International Journal of Mathematical Archive-3(7), 2012, 2520-2523
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ON THE POSSIBILITY OF N-TOPOLOGICAL SPACES

Kamran Alam Khan*
Deptt. of Mathematics, Hemwati Nandan Bahuguna Govt. P. G. College, Naini, Allahabad (U.P.)-India

(Received on: 23-06-12; Accepted on: 10-07-12)

ABSTRACT

The notion of a bitopological space as a triple $(X, \mathfrak{I}_1, \mathfrak{I}_2)$, where X is a set and $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are topologies on X, was first formulated by J.C.Kelly [5]. In this paper our aim is to introduce and study the notion of an N-topological space $(X, \mathfrak{I}_1, \mathfrak{I}_2, \dots \dots \dots \mathfrak{I}_N)$. We first generalize the notion of an ordinary metric to n variables. This metric will be called K-metric. Then the notion of a quasi-pseudo-K-metric will be introduced. We then follow the approach of Kelly to introduce and study the notion of an N-topological space. An example for such a space is produced using chain topology. And finally we define and study some of the possible separation properties for N-topological spaces.

Keywords: *Bitopological Space, Quasi-Pseudo metric, Generalized Metric Space, N-topological Space.*

AMS Subject Classification: *54E55, 54A10, 54E35, 54D10.*

1. INTRODUCTION

Let (X, d) be a metric space. The open d-spheres form a base for a topology $\mathfrak{I}$ for X. When we omit the condition of symmetry from the definition of a metric, we call it a quasi-metric. Such metrics have been studied by Wilson [11], Ribeiro [9] and others. J.C. Kelly [5] noticed the fact that if we study the two topologies $\mathfrak{I}_1$ and $\mathfrak{I}_2$ determined for X by quasi-metric p and its conjugate q, some of the symmetry of the classical metric situation is regained and we can obtain systematic generalizations of standard results such as Urysohn's lemma, Urysohn's metrization theorem, Tietze's extension theorem and the Baire category theorem. He then generalized this structure by defining a bitopological space, i.e. a space with two arbitrary topologies.

1.1 Definition ([5]): A space X on which are defined two (arbitrary) topologies $\mathfrak{I}_1$ and $\mathfrak{I}_2$ is called a bitopological space and denoted by$(X, \mathfrak{I}_1, \mathfrak{I}_2)$.

If we follow this approach to define an N-topological space, i.e. a space with N arbitrary topologies, we need to generalize the concept of a metric (quasi-metric) to n variables such that $N \leq n!$.

K. Menger [6] introduced the notion of a generalized metric. He discussed the logical generalization of metric spaces; *viz.* n-metric spaces. But a new development began with the work of Gähler ([2], [3]) who introduced the concept of 2-metric space. Gähler claimed that 2-metric function is a generalization of an ordinary metric function, but different authors proved that there is no relation between these two functions. Ha et al. [4] showed that a 2-metric is not a continuous function of its variables, whereas an ordinary metric is. It was mentioned by Gähler [2] that the notion of a 2-metric is an extension of an idea of ordinary metric and geometrically $d(x, y, z)$ represents the area of a triangle formed by the points x, y and z in X as its vertices. But this is not always true. Sharma [10] showed that $d(x, y, z) = 0$ for any three distinct points $x, y, z \in \mathbb{R}^2$.

All these considerations and natural generalization of an ordinary metric led B. C. Dhage to introduce a new structure of a generalized metric space called D-metric space in his Ph.D. thesis [1].

In a subsequent series of papers, Dhage attempted to develop topological structures in such spaces. He claimed that D-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results.

In 2003, Mustafa and Sims [7] demonstrated that most of the claims concerning the fundamental topological properties of D-metrics are incorrect. For instance, a D-metric need not be a continuous function of its variables and despite Dhage's attempts to construct, such a topology, D-convergence of a sequence $\{x_n\}$ to x, in the sense that $D(x_m, x_n, x) \to 0$ as $n, m \to \infty$, need not correspond to convergence in any topology. These considerations lead

Corresponding author: *Kamran Alam Khan**
Deptt. of Mathematics, Hemwati Nandan Bahuguna Govt. P. G. College, Naini, Allahabad (U.P.)-India

Mustafa and Sims [8] to seek a more appropriate notion of generalized metric space. They introduced the notion of G-metric space [11], in which the tetrahedral inequality is replaced by an inequality involving repetition of indices.

Now we propose a generalization of G-metric space with G-metric as a function of n variables.

