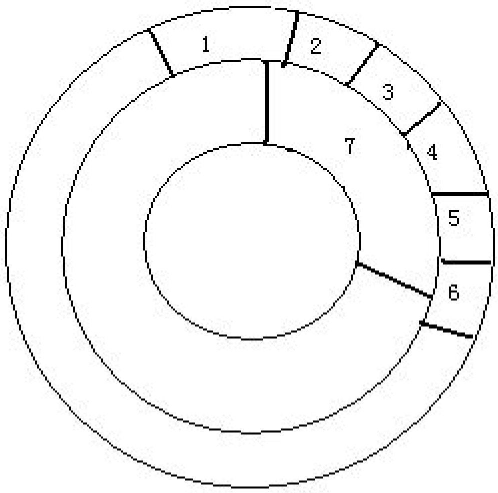
Boundary adjacent regions as those that share a common boundary of non-zero length.

For example: 1, 2, 3 are all boundary adjacent with other 2 countries in below circle map.



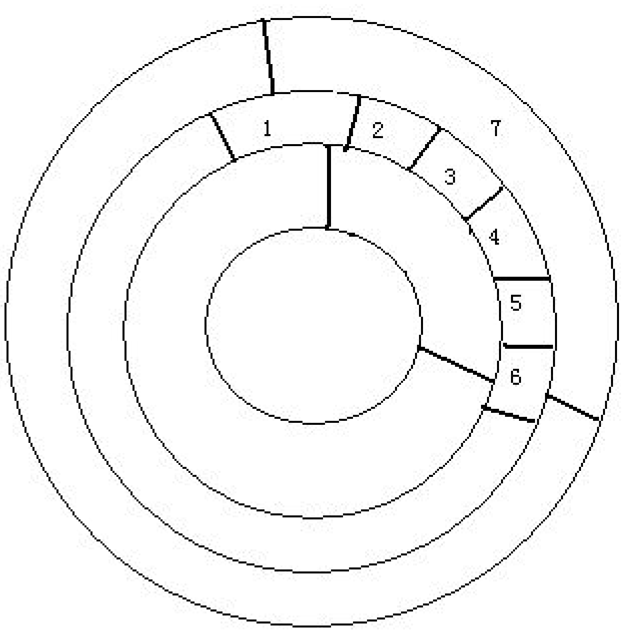
Covered is that the least countries in upper ring have covered one nation in lower ring. Especially, the least N countries in upper ring covering 1 nation in lower ring calls *N Covered*, all the countries in upper ring covering 1 nation in lower ring calls *full Covered.*

For example: 1, 2, 3, 4, 5, 6 are covering 7 in below circle map, that is 6 covering.



Supported is that the least countries in lower ring have covered one nation in upper ring. Especially, the least N countries in lower ring covering 1 nation in upper ring calls *N Supported*, all the countries in lower ring covering 1 nation in upper ring calls *full Supported.*

For example: 1, 2, 3, 4, 5, 6 are supporting 7 in below circle map, that is 6 supporting.

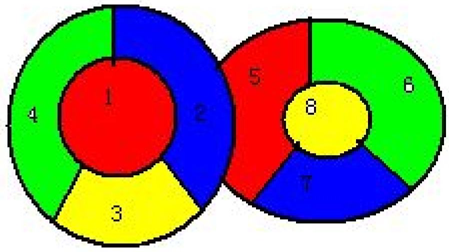


Color is to color country by one or more than one colors. It is recoded as *Color(country) = { color}*. If a country can be colored by more than 1 colors, it can be recoded as *Color(country) = {color1/color2…/colork}*. Such as *Color(3) = {yellow/green/gray}*. Country 3 is colored by *{yellow}* now, but Country 3 has the freedom to color by *{ gree}* or *{gray}*.

Main color is *{color1}* in *Color(country) = {color1/color2…/colork}*, which color the country in fact.

Backup color is *{color2…/colork}* in *Color(country) = {color1/color2…/colork}*, which doesn't color the country in fact, but which has the freedom to color by *{color2…/colork}.*

Dependent color is a country can be colored by more than 1 color, and depends on another multiple color country. It is recoded as *Color(country) = {color1/(dependent country)}*. For example:



Country 5 can be colored by *{red/green/yellow}*. Country 6,7,8 depend on the color of country 5. It is record as *Color(5)= {red/green/yellow},* *Color(6)= {green/(5)}* , *Color(7)= {blue/(5)}* , *Color(8)= {yellow/(5)}* .

Major color is to color countries in a ring with 2 alternate colors. But if there is odd number of countries in a ring, the head and tail are the same color.

