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PROOF OF FERMAT’S THEOREM

(www.fermat-theorem.net)

In the article showed that the equation of Fermat’s theorem is a transcendental equation. This transcendental equation has no solution in integers. Therefore, Fermat's Last Theorem is true.

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1. EQUATION $(V+y)^p - y^p = A^p$

1. We consider the equation:

$$x^p + y^p = z^p, \quad p \geq 3. \tag{1}$$

Fermat’s theorem: the equation (1) has no solutions in integers (x, y, z), if integer $p \geq 3$, [1, 2].

If the number is not an integer, then it is either a rational fraction, or irrational value.

Let the numbers (x, y, z) are the rational fractions, then the relation (1) we can lead to a common denominator. In this case, the equation (1) will have solutions in integers, contradicting Fermat’s theorem.

Fermat’s theorem we state as follows: if an integer $p \geq 3$, then in the equation (1) at least one of the numbers (x, y, z) is an irrational number.

We consider the equation (1), provided that the variables (x, y, z) are positive integers.

2. Assuming that $x = y$, the equation (1) becomes:

$$2y^p = z^p, \quad p \geq 3. \tag{2}$$

From (2), we have the expression:

$$z = \sqrt[p]{2} \cdot y, \quad p \geq 3. \tag{3}$$

For integers (y, z), the equality (3) is not feasible, since the right-hand side is an irrational number by definition.

Consequently, at least one of the numbers (y, z) is an irrational number.

3. Provided $0 < x < y < z$, we write equation (1) as follows:

$$\begin{aligned} & z^p - y^p = x^p, \quad p \geq 3, \\ & z > y > x > 0. \end{aligned} \tag{4}$$

If x is an integer, then by Fermat's theorem, at least one of the numbers (y, z) is an irrational number.

Let $x = A$, where a positive integer is A.

In this case, the equation (4) we write as:

$$\begin{aligned} & z^p - y^p = A^p, \quad p \geq 3, \\ & z > y > A > 0. \end{aligned} \tag{5}$$

Provided $0 < y < x < z$ ($y = A$), a problem also reduces to the equation (5) in the form:

$$\begin{aligned} & z^p - x^p = A^p, \quad p \geq 3, \\ & z > x > A > 0. \end{aligned} \tag{6}$$

4. Any composite integer can be uniquely represented as a product of prime factors.

We represent the integer A^p, $(p \geq 3)$ as an expansion in integer factors (V, U) provided that $U > V \geq 1$:

$$A^p = V \cdot U, \quad U > V \geq 1, \quad p \geq 3, \tag{7}$$

$$V \cdot U = \{V_k \cdot U_k\}, \quad k = (1, N), \tag{8}$$

$$V_1 \cdot U_1 \ = ... = \ V_k \cdot U_k \ = ... = \ V_N \cdot U_N \ , \quad U_k > V_k \ , \ V_1 = 1 \ , \ U_1 = U \ .$$

For an odd number A , integer multipliers (V, U) are odd numbers. For an even number A , integer multipliers (V, U) are either even integers or integers of different parity.

5. Consider the equation (5).

In view of (7), we represent the equation (5) in the form of decomposition on the multipliers:

$$(z-y)\cdot(z^{p-1}+z^{p-2}\cdot y+...+z\cdot y^{p-2}+y^{p-1}) \ = \ V\cdot U \ , \qquad (9)$$

$$p=2n\,, \quad n\geq 2 \ , \quad p=2n+1\,, \quad n\geq 1 \ .$$

Since (y, z, V, U) - integers, $z>y>0 \quad U>V>0$ then the equation (9) can be represented as a system of equations:

$$z \ - \ y \ = \ V \ , \qquad (10)$$

$$z^{p-1} \ + \ z^{p-2}\cdot y \ + ... + \ z\cdot y^{p-2} \ + \ y^{p-1} \ = \ U \ , \qquad (11)$$

$$V\equiv V_k, \ U\equiv U_k, \quad y\equiv y_k, \quad z\equiv z_k \ , \ k=(1, N) \ .$$

In view of (10), the equation (5) takes the following form:

$$(V+y)^p - y^p = A^p \ , \quad p\geq 3 \ ,$$

$$(V+y)>A\,, \quad y>A \ . \qquad (12)$$

In the equation (12) (V, A) are integers of the same parity, by definition.

6. For equation (5) we have the condition:

$$z \ < \ A \ + \ y \ . \qquad (13)$$

In view of (10), the inequality (13) has the form:

$$V \ < \ A \ . \qquad (14)$$

Therefore, under the condition $V \geq A$, the equation (12) has no solution in the integers (y, V, A) .