This metric will be called as *K-metric*.

2. MAIN RESULTS

2.1 Definition: Let X be a non-empty set, and $\mathbb{R}^+$ denote the set of non-negative real numbers. Let $K: X^n \to \mathbb{R}^+$ be a function satisfying the following properties:

[K 1]: $K(x_1, x_2, \dots \dots . x_n) = 0$ if $x_1 = x_2 = \cdots = x_n$
[K 2]: $K(x_1, x_1, \dots \dots . x_1, x_2) > 0$ for all $x_1, x_2 \in X$ with $x_1 \neq x_2$
[K 3]: $K(x_1, x_1, \dots \dots . x_1, x_2) \leq K(x_1, x_2, \dots \dots . x_n)$ for all $x_1, x_2, \dots \dots, x_n \in X$ with the condition that any two of the points $x_2, \dots \dots, x_n$ are distinct.
[K 4]: $K(x_1, x_2, \dots \dots . x_n) = K\left(x_{\pi^r(1)}, x_{\pi^r(2)}, \dots \dots . x_{\pi^r(n)}\right)$ for all $x_1, x_2, \dots \dots, x_n \in X$ and every permutation π of $\{1,2, \dots ., n\}$ such that $\pi(s) = s + 1$ for all $1 \leq s < n$, $\pi(n) = 1$ and for all $r \in \mathbb{N}$
[K 5]: $K(x_1, x_2, \dots \dots ., x_n) \leq K(x_1, x_{n+1}, \dots \dots . x_{n+1}) + K(x_{n+1}, x_2, \dots \dots ., x_n)$ for all $x_1, x_2, \dots \dots ., x_n, x_{n+1} \in X$ then the function K is called a *K-metric* on X, and the pair (X, K) a *K-metric space.*

Example 2.1: Let $\mathbb{R}$ denote the set of all real numbers. Define a function $\rho: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots \dots \times \mathbb{R} \to \mathbb{R}$ by $\rho(x_1, \dots, x_n) = max\{|x_1 - x_2|, \dots \dots |x_{n-1} - x_n|, |x_n - x_1|\}$

for all $x_1, x_2, \dots, x_n \in X$. Then $(\mathbb{R}, \rho)$ is a K-metric space.

Example 2.2: For any metric space (X, d), the following metrics define K-metrics on X:

(1) $K_s(d)(x_1, x_2, \dots, x_n) = \frac{1}{n}\left[\sum_{r=1}^{n-1} d\left(x_r, x_{r+1}\right) + d\left(x_n, x_1\right)\right]$
(2) $K_m(d)(x_1, \dots, x_n) = max\left\{d\left(x_1, x_2\right), \dots \dots d\left(x_{n-1}, x_n\right), d\left(x_n, x_1\right)\right\}$

2.2 Definition: Let (X, K) be a K-metric space then for $x_0 \in X, r > 0$, the *K-ball* with centre x_0 and radius r is

$$B_K(x_0, r) = \{y \in X: K(x_0, y, y, \dots ., y) < r\}$$

Proposition 2.1 Let (X, K) be a K-metric space, then for $x_0 \in X$, and $r > 0$ the K-ball $B_K(x_0, r)$ is open in X.

Proof: Using the definition 2.1 it is easy to show that to each point $y \in B_K(x_0, r)$ there exists a K-ball centered at y and contained in $B_K(x_0, r)$. It follows that the K-ball $B_K(x_0, r)$ is open in X.
Hence the collection $\{B_K(x, r): x \in X\}$ of all such balls in X is closed under arbitrary union and finite intersection. Hence the collection $\{B_K(x, r): x \in X\}$ of all open balls induces a topology on X called the K-metric topology generated by the K-metric on X.

2.3 Definition: Let X be a non-empty set, and $\mathbb{R}^+$ denote the set of non-negative real numbers. Let $\rho: X^n \to \mathbb{R}^+$ be a function satisfying the following properties:

[P 1]: $\rho(x_1, x_2, \dots \dots . x_n) = 0$ if $x_1 = x_2 = \cdots = x_n$
[P 5]: $\rho(x_1, x_2, \dots \dots ., x_n) \leq \rho(x_1, x_{n+1}, \dots \dots . x_{n+1}) + \rho(x_{n+1}, x_2, \dots \dots ., x_n)$ for all $x_1, x_2, \dots \dots ., x_n, x_{n+1} \in X$ then ρ is called a *quasi-pseudo-K-metric* on the non-empty set X.