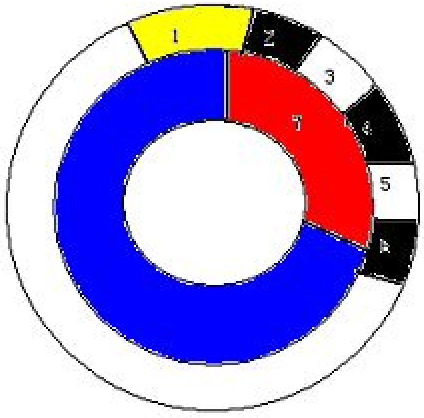
Isolating color is to isolate the same major color with 1 another color in a ring, which is of odd number of countries.

For example: We can see the example in below circle map.

Major color of ring 1 is white color and no isolate color, record as *Major(1) = {white}, Isolating(1) = {};*

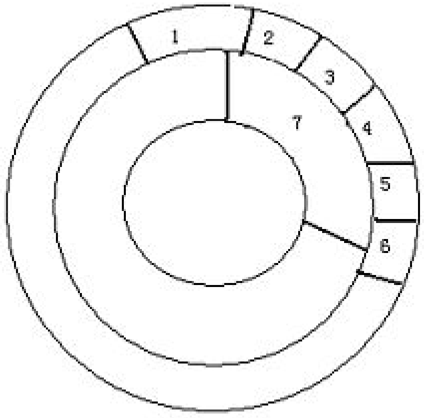
Major color of ring 2 is red and blue color and no isolate color, record as *Major(1) = {read, blue}, Isolating(1) = {};*

Major color of ring 3 is black and white color and isolate color is yellow color, record as *Major(3) = {white, black}, Isolating(3) = {yellow}.*



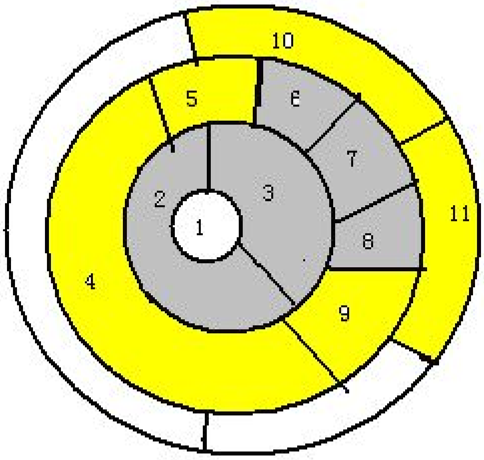
Country number is the total country number of a ring.

For example: country number of ring 3 is 7 in below circle map. Record as *Country(3) = 7*.



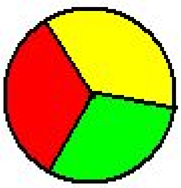
Border countries are all the boundary countries between colored and uncolored. It’s the frontier of countries colored.

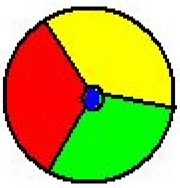
Home countries are all the boundary countries closed by *Border countries*. It’s the home of countries colored. For example:



Countries 1 to 11 are colored, the *Border countries* are marked as yellow color, which are close border to seal all countries colored. *Home countries* are marked as gray color.

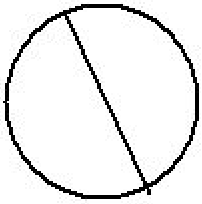
Empty country is a point, which is not a real country, only a proving tool. For example, 3 countries is equivalent to 3 countries and a *Empty country.*



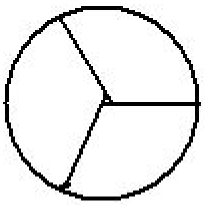


Kn countries are k countries are all adjacent. Anyone of k country is adjacent with other k-1 countries.

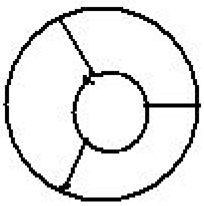
For example: *k2 countries* are in below circle map.



k3 countries are in below circle map. Anyone country is adjacent with other two countries.



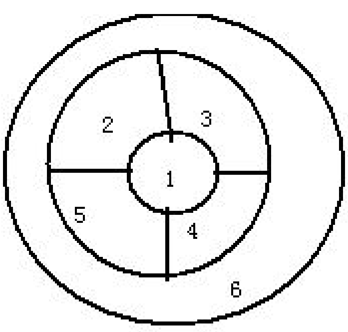
K4 countries are in below circle map. Anyone country is adjacent with other three countries.



