Gram-Schmidt Orthogonalization in Geometric Algebra

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Let \( \mathbb{R}_n \) be the geometric algebra of the real linear space \( \mathbb{R}^n \). Let \( \{a_1, a_2, \ldots, a_r\} \), \( r \leq n \) be a set of \( r \) linearly independent vectors. Then the \( r \)-multivector \( A_r = a_1 \wedge a_2 \wedge \ldots \wedge a_r \) will necessarily be different from zero; \( A_r \neq 0 \), and vice versa, because the \( r \)-volume defined by \( A_r \) will be different from zero.

The linearly independent set of vectors \( \{a_1, a_2, \ldots, a_r\} \), \( r \leq n \) can be systematically orthogonalized. We construct the graded sequence of multivectors

\[
A_0 = 1, \quad A_1 = a_1, \quad A_2 = a_1 \wedge a_2, \ldots, \quad A_r = a_1 \wedge a_2 \wedge \ldots \wedge a_r.
\]

We can use \( A_0, A_1, \ldots, A_r \) in order to define a new set of vectors

\[
e_k = \tilde{A}_{k-1} A_k = \tilde{A}_{k-1} \bigwedge A_k, \quad k = 1, \ldots, r,
\]

where the tilde over \( A_{k-1} \) means to reverse the order of vector factors. E.g. \( \tilde{A}_2 = a_2 \wedge a_1 = -A_2 \), \( \tilde{A}_3 = a_3 \wedge a_2 \wedge a_1 = -A_3 \), etc. The geometric product \( \tilde{A}_{k-1} A_k \) can be replaced by the left contraction, because by construction, the \((k-1)\)-subspace defined by \( A_{k-1} \) is fully contained in the \( k \)-subspace defined by \( A_k \). Let us remember the meaning of the left contraction: \( \tilde{A}_{k-1} A_k \) results in an \( k-(k-1) = 1 \) dimensional subspace of the \( k \)-subspace defined by \( A_k \), which is orthogonal to the \((k-1)\)-subspace defined by \( A_{k-1} \). Therefore the set of \( r \) vectors \( e_k \), \( k = 1, \ldots, r \) must be an orthogonal set, and span the \( r \)-subspace defined by \( A_r \). The last property, can be easily verified by calculating the geometric product of all \( e_k \), \( k = 1, \ldots, r \):

\[
\begin{align*}
\mathbf{e}_1 \mathbf{e}_2 \ldots \mathbf{e}_r &= 1 A_1 \tilde{A}_1 A_2 \tilde{A}_2 \ldots A_{r-1} \tilde{A}_{r-1} A_r \\
&= A_1 \ast \tilde{A}_1 \ast A_2 \ast \tilde{A}_2 \ast \ldots \ast A_{r-1} \ast \tilde{A}_{r-1} A_r \\
&= |A_1|^2 |A_2|^2 \ldots |A_{r-1}|^2 |A_r|
\end{align*}
\]

where the symbol \( \ast \) signifies the scalar product, i.e. the scalar part of the geometric product of two multivectors, and \( |A| \) is the positive scalar magnitude of the multivector \( A \) defined by \( |A|^2 = A \ast A \). Obviously the product \( \mathbf{e}_1 \mathbf{e}_2 \ldots \mathbf{e}_r \) constitutes a factorization of \( A_r \) into a product of orthogonal vectors.

This result fully corresponds to the conventional Gram-Schmidt orthogonalization process in linear algebra.
References