2.4 Definition: Let ρ be a quasi-pseudo-K-metric on a non-empty set X, and let ρ_k be defined by

$$\rho_k(x_1, x_2, \dots \dots . x_n) = \rho\left(x_{\pi_k(1)}, x_{\pi_k(2)}, \dots \dots . x_{\pi_k(n)}\right)$$

Where π_k is some permutation of $\{1,2, \dots ., n\}$.

Then it is trivial matter to verify that ρ_k is a quasi-pseudo-K-metric on X.

If ρ_k is a quasi-pseudo-K-metric on a set X, then $\{y \in X: \rho_k(x, y, y, \dots ., y) < r\}$, the open ρ_k-ball with centre x and radius $r > 0$, is denoted by $B_{\rho_k}(x, r)$. Just as in the classical case, the collection of all open ρ_k-balls forms a base for a topology. The topology determined in this way by ρ_k will be denoted by $\Im_k$ and will be called the quasi-pseudo-K-metric topology of ρ_k.

Let $\rho_1, \rho_2, \dots \dots . . \rho_N$ be the quasi-pseudo-K-metrics defined on any non-empty set X and $\Im_1, \Im_2, \dots \dots \dots \Im_N$ be the corresponding topologies induced by $\rho_1, \rho_2, \dots \dots . . \rho_N$ respectively. Then the natural topological structure associated these quasi-pseudo-K-metrics is that of the set X with N topologies.

2.5 Definition: A non-empty set X equipped with N arbitrary topologies $\Im_1, \Im_2, \dots \dots \Im_N$ is called an *N-topological space* and denoted by $(X, \Im_1, \Im_2, \dots \dots \dots \Im_N)$.

Now we give an interesting example of an N-topological space:

Let $X = \{a_1, a_2, \dots \dots . a_n\}$ be any finite set and $\Im_F^{a_r} (a_r \in X)$ be a collection $\{A_{\alpha_k}^{a_r} : \alpha_k \in \Lambda\}$ of subsets of X including the void set ϕ and the set X such that

$$\phi = A_{\alpha_0}^{a_r} \subset A_{\alpha_1}^{a_r} \subset A_{\alpha_2}^{a_r} \dots \dots . . \subset A_{\alpha_n}^{a_r} = X$$

Where $a_r \in A_{\alpha_k}^{a_r}$ and $\left|A_{\alpha_k}^{a_r}\right| = \text{k}$ for $1 \le k \le n$.

Then obviously $\Im_F^{a_r}$ will be a chain topology on X. We shall call it special finite generalized Sierpiński topology on X with nucleus $A_{\alpha_1}^{a_r} (= \{a_r\})$.For a given singleton nucleus, a total of $(n-1)!$ special finite generalized Sierpiński topologies can be defined on any n-element finite set. Since X contains 'n' elements, hence total number of special finite generalized Sierpiński topologies which can be defined on X will be equal to $n \times (n-1)!$, *i.e.* $n!$.

Let $\Im_1, \Im_2, \dots \dots \dots \Im_N (2 \le N \le n!)$ be the special finite generalized Sierpiński topologies defined on X then obviously $(X, \Im_1, \Im_2, \dots \dots \dots \Im_N)$ will be an N-topological space. These types of special N-topological spaces will be called as N-topological finite generalized Sierpiński spaces.

Now we shall define and study the separation properties of N-topological spaces.

2.6 Definition: An N-topological space $(X, \Im_1, \Im_2, \dots \dots \dots \Im_N)$ is said to be *N-wise* T_0 if for every pair of distinct points of X there exists a $\Im_i$-open set ($i \in \{1,2, \dots, N\}$) containing one of the points and not the other.

2.7 Definition: An N-topological space $(X, \Im_1, \Im_2, \dots \dots \dots \Im_N)$ is said to be *N-wise* T_1 if for every pair of distinct points $x, y \in X$ there exist $U_i \in \Im_i, U_j \in \Im_j$ for some $i, j \in \{1,2, \dots, N\}, i \ne j$, such that $x \in U_i$, $y \notin U_i$ and $y \in U_j$, $x \notin U_j$.

2.8 Definition: An N-topological space $(X, \Im_1, \Im_2, \dots \dots \dots \Im_N)$ is said to be *N-wise* T_2 if for every pair of distinct points $x, y \in X$ there exist $U_i \in \Im_i, U_j \in \Im_j$ for some $i, j \in \{1,2, \dots, N\}$, $i \ne j$, such that

$$x \in U_i , y \in U_j, U_i \cap U_j = \phi$$

Example: 2.3: An N-topological special finite generalized Sierpiński space is N-wiseT_0.