Graph theory has proven *K4 countries* are the max adjacent relationship in planar graph.

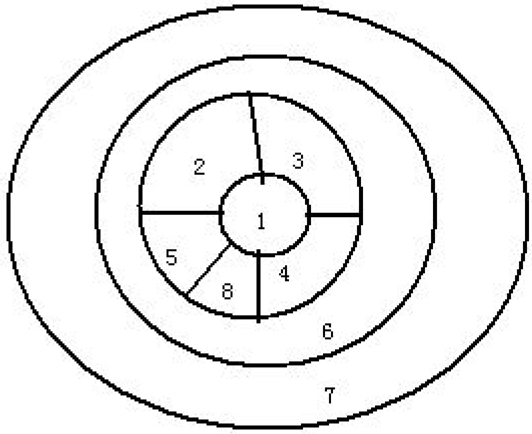
NKn countries are k countries are all non-adjacent. Anyone of k country is non-adjacent with other k-1 countries.

For example: *Nk2 countries* are in below.



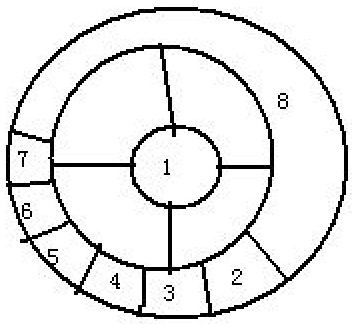
Country 1 and 6, 2 and 4, 3 and 5 are all *Nk2 countries*

Nk3 countries are in below.



Country (2,4 ,7) or (3,8,7) are all *Nk3 countries*

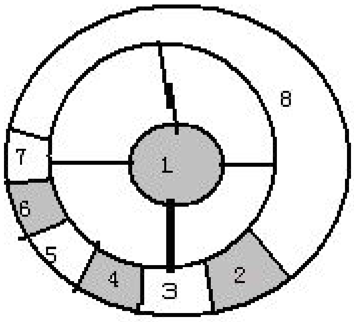
Nk3 countries are in below.



Country (1,2,4,6) are *Nk4 countries*

Any *NKx countries* can divide into (x-1) *NK2 countries chains.*

For example: *Nk4 countries* (1,2,4,6) = (1,2) + (2,4) + (4,6) = 3 *NK2 countries*



6. Preliminary theorem

(6.1) *K4 countries* have only 3 scenarios in circle map.

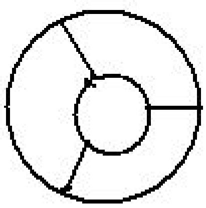
Because in a ring, one country can only adjacent with 2 countries at most => there are at most 3 countries in a ring, but *K4 countries* have 4 countries > 3 => *K4 countries* are at least in 2 rings.

If total of rings >=3, there must be one ring insulating another ring. =>there must be 2 countries being non-adjacent.=> total of rings <=2,

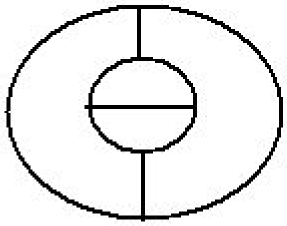
Because total of rings >=2 and total of rings < 2 => total of rings =2.

Total of rings =2 and *K4 countries* have 4 countries => *K4 countries* have only 3 scenarios in circle map. I.e.

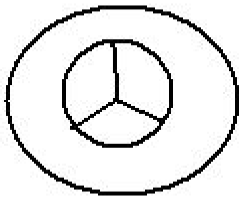
(6.1.1) *country(1) = 1, country (2) = 3.*



(6.1.2) *country(1) = 2, country (2) = 2.*



(6.1.3) *country(1) = 3, country (2) = 1*.

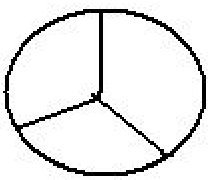


Is there any country across rings? No. Because there is only 2 adjacent rings, countries can keep adjacent relationship in adjacent rings, don’t’ need to cross rings.

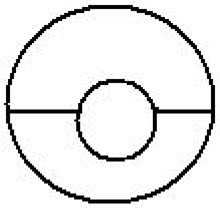
(6.2) *K3 countries* have only 3 scenarios in circle map.

Similarly, we can get 3 scenarios.

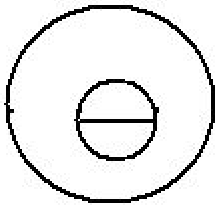
(6.2.1) *country (1) = 3*.



(6.2.2) *country(1) = 1, country (2) = 2*.



