Strong relationship between prime numbers and sporadic groups

by Klaus Lange

klauslange.mathematics@email.de

Abstract: This paper shows a strong relationship between sporadic groups and prime numbers. It starts with new properties for the well known supersingular prime numbers of the moonshine theory. These new properties are only a preparation for the main result of this paper to show that the sporadic groups are strongly connected to prime numbers.

In moonshine theory a super singular prime is a prime divisor of the order of the Monster group.

So we have exactly 15 supersingular primes. Be $\mathfrak{S}$ the set of all supersingular primes then we can write

Definition 1: Set of supersingular primes

The set $\mathfrak{S}$, with

$$\mathfrak{S} = \{2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 41; 47; 59; 71\}$$

is called the set of supersingular primes.

Lemma 1: First property

(i) Be $\mathfrak{s}_i \subset \mathfrak{S}$ the i^{th} element of $\mathfrak{S}$ then it follows $\prod_{i=1}^{15} \mathfrak{s}_i + 1 \in \mathbb{P}$

(ii) $\prod_{i=1}^{15} \mathfrak{s}_i - 1 \notin \mathbb{P}$

Proof:

Simple calculation shows that

(i) $2*3*5*7*11*13*17*19*23*29*31*41*47*59*71+1 = 1618964990108856391 \in \mathbb{P}$

(ii) $2*3*5*7*11*13*17*19*23*29*31*41*47*59*71-1 = 1618964990108856389 = 53*30546509247336913$ ■

Another property belongs to the gaps of supersingular primes: Between 2 and the largest supersingular prime 71 some prime numbers are not an element of $\mathfrak{S}$. For these prime numbers we can define a new set as a sort of an inverse set from supersingular primes. This idea leads to

Definition 2: Inverse set of supersingular primes

The set $\mathfrak{S}^{-1}$, with

$$\mathfrak{S}^{-1} = \{37; 43; 53; 61; 67\}$$

is called the inverse set of supersingular primes and harbors all prime numbers between 2 and the largest supersingular prime 71, that are not element of $\mathfrak{S}$.

Lemma 2: Second property

(i) $\mathfrak{S}^{-1}$ has at least one element. Be $t_i \in \mathfrak{S}^{-1}$ the i^{th} element of $\mathfrak{S}^{-1}$ then it follows

$$\prod_{i=1}^{5} t_i + 2 \in \mathbb{P}.$$

(ii) $\prod_{i=1}^{5} t_i + 2$ is the smallest prime number of this kind, that means all numbers $0 < \prod_{i=1}^{5} t_i - 2^k$ with $k \in \mathbb{N}$, are not prime numbers:

$$0 < \prod_{i=1}^{5} t_i - 2^k \notin \mathbb{P}$$

Proof:

(i) $37 \in \mathfrak{S}^{-1}$. Simple calculation shows that $37*43*53*61*67+2 = 344628103 \in \mathbb{P}$.

(ii) The largest k that satisfy $0 < \prod_{i=1}^{5} t_i - 2^k$ is given by k = 28. With simple computations it follows $\prod_{i=1}^{5} t_i - 2^k \notin \mathbb{P}$ for all $0 < k < 29$, $k \in \mathbb{N}$ ■

Remark:

The next k > 1 that satisfy $\prod_{i=1}^{5} t_i + 2^k \in \mathbb{P}$ is k = 5.

The two given simple prime number properties will show in a new way, why the Monster group is separated from all other sporadic simple groups.

Corollary 1:

Only the Monster group satisfy both properties given by Lemma 1 and Lemma 2. All other sporadic groups satisfy at most only one of the properties given by Lemma 1 and Lemma 2 in analogical usage.

Remark:

Before we can proof Theorem 1 we have to clarify what that mean to use Lemma 1 and Lemma 2 in an analogical way. For this we need two more definitions.

Definition 3: Set of prime divisors of the order from a sporadic group

Be G a sporadic group with order o and $\mathfrak{d}_i$ are all prime devisors of order o, then the set $\mathfrak{D}$, with

$$\mathfrak{D} = \{\mathfrak{d}_1; \mathfrak{d}_2; \ldots; \mathfrak{d}_i; \ldots; \mathfrak{d}_n\}$$

is called the set of prime devisors of the order o of group G.