Example: 2.4: Let $X = \{a_1, a_2, \dots \dots . a_n\}$ and $\Im_1^{a_1}, \Im_2^{a_2}, \dots . \Im_n^{a_n}$ be the special finite generalized Sierpiński topologies with nuclei $a_1, a_2, \dots \dots . a_n$ respectively. Then the n-topological special finite generalized Sierpiński space $\left(X, \Im_1^{a_1}, \Im_2^{a_2}, \dots . \Im_n^{a_n}\right)$ is an n-wise T_2 space.

Proposition 2.2: An N-topological space $(X, \Im_1, \Im_2, \dots \dots \dots \Im_N)$ is *N-wise* T_0 if any one of the topological spaces $(X, \Im_i), (i \in \{1,2, \dots, N\})$ is T_0.

Proof: Suppose there exists $i \in \{1,2, \dots, N\}$ such that $(X, \Im_i)$ isT_0. Let x, y be two distinct points of X. Then there is a $\Im_i$-open set containing one of the points and not the other. Thus $(X, \Im_1, \Im_2, \dots \dots \dots \Im_N)$ is *N-wise* T_0.

Proposition 2.3: An N-topological space $(X, \Im_1, \Im_2, \dots \dots \dots \Im_N)$ is *N-wise* T_0 if and only if given two distinct points of $x, y \in X$ there exists $i \in \{1,2, \dots, N\}$ such that $\Im_i - Cl\{x\}$ and $\Im_i - Cl\{y\}$ are distinct.

Proof: Let x, y be two distinct points of X. Suppose there exists $i \in \{1,2, \dots, N\}$ such that $\Im_i - Cl\{x\} \ne \Im_i - Cl\{y\}$. Hence there is a point z of Xsuch that $z \in \Im_i - Cl\{y\}$ and $z \notin \Im_i - Cl\{x\}$. If $y \in \Im_i - Cl\{x\}$ then $\Im_i - Cl\{y\} \subset \Im_i - Cl\{x\}$, so that $z \in \Im_i - Cl\{y\} \subset \Im_i - Cl\{x\}$. Which contradicts the fact that $z \notin \Im_i - Cl\{x\}$.

Hence we have $y \notin \Im_i - Cl\{x\}$. Thus $U = X - [\Im_i - Cl\{x\}]$ is a $\Im_i$-open set containing y and not containing x so that there is a $\Im_i$-open set containing one of the points and not the other. Thus $(X, \Im_1, \Im_2, \ldots \ldots \Im_N)$ is *N-wise* T_0.

Conversely, let $(X, \Im_1, \Im_2, \ldots \ldots \ldots \Im_N)$ be *N-wise* T_0. Let x, y be two distinct points of X. Then there exists $i \in \{1,2, \ldots, N\}$ such that there is a $\Im_i$-open set U containing x and not y. Then $X - U$ is a $\Im_i$-closed set containing y but not x. Therefore, $y \in \Im_i - Cl\{y\} \subset X - U$ and so $x \notin \Im_i - Cl\{y\}$, since $x \notin X - U$. Hence $\Im_i - Cl\{x\} \neq \Im_i - Cl\{y\}$.

2.9 Definition: An N-topological space $(X, \Im_1, \Im_2, \ldots \ldots \ldots \Im_N), |X| \geq k$, is said to be *N-wise k-point Hausdorff* $(2 \leq k \leq N)$ if for given any k-tuple of distinct points $x_1, x_2, \ldots . x_k$ of X there exist $U_{i_1} \in \Im_{i_1}, U_{i_2} \in \Im_{i_2}, \ldots . . \Im_{i_k}$, $i_1, i_2, \ldots ., i_k \in \{1,2 \ldots, N\}, i_1 \neq i_2 \neq \ldots \ldots \neq i_k$ such that

$$x_1 \in U_{i_1}, x_2 \in U_{i_2}, \ldots \ldots . . x_k \in U_{i_k}$$

and $$U_{i_1} \cap U_{i_2} \cap \ldots \ldots . \cap U_{i_k} = \phi$$

Example: 2.5: Let $X = \{a_1, a_2, \ldots \ldots . a_n\}$ and $\Im_1^{a_1}, \Im_2^{a_2}, \ldots . \Im_n^{a_n}$ be special finite generalized Sierpiński topologies with nuclei $a_1, a_2, \ldots \ldots . a_n$ respectively. Then the n-topological special finite generalized Sierpiński space $\left(X, \Im_1^{a_1}, \Im_2^{a_2}, \ldots . \Im_n^{a_n}\right)$ is an n-wise k-point Hausdorff space for all $2 \leq k \leq n$.