(6.2.3) *country(1) = 2, country (2) = 1.*



To prove conveniently, we can unify (6.1) and (6.2). For *K3 countries,* we regard there is a *empty country* in *home countries*. Then *K3 countries* become *K4 countries*

(6.2.1) *country (1) = 1 empty country , country (2) = 3.*

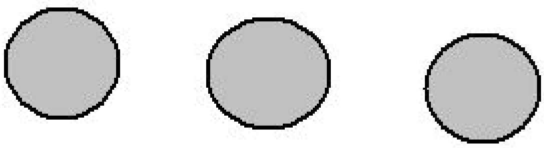
(6.2.2) *country (1) = 1+ 1 empty country, country (2) = 2.*

(6.2.3) *country (1) = 2+ 1 empty country, country (2) = 1.*

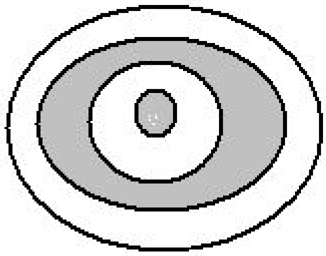
7. Four color theorem

There are N countries. Firstly, search max adjacent relationship.

If no *K2 countries,* all countries are *non-adjacent,* one color is sufficient.



Max adjacent relationship is *K2 countries,* all countries are at most adjacent with one country, and two colors are sufficient.



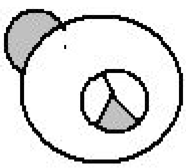
Max adjacent relationship is *K3 countries,* we can add a *empty country* in ring 1, it becomes *K4 countries. border countries* are formed.

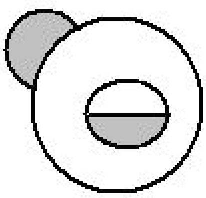
Max adjacent relationship is *K4 countries,* because *K4 countries* have two rings, *border countries* are formed now.

Next we select the 5th adjacent country to color. There are 3 scenarios:

Scenario 1. There is only 1 country in *border countries*

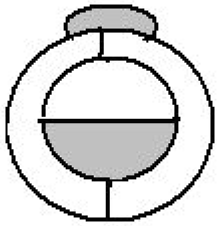
We can always find one non-adjacent countries in *border countries,* because there are 3 countries in home of *K4 countries,* and there are 2 countries in home of *K3 countries* .

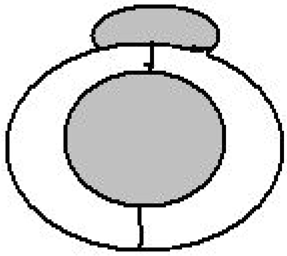




Scenario 2. There are 2 countries in *border countries*

We can always find one non-adjacent country in *border countries* or *home countries,* because there are 2 countries in home of *K4 countries* ,and there is 1 country in home of *K3 countries.*

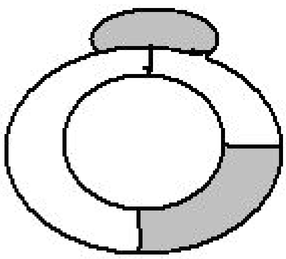


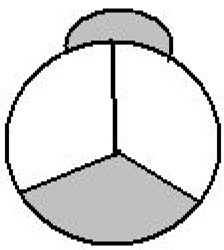


Scenario 3. There are 3 countries in *border countries.*

Scenario 3.1. the 5th country is adjacent with less than 3 countries in *border countries.*

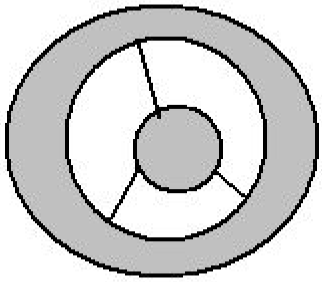
We can always find one non-adjacent country in *home countries.*

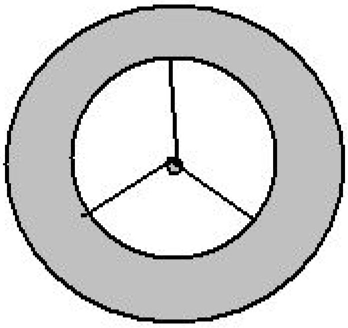




Scenario 3.2. the 5th country is adjacent with 3 countries in *border countries.*