Definition 4: Inverse set of non prime devisors

Be set $\mathfrak{D}$ given according to Definition 3. All prime numbers $\mathfrak{q}_i < \mathfrak{d}_n$ which are not element of $\mathfrak{D}$ are elements of the set $\mathfrak{D}^{-1}$, with

$$\mathfrak{D}^{-1} = \{\mathfrak{q}_1; \mathfrak{q}_2; \ldots; \mathfrak{q}_i; \ldots; \mathfrak{q}_n\}$$

is called the inverse set of $\mathfrak{D}$.

If all prime numbers $\mathfrak{q}_i < \mathfrak{d}_n$ are elements of $\mathfrak{D}$ then it is

$$\mathfrak{D}^{-1} = \{\ \}.$$

Proof of Corollary 1:

(i) The first and second property for the Monster group is proven by Lemma 1 and Lemma 2.

(ii) To check the properties of Lemma 1 and Lemma 2 for all other sporadic groups we use for set $\mathfrak{S}$ the set $\mathfrak{D}$ and for set $\mathfrak{S}^{-1}$ the set $\mathfrak{D}^{-1}$ according to Definition 3 and Definition 4.

- *The Lyons group*

  The prime devisors set of the order of the Lyons group is given with

  $\mathfrak{D} = \{\ 2;\ 3;\ 5; 7; 11; 31; 37; 67\}$

  and it follows

  $\mathfrak{D}^{-1} = \{\ 13; 17; 19; 23; 29; 41; 43; 47; 53; 59; 61\}$.

  Check of first property:

  $$2 * 3 * 5 * 7 * 11 * 31 * 37 * 67 + 1 = 177521191 = 17 * 1637 * 6379$$
  $$2 * 3 * 5 * 7 * 11 * 31 * 37 * 67 - 1 = 177521189 \in \mathbb{P}$$

  Check of second property:

  $$13 * 17 * 19 * 23 * 29 * 41 * 43 * 47 * 53 * 59 * 61 + 2 = 44266949489693413$$
  $$= 7 * 433 * 4243 * 8707 * 395323$$

  Both properties are not satisfied!

- *The Baby Monster group*

  The prime devisors set of the order of the Baby Monster group is given with

  $\mathfrak{D} = \{\ 2;\ 3;\ 5; 7; 11; 13; 17; 19; 23; 31; 47\}$

  and it follows

  $\mathfrak{D}^{-1} = \{\ 29; 37; 41; 43\}$.

  Check of first property:

  $2 * 3 * 5 * 7 * 11 * 13 * 17 * 19 * 23 * 31 * 47 + 1 = 325046311591 = 97 * 3350992903$

  $$2 * 3 * 5 * 7 * 11 * 13 * 17 * 19 * 23 * 31 * 47 - 1 = 325046311589$$
  $$= 272299 * 1193711$$

  Check of second property:

  $$29 * 37 * 41 * 43 + 2 = 1891701 = 3 * 3 * 3 * 7 * 10009$$

  Both properties are not satisfied!

- *The Janko group J_4*

  The prime devisors set of the order of the Janko group J_4 is given with

  $\mathfrak{D} = \{ 2; 3; 5; 7; 11; 23; 29; 31; 37; 43\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 13; 17; 19; 41\}$.

  Check of first property:

  $$2 * 3 * 5 * 7 * 11 * 23 * 29 * 31 * 37 * 43 + 1 = 75992317171 = 95273 * 797627$$

  $$2 * 3 * 5 * 7 * 11 * 23 * 29 * 31 * 37 * 43 - 1 = 75992317169 = 81749 * 929581$$

  Check of second property:

  $$13 * 17 * 19 * 41 + 2 = 172161 = 3 * 3 * 11 * 37 * 47$$

  Both properties are not satisfied!

- *The Thompson group*

  The prime devisors set of the order of the Thompson group is given with

  $\mathfrak{D} = \{ 2; 3; 5; 7; 13; 19; 31\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 11; 17; 23; 29\}$.

  Check of first property:

  $$2 * 3 * 5 * 7 * 13 * 19 * 31 + 1 = 1607971 = 73 * 22027$$

  $$2 * 3 * 5 * 7 * 13 * 19 * 31 - 1 = 1607969 = 11 * 11 * 97 * 137$$

  Check of second property:

  $$11 * 17 * 23 * 29 + 2 = 124731 = 3 * 3 * 13859$$

  Both properties are not satisfied!