Proposition 2.4: A k-topological space $(X, \Im_1, \Im_2, \ldots \ldots \ldots \Im_k)$ is k-wise k-point Hausdorff $(|X| \geq k)$ if the diagonal $\Delta = \{(x_1, x_2, \ldots . x_k) : x_1, x_2, \ldots . x_k \in X \text{ and } x_1 = x_2 = \cdots = x_k\}$ is closed in the product space

$(X \times X \times \ldots . . \times X, \Im_1 \times \Im_2 \times \ldots \ldots \ldots \times \Im_k)$.

Proof: Let $x_1, x_2, \ldots . x_k$ are distinct points of the space $X, (|X| \geq k)$. Then

$$(x_1, x_2, \ldots . x_k) \in X \times X \times X \ldots . . \times X - \Delta = \Delta'$$

Since Δ is closed in $X \times X \times \ldots . . \times X$(k times), hence $\Delta'$is open in $(X \times X \times \ldots . . \times X, \Im_1 \times \Im_2 \times \ldots \ldots \ldots \times \Im_k)$. Thus there exist $\Im_1$-open set U_1, $\Im_2$-open set $U_2 \ldots \ldots \Im_k$-open set U_k such that $(x_1, x_2, \ldots . x_k) \in U_1 \times U_2 \times \ldots \ldots \times U_k \subset \Delta'$.

But $U_1 \times U_2 \times \ldots \ldots \times U_k \subset \Delta'$ implies that $U_1 \times U_2 \times \ldots \ldots \times U_k$ contains no element of the form $(x, x, \ldots . . . x)$ and so we have $U_1 \cap U_2 \cap \ldots \ldots \cap U_k = \phi$.

Again $(x_1, x_2, \ldots x_k) \in U_1 \times U_2 \times \ldots \times U_k \Longrightarrow x_1 \in U_1, x_2 \in U_2 \ldots \ldots, x_k \in U_k$

Thus to every k-tuple of distinct points $x_1, x_2, \ldots . x_k$ of X there exist $\Im_1$-open set U_1, $\Im_2$-open set $U_2 \ldots \ldots \ldots \Im_k$-open set U_k such that such that $x_1 \in U_1, x_2 \in U_2 \ldots \ldots, x_k \in U_k$ such that $U_1 \cap U_2 \cap \ldots \ldots \cap U_k = \phi$. It follows that X is k-wise k-point Hausdorff.

REFERENCES

[1] Dhage, B.C., A study of some fixed point theorem, Ph.D. Thesis, Marathwada Univ. Aurangabad, (1984).

[2] Gähler, S., 2-metrische räume und ihre topologische struktur, Math. Nachr., 26 (1963), 115-148.

[3] Gähler, S., Zur geometric 2-metrische räume, Rev. Roum. Math. Pures et Appl., 11 (1966), 664-669.

[4] Ha, K. S., Cho, Y. J., and White, A., strictly convex and 2-convex 2-normed spaces, Math. Japonica, 33 (1988), No.3, 375-384.

[5] Kelly, J.C., Bitopological spaces, Proc. London Math. Soc., 13 (1963), No.3, 71-89.

[6] Menger, K. Untersuchungen ueber allgeine Metrik, Math. Ann., 100 (1928), 75-163.

[7] Mustafa, Z., and Sims, B., Some remarks concerning D-metric spaces, Proceedings of the International Conferences on Fixed Point Theory and Applications, Valencia (Spain), July (2003), 189-198.

[8] Mustafa, Z., Sims, B., A new approach to generalized metric spaces, J. Of Nonlinear and Convex Analysis, 7 (2006), No.2, 289-297.

[9] Ribeiro, H., sur les espaces a metrique faible, Portugaliae Math., 4 (1943), 21-40.

[10] Sharma, A. K., A note on fixed points in 2-metric spaces, Indian J. Pure Appl. Math., 11 (1980), No.2, 1580-1583.

[11] Wilson, W.A. on quasi-metric spaces, American J. Math., 53 (1931), 675-684.