7. We consider the special solutions of equation (1).

Assuming that $y = 1$, we have the solution:

$$1 + V = (1 + A^p)^{1/p}, \quad p \geq 3. \tag{15}$$

Assuming that $y = A$, we have the solution:

$$V = \left(\sqrt[p]{2} - 1\right) \cdot A, \quad p \geq 3. \tag{16}$$

Assuming that $y = V$, we have the solution:

$$A = V \cdot \left(2^p - 1\right)^{1/p}, \quad p \geq 3. \tag{17}$$

Provided that $V = A$, we have the inequality:

$$\left(A + y\right)^p > A^p + y^p, \quad p \geq 2.$$

Since for the integers (y, V, A) and subject to $p \geq 3$, the equalities (15) – (17) are not feasible, then for the equation (12) we have the following inequalities:

$$\begin{aligned} &A < y < V, \\ &1 < y < A < V, \end{aligned} \tag{18}$$

$$\begin{aligned} &A < V < y, \\ &V < A < y, \end{aligned} \tag{19}$$

$$\begin{aligned} &V < y < A, \\ &1 < y < V < A. \end{aligned} \tag{20}$$

The inequalities (20) are in contradiction to equation (12), since by condition $y > A$.

8. Consider the equation (12) subject to (18).

We represent the equation (12) in the form:

$$\left(\frac{V}{A}\right)^p \cdot \left(\left(1+\frac{y}{V}\right)^p - \left(\frac{y}{V}\right)^p\right) = 1, \quad p \geq 3. \qquad (21)$$

For the equality (21) we have the following inequality:

$$\left(1+\frac{y}{V}\right)^p - \left(\frac{y}{V}\right)^p > 1, \quad p \geq 2.$$

So, for the integers (y , p , V , A) and under the condition $V > A$, the equality (21) is impossible: the left side is greater than 1.

Therefore, under condition (18), the equation (12) has no solutions in integers.

Consider the equation (12) subject to (19). We write the equation (12) as:

$$\left(\frac{y}{A}\right)^p \cdot \left(\left(1+\frac{V}{y}\right)^p - 1\right) = 1, \quad p \geq 3. \qquad (22)$$

Since $y > A$, then from (22) we have the inequality:

$$\left(1+\frac{V}{y}\right)^p - 1 < 1, \quad p \geq 3. \qquad (23)$$

From (23), we obtain the condition for integers (y , p , V) :

$$\frac{pV}{y} < p \cdot \left(\sqrt[p]{2} - 1\right) \leq 3\left(\sqrt[3]{2} - 1\right) < 1, \; p \geq 3. \qquad (24)$$

In the expression (22), we separate the factor (pV) :

$$\left(\frac{y}{A}\right)^{p-1} \cdot \left(\frac{pV}{A}\right) \cdot f\left(\frac{y}{V}\right) = 1, \quad p \geq 3, \qquad (25)$$

$$f\left(\frac{y}{V}\right) = \frac{y}{pV} \cdot \left(\left(1+\frac{V}{y}\right)^p - 1\right), \; f > 1. \qquad (26)$$

So, for the integers (y, V, A) and subject to $A < V < y$, the equality (25) is impossible: the left-hand side is greater than 1.

Thus, under the condition $y > V > A$, equation (12) has no solutions in integers.

9. Consider the equation (25) provided $y > A > V$.

If $p \cdot V \geq A$, then the equality (25) is impossible: the left-hand side is greater than 1.

Provided $A > p \cdot V$, from (11), we have the following inequality:

$$\frac{pV}{A} \cdot f\left(\frac{y}{V}\right) < 1, \quad pV < A < y. \qquad (27)$$

In view of (26), from (27), we have the inequality (23) and the condition (24).

Thus, for equation (12), we have the condition (24) and the following inequality:

$$y \ > \ A \ > \ p \cdot V, \quad p \geq 3. \qquad (28)$$

10. We represent the equation (12) as follows:

$$\left(1 + \frac{v}{p}\right)^p - 1 = u, \quad p \geq 3,$$

$$v = \frac{pV}{y}, \quad u = \left(\frac{A}{y}\right)^p. \qquad (29)$$

We use the binomial formula:

$$\left(\frac{v}{p} + 1\right)^p = \sum_{k=0}^{p} C_p^k \cdot \left(\frac{v}{p}\right)^{p-k}. \qquad (30)$$