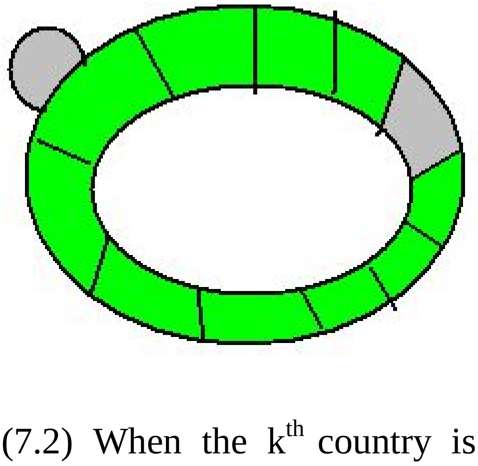
We can always find one non-adjacent country in *home countries.* For *K3* *countries,* we can find the *empty countries.*



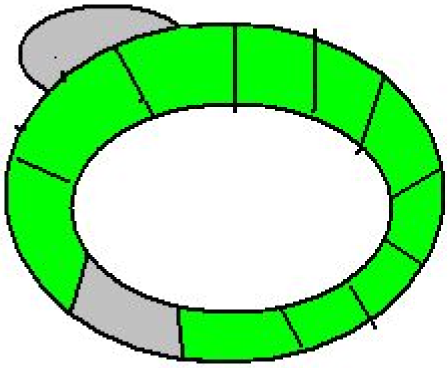


The *border countries* become longer and longer. When *border countries* are more than 3 countries, there are 3 scenarios also. Assume to color the k^{th} country and *border countries* are above three.

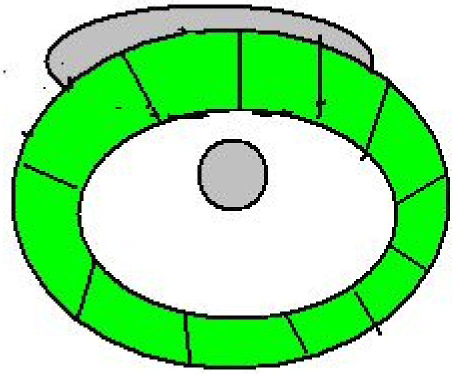
(7.1) When the k^{th} country is adjacent with one country in *border countries.* We can always find at least two non-adjacent countries in *border countries.*



(7.2) When the k^{th} country is adjacent with two countries in *border* *countries.* We can always find at least one non-adjacent country in *border countries.*



(7.3) When the k^th^ country is adjacent with 3 or more than 3 countries in *border countries.* We can always find at least one non-adjacent country in *home countries.*

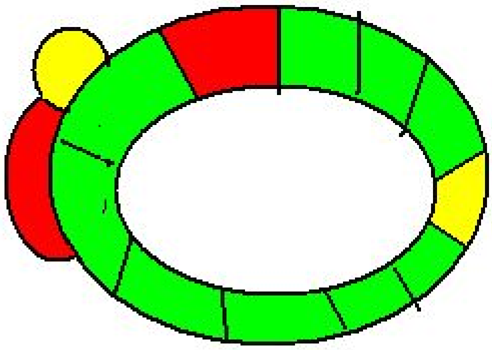


Above proof indicates every country can find *non-adjacent country*, every country is in *NK2 countries* except countries in ring 1 and ring 2.

Next to prove all the *NK2 countries* are not overlapped.

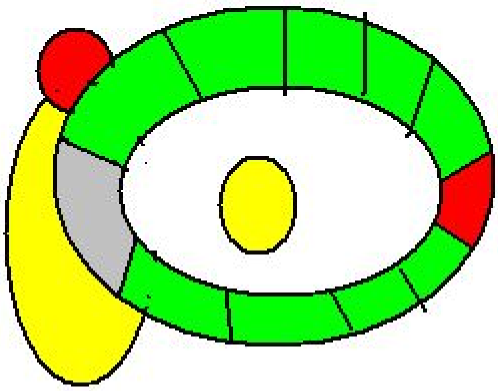
Assume to color the $(k+1)^{th}$ country

In scenario (7.1) (7.2), the $(k+1)^{th}$ country is adjacent with less than 3 countries of *border countries and* not adjacent with all *border countries.* We can always find another non-adjacent country in *border countries,* because *border countries* are above three. The 2 *NK2 countries* are not overlapped.