- *The O'Nan group*

  The prime devisors set of the order of the O'Nan group is given with

  $\mathfrak{D} = \{ 2; 3; 5; 7; 11; 19; 31\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 13; 17; 23; 29\}$.

  Check of first property:

  $$2 * 3 * 5 * 7 * 11 * 19 * 31 + 1 = 1360591 \in \mathbb{P}$$

  But we have $2 * 3 * 5 * 7 * 11 * 19 * 31 - 1 = 1360589 \in \mathbb{P}$ in contradiction to Lemma 1, (i)

  Check of second property:

  $$13 * 17 * 23 * 29 + 2 = 147409 \in \mathbb{P}$$

  But we have $13 * 17 * 23 * 29 - 2^4 = 147391 \in \mathbb{P}$ in contradiction to Lemma 2, (ii).

  Both properties are not satisfied!

- *The Fischer group F_{24}*

  The prime devisors set of the order of the Fischer group F_{24} is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 13; 17; 23; 29\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 19\}$.

  Check of first property:

$$2*3*5*7*11*13*17*23*29+1 = 340510171 \in \mathbb{P}$$

$$2*3*5*7*11*13*17*23*29-1 = 340510169 = 31*31*354329$$

  Check of second property:

$$19+2 = 21 = 3*7$$

  Only the second property is not satisfied!

- *The Rudvalis group*

  The prime devisors set of the order of the Rudvalis group is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 13; 29\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 11; 17; 19; 23\}$.

  Check of first property:

$$2*3*5*7*13*29+1 = 79171 = 41*1931$$

$$2*3*5*7*13*29-1 = 79169 = 17*4657$$

  Check of second property:

$$11*17*19*23+2 = 81721 = 71*1151$$

  Both properties are not satisfied!

- *The Fischer group F_{23}*

  The prime devisors set of the order of the Fischer group F_{23} is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 13; 17; 23\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 19\}$.

  Check of first property:

$$2*3*5*7*11*13*17*23+1 = 11741731 = 3209*3659$$

$$2*3*5*7*11*13*17*23-1 = 11741729 \in \mathbb{P}$$

  Check of second property:

$$19+2 = 21 = 3*7$$

  Both properties are not satisfied!

- *The Conway group Co_1*

  The prime devisors set of the order of the Conway group Co_1 is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 13; 23\}$

and it follows

$\mathfrak{D}^{-1} = \{ 17; 19\}$.

Check of first property:

$$2 * 3 * 5 * 7 * 11 * 13 * 23 + 1 = 690691 = 139 * 4969$$
$$2 * 3 * 5 * 7 * 11 * 13 * 23 - 1 = 690689 \in \mathbb{P}$$

Check of second property:

$$17 * 19 + 2 = 325 = 5 * 5 * 13$$

Both properties are not satisfied!

- *The Conway groups Co_2 and Co_3 and the Mathieu groups M_{23} and M_{24}*

  The prime devisors set of the order of the Conway groups Co_2 and Co_3 and the Mathieu groups M_{23} and M_{24} are given with

  $\mathfrak{D} = \{ 2; 3; 5; 7; 11; 23\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 13; 17; 19\}$.

  Check of first property:

  $$2 * 3 * 5 * 7 * 11 * 23 + 1 = 53131 = 13 * 61 * 67$$
  $$2 * 3 * 5 * 7 * 11 * 23 - 1 = 53129 \in \mathbb{P}$$

  Check of second property:

  $$13 * 17 * 19 + 2 = 4201 \in \mathbb{P}$$

  But it is $13 * 17 * 19 - 2^8 = 3943 \in \mathbb{P}$ in contradiction to Lemma 2,(ii)

  Both properties are not satisfied!

- *The Harada-Norton group and Janko Group J_1*

  The prime devisors set of the order of the Harada-Norton group and Janko group J_1 are given with

  $\mathfrak{D} = \{ 2; 3; 5; 7; 11; 19\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 13; 17\}$.

