Determinants in Geometric Algebra

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1 Definition

Let $f$ be a linear map\(^1\) of a real linear vector space $\mathbb{R}^n$ into itself, an endomorphism

$$f : \mathbf{a} \in \mathbb{R}^n \rightarrow \mathbf{a}' \in \mathbb{R}^n.$$  

This map is extended by outermorphism (symbol $\mathcal{f}$) to act linearly on multivectors

$$f(\mathbf{a}_1 \wedge \mathbf{a}_2 \ldots \wedge \mathbf{a}_k) = f(\mathbf{a}_1) \wedge f(\mathbf{a}_2) \ldots \wedge f(\mathbf{a}_k), \quad k \leq n.$$  

By definition $f$ is grade-preserving and linear, mapping multivectors to multivectors. Examples are the reflections, rotations and translations described earlier. The outermorphism of a product of two linear maps $fg$ is the product of the outermorphisms $\mathcal{f}\mathcal{g}$

$$f[g(\mathbf{a}_1)] \wedge f[g(\mathbf{a}_2)] \ldots \wedge f[g(\mathbf{a}_k)] = f[g(\mathbf{a}_1) \wedge g(\mathbf{a}_2) \ldots \wedge g(\mathbf{a}_k)] = \mathcal{f}\mathcal{g}.$$

with $k \leq n$. The square brackets can safely be omitted.

The $n$–grade pseudoscalars of a geometric algebra are unique up to a scalar factor. This can be used to define the determinant\(^2\) of a linear map as

$$\det(f) = \mathcal{f}(I)I^{-1} = \mathcal{f}(I) \circ I^{-1} \quad \text{and therefore} \quad \mathcal{f}(I) = \det(f)I.$$  

For an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ the unit pseudoscalar is $I = \mathbf{e}_1 \mathbf{e}_2 \ldots \mathbf{e}_n$ with inverse $I^{-1} = (-1)^n \mathbf{e}_n \mathbf{e}_{n-1} \ldots \mathbf{e}_1 = (-1)^n (-1)^{n(n-1)/2} I$, where $g$ gives the number of basis vectors, that square to $-1$ (the linear space is then $\mathbb{R}^{p,q}$).

According to Grassmann $n$–grade vectors represent oriented volume elements of dimension $n$. The determinant therefore shows how these volumes change under linear maps. Composing two linear maps gives the product of these volume factors

$$\mathcal{f}\mathcal{g}(I) = \mathcal{f}[\det(g)I] = \det(g) \mathcal{f}(I) = \det(g) \det(f)I.$$  

Therefore

$$\det(fg) = \det(g) \det(f).$$

\(^1\)The treatment in this section largely follows [1].

\(^2\)The symbol $(\cdot)$ means the (symmetric) scalar product of two multivectors, i.e. the scalar (◦–grade) part of their geometric product.
2 Adjoint and Inverse Linear Maps

For every linear map \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) exists\(^3\) a unique adjoint linear map \( \tilde{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n \), such that

\[
b \ast \tilde{f}(a) = \tilde{f}(b) \ast a, \quad \forall a, b \in \mathbb{R}^n.
\]

The adjoint linear map extends again via outer-morphism

\[
\tilde{f}(a_1 \wedge a_2 \ldots \wedge a_k) = \tilde{f}(a_1) \wedge \tilde{f}(a_2) \ldots \wedge \tilde{f}(a_k), \quad k \leq n.
\]

In general we have for multivectors \( A, B \) that

\[
B \ast \tilde{f}(A) = \tilde{f}(B) \ast A,
\]

which can be applied to the defining\(^4\) relationship[3] for the (right) contraction

\[
(C \mathcal{L} A) \ast B = C \ast (A \wedge B), \quad \forall \text{ multivectors} \ A, B, C.
\]

For simple grade \( c \)-vectors \( C \) and \( a \)-vectors \( A \), the right contraction \( (C \mathcal{L} A) \) is a grade \( c - a \) sub-space multivector of \( C \) perpendicular to \( A \). We now get \( \forall A, B, C \)

\[
\tilde{f}(C \mathcal{L} A) \ast B = (C \mathcal{L} A) \ast \tilde{f}(B) = C \ast \left( (A \wedge f(B)) \right) = C \ast \left( f^{-1}(A) \wedge \tilde{f}(B) \right) = C \ast \left( \tilde{f}^{-1}(A) \wedge f\right) = \tilde{f}(C) \ast \left( \tilde{f}^{-1}(A) \wedge f\right) = \tilde{f}(C \mathcal{L} \tilde{f}^{-1}(A)) \ast B,
\]

and therefore

\[
\tilde{f}(C \mathcal{L} A) = \tilde{f}(C) \mathcal{L} \tilde{f}^{-1}(A).
\]

By substituting the pseudoscalar \( I \) for \( C \) and left multiplying with the inverse \( I^{-1} \) we get a general formula for calculating the inverse of \( \tilde{f} \)

\[
I^{-1} \tilde{f}(IA) = I^{-1}(\tilde{f}(I) \mathcal{L} \tilde{f}^{-1}(A)) = I^{-1} \tilde{f}(I) \tilde{f}^{-1}(A) = \det(f) \tilde{f}^{-1}(A),
\]

where we used the fact that right contraction with a pseudoscalar is nothing but the geometric product and that \( f \) is grade preserving.

In the derivation of \( \tilde{f}^{-1} \) we tacitly used the following property of the determinant obtained by applying \( B \ast \tilde{f}(A) = \tilde{f}(B) \ast A \)

\[
\det(f) = \tilde{f}(I) \ast I^{-1} = I \ast \tilde{f}(I^{-1}) = \tilde{f}(I) \ast I^{-1} = \det(\tilde{f}),
\]

because of the symmetry of the scalar product and because \( I^{-1} = (-1)^{n(n-1)/2} I \).

An analogous explicit expression can be derived for \( \tilde{f}^{-1} \)

\[
\tilde{f}^{-1}(A) = \det(f)^{-1} \tilde{f}(fA) I^{-1}, \quad \tilde{f}^{-1}(A) = \det(f)^{-1} \tilde{f}(fA) I^{-1}.
\]

These formulas are very compact and computationally efficient. They show that the inverse mappings can be constructed as double-dualities. Duality here means multiplication with the pseudoscalar \( I \) or \( I^{-1} \).

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\(^3\)An explicit definition for the adjoint linear map can be given as \( \tilde{f}(a) = e^k (f(e_l) \ast a) \), with \( e^k \ast e_l = \delta^{k}_{l} \) (the Kronecker delta symbol), where \( 1 \leq k, l \leq n \). Here the vectors \( \{e_1, e_2, \ldots, e_n\} \) form (a not necessarily orthonormal nor orthogonal) basis of \( \mathbb{R}^n \).

\(^4\)The symbols \( (\ast) \) and \( (\wedge) \) denote the (symmetric) scalar and the antisymmetric outer product parts of the geometric product of multivectors.
References