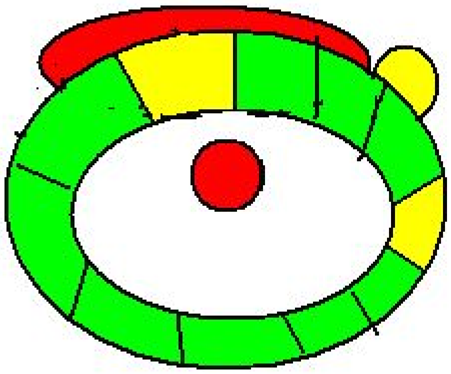
In scenario (7.1) (7.2), the (k+1)th country is adjacent with 3 or more than 3 of *border countries* or is adjacent with all *border countries*

We can always find a non-adjacent country in *home countries.*



How to ensure the *home countries* are sufficient? Because the (k+1)th country is adjacent with 3 or more than 3 of *border countries* or is adjacent with all *border countries,* it must *full cover* at least one country(gray color country). It means that one country is from *border countries* to *home countries.* Though we have consumed one home country, we add at least one home country also. The *home countries* do not decrease.

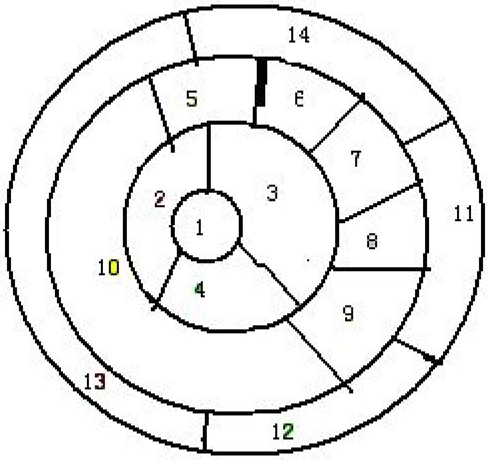
In scenario (7.3), the $(k+1)^{th}$ country is always non-adjacent with the countries which are *full covered* by the k^{th} country or in *border countries*.



All countries are in *NK2 countries(*except ring 1 and ring 2 countries*)*, these *NK2 countries* are connected with header and tail, and become *NK2 countries chains*. The header of every *NK2 countries chain* is in the first *K4 countries.* Ring 1 and ring 2 countries are perhaps in *NK2 countries chain*, perhaps not. There are at most 4 *NK2 countries chains.* Every *NK2 countries* chain is colored by one color. Every *NK2 countries chain* is at most adjacent with 3 others *NK2 countries chains.*

(7.4) Any map can divide into one complete graph *Kn countries*(n<=4) and several (<=4) *NK2 countries chains.*

For example:



First to search *K4 countries,* (1,2,3,4).

Next to find *NKn countries (NK2 countries chains)* .

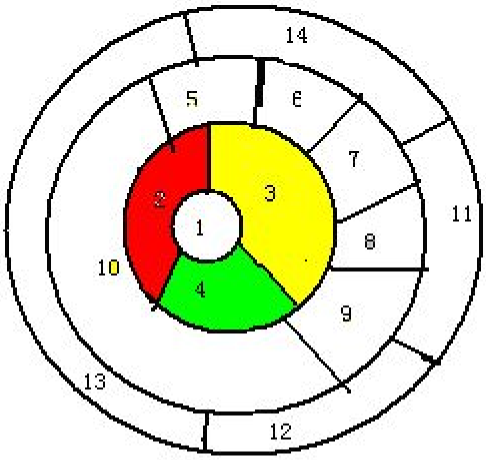
NK4 countries (4,5,7,12) = (4,5)+(5,7)+(7,12);

NK3 countries (1,9,14) = (1,9)+(9,14);

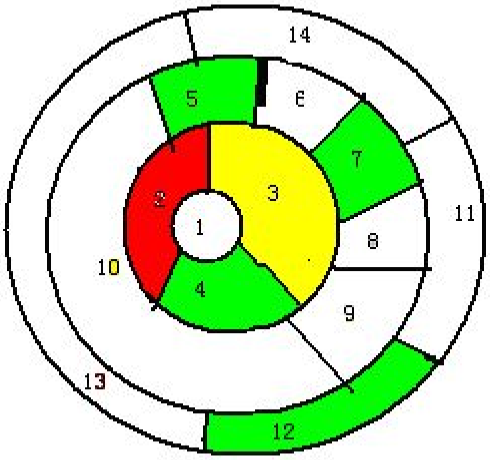
NK4 countries (2,6,8,13) = (2,6)+(6,8)+(8,13);

NK3 countries (3,10,11) = (3,10)+(3,10)+(10,11);

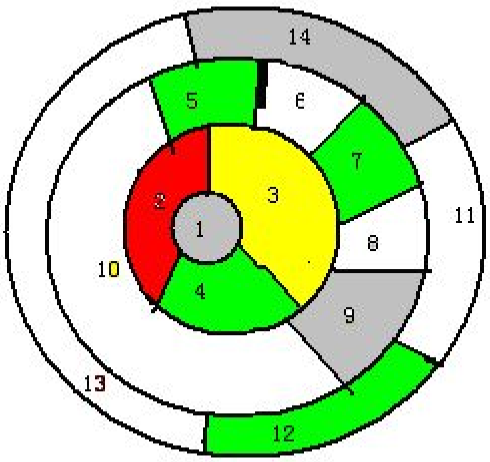
Next to color *K4 countries* by 4 colors.