  Check of first property:

  $$2 * 3 * 5 * 7 * 11 * 19 + 1 = 43891 \in \mathbb{P}$$
  $$2 * 3 * 5 * 7 * 11 * 19 - 1 = 43889 \in \mathbb{P}, \text{ in contradiction to Lemma 1,(ii)}$$

  Check of second property:

  $$13 * 17 + 2 = 223 \in \mathbb{P}$$

  But $13 * 17 - 2^6 = 157 \in \mathbb{P}$ in contradiction to Lemma 2, (ii)

  Both properties are not satisfied!

- *The Janko group J_3*

  The prime devisors set of the order of the Janko group J_3 is given with

  $\mathfrak{D} = \{ 2; 3; 5; 17; 19\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 7; 11; 13\}$.

  Check of first property:

$$2 * 3 * 5 * 17 * 19 + 1 = 9691 = 11 * 881$$
$$2 * 3 * 5 * 17 * 19 - 1 = 9689 \in \mathbb{P}$$

  Check of second property:

$$7 * 11 * 13 + 2 = 1003 = 17 * 59$$

  Both properties are not satisfied!

- *The Held group*

  The prime devisors set of the order of the Held group is given with

  $\mathfrak{D} = \{ 2; 3; 5; 7; 17\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ 11; 13\}$.

  Check of first property:

$$2 * 3 * 5 * 7 * 17 + 1 = 3571 \in \mathbb{P}$$
$$2 * 3 * 5 * 7 * 17 - 1 = 3569 = 43 * 83$$

  Check of second property:

$$11 * 13 + 2 = 145 = 5 * 29$$

  Both properties are not satisfied!

- *The Fischer group Fi_{22} and the Suzuki sporadic group*

  The prime devisors set of the order of the Fischer group F_{22} and the Suzuki sporadic group is given with

  $\mathfrak{D} = \{ 2; 3; 5; 7; 11; 13\}$

  and it follows

  $\mathfrak{D}^{-1} = \{ \}$.

  Check of first property:

$$2 * 3 * 5 * 7 * 11 * 13 + 1 = 30031 = 59 * 509$$
$$2 * 3 * 5 * 7 * 11 * 13 - 1 = 30029 \in \mathbb{P}$$

  Check of second property:

  $\mathfrak{D}^{-1} = \{ \}$ in contradiction to Lemma 2, (i)

  Both properties are not satisfied!

- *The McLaughlin group, the Higman-Sims group and the Mathieu group M_{22}*
  The prime devisors set of the order of the McLaughlin group, the Higman-Sims group and the Mathieu group M_{22} is given with
  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11\}$
  and it follows
  $\mathfrak{D}^{-1} = \{ \}$.
  Check of first property:
  $$2 * 3 * 5 * 7 * 11 + 1 = 2311 \in \mathbb{P}$$
  $$2 * 3 * 5 * 7 * 11 - 1 = 2309 \in \mathbb{P}$$
  Check of second property:
  $\mathfrak{D}^{-1} = \{ \}$ in contradiction to Lemma 2, (i)
  Both properties are not satisfied!

- *The Mathieu groups M_{11} and M_{12}*
  The prime devisors set of the order of the Mathieu groups M_{11} and M_{12} is given with
  $\mathfrak{D} = \{ 2;\ 3;\ 5; 11\}$
  and it follows
  $\mathfrak{D}^{-1} = \{ 7\}$.
  Check of first property:
  $$2 * 3 * 5 * 11 + 1 = 331 \in \mathbb{P}$$
  $$2 * 3 * 5 * 11 - 1 = 329 = 7 * 47$$
  Check of second property:
  $$7 + 2 = 9 = 3 * 3$$
  Only the second property is not satisfied!

- *The Janko group J_2*
  The prime devisors set of the order of the Janko group J_2 is given with
  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7\}$
  and it follows
  $\mathfrak{D}^{-1} = \{\}$.
  Check of first property:
  $$2 * 3 * 5 * 7 + 1 = 211 \in \mathbb{P}$$
  $$2 * 3 * 5 * 7 - 1 = 209 = 11 * 19$$
  Check of second property:
  $\mathfrak{D}^{-1} = \{ \}$ in contradiction to Lemma 2, (i)
  Only the second property is not satisfied!

Result: Only the Monster group satisfies the properties given by Lemma 1 and Lemma 2 ■

Remarks

(i) It is interesting that only the supersingular primes separates the Monster group from all other sporadic groups by these two simple prime number properties.