C_p^k - Binomial coefficient:

$$C_p^k = \frac{p!}{k!\,(p-k)!}, \quad p \geq k \geq 0,$$

$$C_p^0 = C_p^p = 1, \quad C_p^1 = C_p^{p-1} = p, \quad 0! = 1! = 1.$$

With (30), we write (29) as follows:

$$v \cdot \left(1 + \frac{1}{p} \cdot \sum_{k=0}^{p-2} C_p^k \cdot \left(\frac{v}{p} \right)^{p-k-1} \right) = u , \quad v < p , \quad p \geq 3. \qquad (31)$$

Under the condition $v \geq u$, equality (31) is impossible: the left-hand side of equality more the right-hand, by definition. Consequently, for equation (12) we have the following condition $1 > u > v$:

$$1 > \left(\frac{A}{y} \right)^p > \frac{pV}{y} , \quad y > A > pV , \quad p \geq 3 . \qquad (32)$$

11. Consider the equation (1.12) under the condition $A = a$, a - a prime number, ($a \geq 2$).

In this case, equation (12) has the form:

$$(V + y)^p - y^p = a^p , \quad a \geq 2 , \ p \geq 3 . \qquad (33)$$

In equation (33), (a , V) are integers of the same parity.

In view of (14), for the equation (33), we have the following conditions:

$$(V + y) > a , \quad y > a > V , \qquad (34)$$

$$(V + y) > a , \quad a > y , \ a > V . \qquad (35)$$

We imagine a whole number a^p as an expansion in integer factorization (V , U) :

$$a^p = V \cdot U , \quad U > V \geq 1 , \quad p \geq 3 . \qquad (36)$$

Under the condition $p = 2n , \ n \geq 2$, the factors (V , U) are:

$$V = a^{n-i} , \quad U = a^{n+i} , \quad i = (1 , n) . \qquad (37)$$

Under the condition $p = 2n + 1 , \ n \geq 1$, the factors (V , U) are:

$$V = a^{n-i} , \quad U = a^{n+i+1} , \quad i = (0 , n) . \qquad (38)$$

In view of (37) and (38), inequality $a > V$ has the form:

$$a > a^{n-i}, \quad i \le n. \tag{39}$$

The inequality (39) is feasible only if $i = n$.

In this case, $V = 1$ and the conditions (34) - (35), respectively, take the following form:

$$y > a - 1, \quad a \ge 2. \tag{40}$$

$$1 + y > a > y, \quad a \ge 2. \tag{41}$$

For integers (a, y) , the condition (41) is not feasible.

Consequently, the expansion (36) can be represented only in the form:

$$a^p = V \cdot U, \quad V = 1, \quad U = a^p, \quad p \ge 3. \tag{42}$$

Thus, equation (33) has the form:

$$\begin{gathered}(1 + y)^p - y^p = a^p, \quad p \ge 3, \\ y > a - 1, \quad a \ge 2.\end{gathered} \tag{43}$$

Under the condition $a = 2$, equation (33) is:

$$(1 + y)^p - y^p = 2^p, \quad p \ge 3. \tag{44}$$

For an integer y , any parity, equality (44) cannot be satisfied: the left hand side is odd. Consequently, for the equation (33) we have the condition: a prime number $a \ge 3$.

Under the condition $y = a$, equation (33) has a solution:

$$\left(\sqrt[p]{2} - 1\right) \cdot a = 1 . \tag{45}$$

For an integer a , equality (45) is not feasible.

Thus, for the equation (33) we have the condition:

$$y > a , \quad a \geq 3 . \tag{46}$$

Equation (33), we will present in the form:

$$\left(1 + \frac{1}{y} \right)^{p} - 1 = \left(\frac{a}{y} \right)^{p} , \quad a \geq 3 , \; p \geq 3 . \tag{47}$$

According to (24) and (32), provided that $V = 1 , \; A = a$, for the equation (47) we have the following inequalities:

$$\frac{p}{y} < p \cdot \left(\sqrt[p]{2} - 1 \right) < 1 , \; p \geq 3 , \tag{48}$$

$$\frac{p}{y} < \left(\frac{a}{y} \right)^{p} < 1 , \quad y > a > p \geq 3 . \tag{49}$$

Equation (47) is a special case of (29), provided that $V = 1 , \; A = a$, a prime number $a > p \geq 3$.