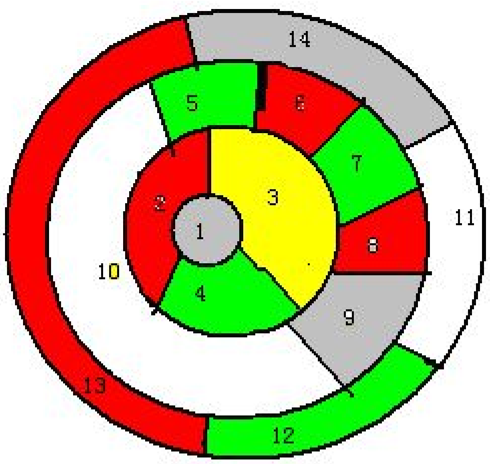
Color *NK4 countries* (4,5,7,12) = (4,5)+(5,7)+(7,12) by 1 color;



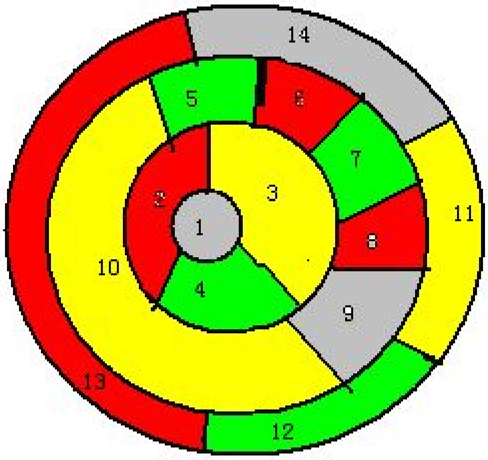
Color *NK3 countries* (1,9,14) = (1,9)+(9,14) by 1 color;



Color *NK4 countries* (2,6,8,13) = (2,6)+(6,8)+(8,13) by 1 color;



Color *NK3 countries* (3,10,11) = (3,10)+(3,10)+(10,11) by 1 color;

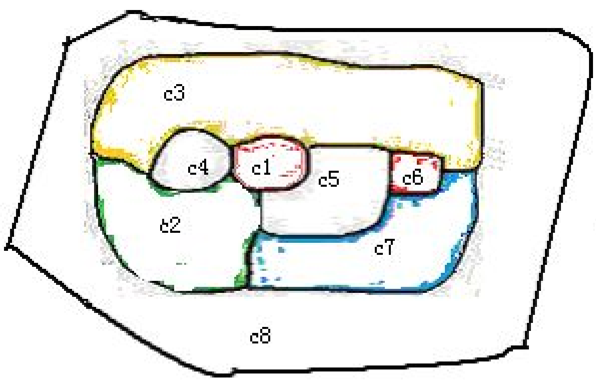


Finally, four color theorem is proven now!

8. Verification and Demo

One example to verify and explain:

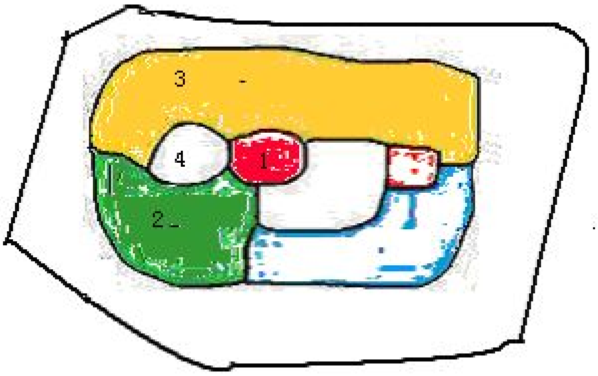
To descript clearly, all countries are marked the number by the color order in advance. The order number (country number) map is below,



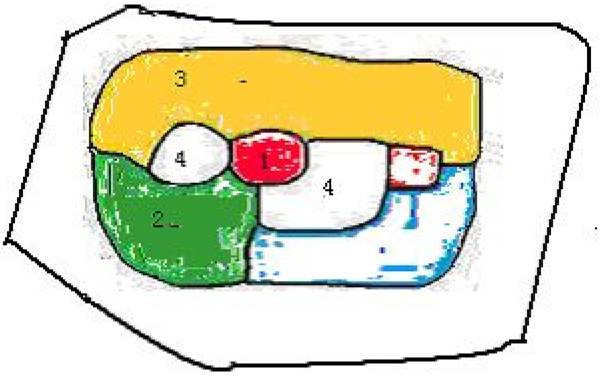
The algorithm bases on (7.4), is simpler than the example in section 7.