(ii) Some of the other groups has same properties e.g. that $\prod_{i=1}^{n} \mathfrak{d}_i + 1 \in \mathbb{P}$ and $\prod_{i=1}^{n} \mathfrak{d}_i - 1 \in \mathbb{P}$. This could be a hint that there are new relationships between those groups.

(iii) On the other hand the question comes: Why does the Monster group is separated so strong by these properties? Because the most sporadic groups are subgroups from the Monster group. Do we have another simple prime number properties to see that the Monster group is a part of the other sporadic groups? Yes!

Lemma 3: Third property

Be $\mathfrak{s}_i \in \mathfrak{S}$ the i^{th} element of $\mathfrak{S}$ then it follows $\sum_{i=1}^{15} \mathfrak{s}_i + 1 \in \mathbb{P}$ or $\sum_{i=1}^{15} \mathfrak{s}_i - 1 \in \mathbb{P}$.

Proof:

$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 41 + 47 + 59 + 71 + 1 = 379 \in \mathbb{P}$$

$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 41 + 47 + 59 + 71 - 1 =$$

$$377 = 13 * 29$$

■

For all other sporadic groups this leads to

Lemma 3a: Third property for sporadic groups

Be $\mathfrak{d}_i \in \mathfrak{D}$ the i^{th} element of $\mathfrak{D} \neq \mathfrak{S}$ then it follows

$$\exists \text{ at least one } \mathfrak{D} \text{ with } \begin{cases} \sum_{i=1}^{n} \mathfrak{d}_i \in \mathbb{P}, \text{if } \sum_{i=1}^{n} \mathfrak{d}_i \equiv 1 \; mod \; 2 \\ \sum_{i=1}^{n} \mathfrak{d}_i \pm 1 \in \mathbb{P}, \quad otherwise \end{cases}$$

Proof:

It is enough to show only for one sporadic group with $\mathfrak{D} \neq \mathfrak{S}$ that Lemma 3a is satisfied. But we check all other 25 sporadic groups.

- *The Lyons group*
  The prime devisors set of the order of the Lyons group is given with

$\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 31; 37; 67\}$

$$2 + 3 + 5 + 7 + 11 + 31 + 37 + 67 = 163 \in \mathbb{P}$$

- *The Baby Monster group*

  The prime devisors set of the order of the Baby Monster group is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 13; 17; 19; 23; 31; 47\}$

$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 31 + 47 + 1 = 179 \in \mathbb{P}$$
$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 31 + 47 - 1 = 177 \equiv 0 \; mod \; 3$$

- *The Janko group J_4*

  The prime devisors set of the order of the Janko group J_4 is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 23; 29; 31; 37; 43\}$

$$2 + 3 + 5 + 7 + 11 + 23 + 29 + 31 + 37 + 43 = 191 \in \mathbb{P}$$

- *The Thompson group*

  The prime devisors set of the order of the Thompson group is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 13; 19; 31\}$

$$2 + 3 + 5 + 7 + 13 + 19 + 31 + 1 = 81 = 3 * 3 * 3 * 3$$

$$2 + 3 + 5 + 7 + 13 + 19 + 31 - 1 = 79 \in \mathbb{P}$$

- *The O'Nan group*

  The prime devisors set of the order of the O'Nan group is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 19; 31\}$

$$2 + 3 + 5 + 7 + 11 + 19 + 31 + 1 = 79 \in \mathbb{P}$$
$$2 + 3 + 5 + 7 + 11 + 19 + 31 - 1 = 77 = 7 * 11$$

- *The Fischer group F_{24}*

  The prime devisors set of the order of the Fischer group F_{24} is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 13; 17; 23; 29\}$

$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 23 + 29 + 1 = 111 = 3 * 37$$

$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 23 + 29 - 1 = 109 \in \mathbb{P}$$

- *The Rudvalis group*

  The prime devisors set of the order of the Rudvalis group is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 13; 29\}$

$$2 + 3 + 5 + 7 + 13 + 29 = 59 \ \in \ \mathbb{P}$$

- *The Fischer group F_{23}*

  The prime devisors set of the order of the Fischer group F_{23} is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 13; 17; 23\}$