2. THE TRANSCENDENTAL EQUATION

1. We consider the equation (1.29):

$$\left(1 + \frac{v}{p}\right)^p - 1 = u \;, \quad p \ge 3 \;, \tag{1}$$

$$0 < v < u < \sqrt[p]{u} < 1. \tag{2}$$

Transform equation (1) to mean:

$$1 + \frac{v}{p} = (1 + u)^{1/p} \,, \quad 0 < u < 1, \quad p \ge 3 \;. \tag{3}$$

We represent the right side of (3) as a power series:

$$(1+u)^{1/p} = 1 + \frac{1}{p} \cdot \sum_{k=1}^{\infty} (-1)^{k+1} \cdot B_k \cdot \frac{u^k}{k} \,. \tag{4}$$

The coefficients B_k , $k \ge 1$ are: $B_1 = 1$, $B_k < 1$, $k \ge 2$

$$B_k = \frac{(p-1)\cdots((k-1)p-1)}{p^{k-1} \cdot (k-1)!} \;, \quad k \ge 2 \;. \tag{5}$$

We use the definition of the function of the natural logarithm:

$$\ln(1+\varepsilon) = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{\varepsilon^k}{k} \,, \quad 0 < \varepsilon < 1 \;. \tag{6}$$

Using (9), we represent (7) as:

$$(1+u)^{1/p} = 1 + \frac{1}{p} \cdot \ln(1+u) + \frac{1}{p} \cdot g(u), \quad 0 < u < 1 \;. \tag{7}$$

The function $g(u)$ is the power series of the form:

$$g(u) = \sum_{k=2}^{\infty}(-1)^k \cdot (1-B_k)\cdot \frac{u^k}{k}. \qquad (8)$$

In view of (7), equation (3) is:

$$\ln(1+u) = v - g(u), \quad 0 < v < u < 1. \qquad (9)$$

For the equality (3), we have the relation:

$$p \cdot \ln\left(1 + \frac{v}{p}\right) = \ln(1 + u). \qquad (10)$$

In view of (10), equation (9) is:

$$\left(1 + \frac{v}{p}\right) - \ln\left(1 + \frac{v}{p}\right) = 1 + \frac{1}{p}\cdot g(u) , \quad p \ge 3. \qquad (11)$$

For the variable v , equation (14) is a transcendental equation of the natural logarithm.

We write the equation (11) for the variables (y, A, V) :

$$\left(1 + \frac{V}{y}\right) - \ln\left(1 + \frac{V}{y}\right) = 1 + \frac{1}{p}\cdot g\left(\frac{A}{y}\right) , \qquad (12)$$

$$y > A > pV, \quad p \ge 3 .$$

For the variable y , the equation (12) is a transcendental equation of the natural logarithm. So, the algebraic equation (1.12) reduces to the transcendental equation However, the transcendental equations can be solved only approximately. In other words, for the equations (12), we cannot find the solutions in integers. Consequently, for the equation (1.12) are also impossible to find the solutions in integers. This in fact means that the equation (1.1) has no solution in integers (x, y, z) .

2. Consider the equation (11).

In view of (6), the left-hand side of equation (11), we represent in the form:

$$\left(1+\frac{v}{p}\right) - \ln\left(1+\frac{v}{p}\right) = 1 + \sum_{k=2}^{\infty}(-1)^k \cdot \frac{1}{k} \cdot \left(\frac{v}{p}\right)^k . \qquad (13)$$

In view of (8) and (13), equation (14) takes the following form:

$$\sum_{k=2}^{\infty}(-1)^k \cdot \frac{1}{k} \cdot \left(\left(\frac{v}{p}\right)^k - \left(1 - B_k\right) \cdot \frac{u^k}{p} \right) = 0 , \quad p \geq 3 . \qquad (14)$$

We introduce the notation:

$$R_k = \left(1 - B_k\right) \cdot p^{k-1} . \qquad (15)$$

From (5) and (15), we have the expressions:

$$B_1 = 1 , \quad B_2 = \frac{p-1}{p} , \quad R_1 = 0 , \; R_2 = 1 ,$$

$$\frac{B_{k+1}}{B_k} = \frac{kp - 1}{kp} , \quad k \geq 1 , \quad p \geq 3 , \qquad (16)$$

$$\frac{R_{k+1}}{R_k} = \left(\frac{1 - B_{k+1}}{1 - B_k} \right) \cdot p , \quad k \geq 2 , \quad p \geq 3 .$$

From (16), we have the following inequalities:

$$1 \geq B_k > B_{k+1} , \quad k \geq 1 ,$$

$$R_{k+1} > R_k \geq 1 , \quad k \geq 2 . \qquad (17)$$

With (15), we write (14) for the variables $(\, y , A , V \,)$:

$$\sum_{k=2}^{\infty}(-1)^k \cdot \frac{1}{k \cdot p^k} \cdot \left(R_k \cdot \left(\frac{A}{y}\right)^{p \cdot k} - \left(\frac{pV}{y}\right)^k \right) = 0 \quad , \qquad (18)$$

$$y > A > pV , \quad p \geq 3 .$$

3. We write the equation (18) as follows:

$$\sum_{k=2}^{\infty} (-1)^k \cdot b_k = 0 , \qquad (19)$$

$$b_k = \frac{1}{k \cdot p^k} \cdot \left(R_k \cdot u^k - v^k \right), \quad b_k > 0 . \qquad (20)$$