1. Search and color max adjacent relationship of complete graph.
2. Find *NK2 countries chains* and colored with more possibilities.

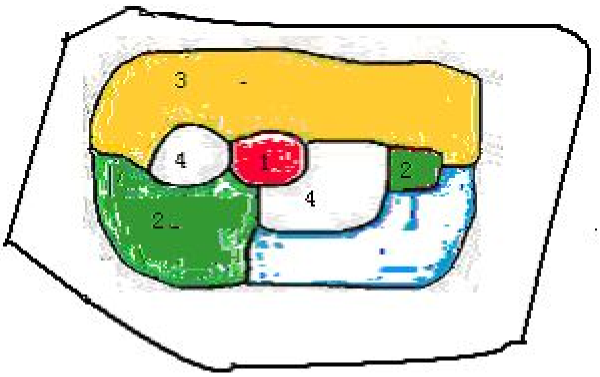
Firstly, search *K4 countries* and colored by {1,2,3,4}



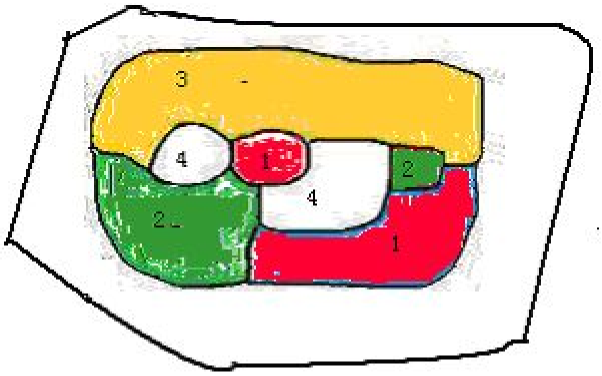
Next, we can select an adjacent country c5, which is adjacent with 3 countries in border. So it is colored by color of home country {4}



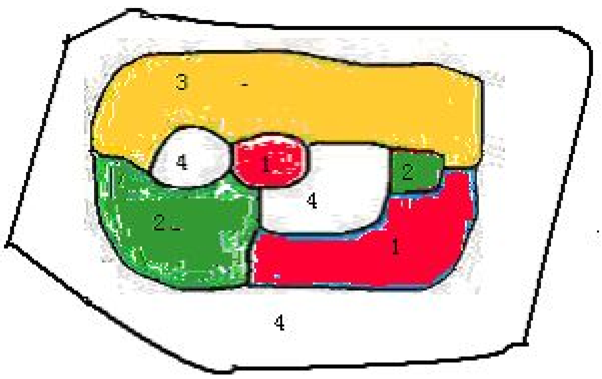
Next, we can select an adjacent country c6, which is adjacent with 2 countries in border. So it's non-adjacent with a country in border and home, can be colored by {2/4}. But color c4 is adjacent with it, so color is {2}.



Next, we can select an adjacent country c7, which is adjacent with 3 countries in border. It can find a non-adjacent country in *home countries*, and color is {1/4}. Because the red country c1 is *full coved*, which become *home country* from *border country* and color c5 is adjacent with it, so color is {1}.



Next, we can select an adjacent country c8, which is adjacent with all border countries. It can find a non-adjacent country in *home countries*. It can be colored by {1/2/4}, and exclude adjacent country c2, c7. Final color is {4}.



The *NK2 countries chains* are in the below table.

First *K4 countries*	***NK2 countries chains***			
C1	C7			
C2	C6			
C3				
C4	C5	C8		

All countries are in the *NK2 countries chains* except first *K4 countries*, which either are the header of *NK2 countries chains*, or are not in the *NK2 countries chains*, e.g. c3.

Every *NK2 countries chain* is at most adjacent with other 3 *NK2 countries chains.* So four colors are sufficient to color any planar or spherical map.

9. Conclusion

Four-color theorem is an interesting phenomenon, but there is a rule hidden the phenomenon. The max adjacent relationship on a surface decides how many colors are sufficient. More than max adjacent countries, almost every country is in a non-adjacent chain. Every non-adjacent chain can decrease color consumption.

Any planar or spherical map is comprised of one max adjacent relationship of complete graph and several non-adjacent relationship chains whose header is in the complete graph.

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