$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 23 = 81 = 3 * 3 * 3 * 3$$

- *The Conway group Co_1*

  The prime devisors set of the order of the Conway group Co_1 is given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 13; 23\}$

$$2 + 3 + 5 + 7 + 11 + 13 + 23 + 1 = 65 = 5 * 13$$

$$2 + 3 + 5 + 7 + 11 + 13 + 23 - 1 = 63 = 3 * 3 * 7$$

- *The Conway groups Co_2 and Co_3 and the Mathieu groups M_{23} and M_{24}*

  The prime devisors set of the order of the Conway groups Co_2 and Co_3 and the Mathieu groups M_{23} and M_{24} are given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 23\}$

$$2 + 3 + 5 + 7 + 11 + 23 = 51 = 3 * 17$$

- *The Harada-Norton group and Janko Group J_1*

  The prime devisors set of the order of the Harada-Norton group and Janko group J_1 are given with

  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 19\}$

$$2 + 3 + 5 + 7 + 11 + 19 = 47 \in \ \mathbb{P}$$

- *The Janko group J_3*

  The prime devisors set of the order of the Janko group J_3 is given with

$\mathfrak{D} = \{ 2;\ 3;\ 5; 17; 19\}$

$$2 + 3 + 5 + 17 + 19 + 1 = 47 \in \mathbb{P}$$

$$2 + 3 + 5 + 17 + 19 - 1 = 45 = 3 * 3 * 5$$

- *The Held group*
  The prime devisors set of the order of the Held group is given with
  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 17\}$

$$2 + 3 + 5 + 7 + 17 + 1 = 35 = 5 * 7$$

$$2 + 3 + 5 + 7 + 17 - 1 = 33 = 3 * 11$$

- *The Fischer group Fi_{22} and the Suzuki sporadic group*
  The prime devisors set of the order of the Fischer group F_{22} and the Suzuki sporadic group is given with
  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11; 13\}$

$$2 + 3 + 5 + 7 + 11 + 13 = 41 \in \mathbb{P}$$

- *The McLaughlin group, the Higman-Sims group and the Mathieu group M_{22}*
  The prime devisors set of the order of the McLaughlin group, the Higman-Sims group and the Mathieu group M_{22} is given with
  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7; 11\}$

$$2 + 3 + 5 + 7 + 11 + 1 = 29 \in \mathbb{P}$$

$$2 + 3 + 5 + 7 + 11 - 1 = 27 = 3 * 3 * 3$$

- *The Mathieu groups M_{11} and M_{12}*
  The prime devisors set of the order of the Mathieu groups M_{11} and M_{12} is given with
  $\mathfrak{D} = \{ 2;\ 3;\ 5; 11\}$

$$2 + 3 + 5 + 11 = 21 = 3 * 7$$

- *The Janko group J_2*
  The prime devisors set of the order of the Janko group J_2 is given with
  $\mathfrak{D} = \{ 2;\ 3;\ 5; 7\}$

$$2 + 3 + 5 + 7 = 17 \in \mathbb{P}$$

$\blacksquare$

The importance of all the calculations in the given proofs will be clear if we compare this to other sets of prime numbers.

For all given sporadic groups we have 18 different sets of $\mathfrak{D}$.

The smallest set was $\mathfrak{D} = \{2;\ 3;\ 5;\ 7\}$.

Starting with this smallest set we now construct 18 prime number basic sets adding for any new set the next prime number. Doing so we have

Definition 5: Basic prime number sets

B_1 = { 2; 3; 5; 7}

B_2 = { 2; 3; 5; 7; 11}

B_3 = { 2; 3; 5; 7; 11; 13}

B_4 = { 2; 3; 5; 7; 11; 13; 17}

B_5 = { 2; 3; 5; 7; 11; 13; 17; 19}

B_6 = { 2; 3; 5; 7; 11; 13; 17; 19; 23}

B_7 = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29}

B_8 = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31}

B_9 = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37}

B_{10} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41}

B_{11} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43}

B_{12} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47}

B_{13} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47; 53}

B_{14} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47; 53; 59}

B_{15} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47; 53; 59; 61}

B_{16} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47; 53; 59; 61; 67}

B_{17} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47; 53; 59; 61; 67; 71}

B_{18} = { 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47; 53; 59; 61; 67; 71; 73}

To compare these basic prime number sets with the sets given by the prime devisors of the order of the sporadic groups we use the properties of Lemma 1 and Lemma 3 in analogical usage.

Lemma 4: Multiplicative comparison

Be $b_i \subset B_j$ the i^{th} element of B

Trying to generate prime numbers with $\prod_{i=1}^{n} b_i + 1 \in \mathbb{P}$ or $\prod_{i=1}^{n} b_i - 1 \in \mathbb{P}$

for each set B_j j=1 to 18 counting less sets B_j that generates a prime number as the sets $\mathfrak{D}$ of all sporadic groups (including the monster group).

Proof:

Simple calculation shows

$2*3*5*7 + 1 \in \mathbb{P}$

$2*3*5*7*11 \pm 1 \in \mathbb{P}$

$2*3*5*7*11*13 - 1 \in \mathbb{P}$

$2*3*5*7*11*13*17 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31 + 1 \in \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41 - 1 \in \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41*43 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41*43*47 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41*43*47*53 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41*43*47*53*59 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41*43*47*53*59*61 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41*43*47*53*59*61*67 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41*43*47*53*59*61*67*71 \pm 1 \notin \mathbb{P}$

$2*3*5*7*11*13*17*19*23*29*31*37*41*43*47*53*59*61*67*71*73 \pm 1 \notin \mathbb{P}$

Only five sets B_j generates prime numbers. But with the sets of the prime devisor of the order of all sporadic groups, we count in proof for Corollary 1 together with proof of Lemma 1 that 14 of these sets generates prime numbers ■

Lemma 5: Additive comparison

Be $b_i \subset B_j$ the i^{th} element of B

Trying to generate prime numbers with $\begin{cases} \sum_{i=1}^{n} b_i \in \mathbb{P}, \text{if } \sum_{i=1}^{n} b_i \equiv 1 \ mod\ 2 \\ \sum_{i=1}^{n} b_i \pm 1 \in \mathbb{P}, \quad otherwise \end{cases}$

for each set B_j j=1 to 18 counting less sets B_j that generates a prime number as the sets $\mathfrak{D}$ of all sporadic groups (including the monster group).

Proof:

Simple calculation shows

$2+3+5+7 \in \mathbb{P}$

$2+3+5+7+11 + 1 \in \mathbb{P}$

$2+3+5+7+11+13 \in \mathbb{P}$

$2+3+5+7+11+13+17 + 1 \in \mathbb{P}$

$2+3+5+7+11+13+17+19 \notin \mathbb{P}$

$2+3+5+7+11+13+17+19+23 + 1 \in \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29 \notin \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31 \pm 1 \notin \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37 \in \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41 + 1 \in \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41+43 \in \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41+43+47 \pm 1 \notin \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41+43+47+53 \notin \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41+43+47+53+59 - 1 \in \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41+43+47+53+59+61 \notin \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41+43+47+53+59+61+67 + 1 \in \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41+43+47+53+59+61+67+71 \notin \mathbb{P}$

$2+3+5+7+11+13+17+19+23+29+31+37+41+43+47+53+59+61+67+71+73 \pm 1 \notin \mathbb{P}$

Ten sets B_j generates prime numbers. But with the sets of the prime devisor of the order of all sporadic groups, we count in proofs for Lemma 3 and Lemma 3a that 13 of these sets generates prime numbers ■

Remark

It is clear that if the numbers decreases the possibility to have a prime number is greater. That's what we see for the basic sets by Definition 5. If we multiply these numbers we have only five prime numbers generated. If we creates sums of the same numbers we generate ten prime numbers. But it isn't so, if we look for the prime devisor sets. The number of generated primes is very stable from 14 to 13. This leads to

Theorem 1: Cross comparison

Be $C_\pi(x \in \mathbb{P})$ the counting function for all prime numbers generated by x then

$$C_\pi\left(\prod_{i=1}^{n} \mathfrak{d}_i \pm 1 \in \mathbb{P}\right) > C_\pi\left(\begin{cases}\sum_{i=1}^{n} \mathrm{b}_i \in \mathbb{P}, \text{if } \sum_{i=1}^{n} \mathrm{b}_i \equiv 1 \; mod \; 2 \\ \sum_{i=1}^{n} \mathrm{b}_i \pm 1 \in \mathbb{P}, \quad otherwise\end{cases}\right)$$

Proof:

This is a result of Lemma 4 und Lemma 5 ■

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