Equation (19), we will present in the form of:

$$\sum_{k=1}^{\infty} \left(b_{2k} - b_{2k+1} \right) = 0 . \qquad (21)$$

So, we have the equality:

$$b_{2k} = b_{2k+1} , \quad k \geq 1 . \qquad (22)$$

In view of (20), the equality (22) has the form:

$$\left(R_{2k} - \frac{2k \cdot u}{(2k+1) \cdot p} \cdot R_{2k+1} \right) \cdot u^{2k} = \left(1 - \frac{2k \cdot \gamma}{(2k+1) \cdot p} \right) \cdot \gamma^{2k} , \qquad (23)$$

$$\frac{R_{2k+1}}{R_{2k}} = p \cdot \left(\frac{1 - B_{2k+1}}{1 - B_{2k}} \right), \quad R_{2k+1} > R_{2k} \geq 1 , \; k \geq 1 , \qquad (24)$$

$$\frac{B_{2k+1}}{B_{2k}} = \frac{2kp - 1}{2kp} , \quad 1 > B_{2k} > B_{2k+1} , \; k \geq 1 . \qquad (25)$$

For the right-hand side of (29), we have the condition:

$$\frac{2k \cdot \gamma}{(2k+1) \cdot p} < 1 , \quad 0 < \gamma < 1 , \; k \geq 1 , \quad p \geq 3 . \qquad (26)$$

According to (26), for equation (29), the following condition is necessary:

$$\frac{2k\cdot u}{(2k+1)\cdot p}\cdot\frac{R_{2k+1}}{R_{2k}} < 1, \quad 0<u<1,\ k\geq 1, \quad p\geq 3. \qquad (27)$$

For the condition (27) we have the inequality:

$$\frac{2k\cdot u}{(2k+1)\cdot p}\cdot\frac{R_{2k+1}}{R_{2k}} < \frac{2k}{(2k+1)\cdot p}\cdot\frac{R_{2k+1}}{R_{2k}}\ , \quad 0<u<1. \qquad (28)$$

Consider the following inequality:

$$\frac{2k}{(2k+1)\cdot p}\cdot\frac{R_{2k+1}}{R_{2k}} \leq 1, \quad k\geq 1, \quad p\geq 3. \qquad (29)$$

In view of (24) and (25), the inequality (29) has the form:

$$B_{2k} \leq \frac{p}{p+1}, \quad k\geq 1, \quad p\geq 3. \qquad (30)$$

Provided $k=1$, we have the expressions:

$$B_2 = \frac{p-1}{p}\ , \quad \frac{p-1}{p} < \frac{p}{p+1}, \quad p\geq 3.$$

Since $B_{2k} > B_{2k+1}$, then condition (30) holds for all $k\geq 1$.
Thus, for the equation (23), we have the inequalities (26) and (27).
The equality (23), we presented in the form:

$$\begin{aligned} & u^{2k}\cdot\left((2k+1)\cdot p\cdot R_{2k} - 2k\cdot u\cdot R_{2k+1}\right) = \\ & = \gamma^{2k}\cdot\left((2k+1)\cdot p - 2k\cdot\gamma\right), \quad k\geq 1. \end{aligned} \qquad (31)$$

The equality (31) must hold for all values of $k\geq 1$.
We represent the expression (31) as follows:

$$\frac{\gamma}{u} = \sqrt[2k]{D_k}\ , \quad k\geq 1. \qquad (32)$$

The function D_k is:

$$D_k = \frac{(2k+1)\cdot p \cdot R_{2k} - 2k \cdot u \cdot R_{2k+1}}{(2k+1)\cdot p - 2k\cdot\gamma}, \quad k \geq 1. \tag{33}$$

Under the condition $k=1$, we have the expressions:

$$R_2 = 1, \quad R_3 = \frac{1}{2}(3p-1) \ ,$$

$$\frac{\gamma}{u} = \left(\frac{3p-(3p-1)\cdot u}{3p-2\gamma}\right)^{\frac{1}{2}}, \quad p \geq 3. \tag{34}$$

Analysis (34) we give below, in Paragraph 3.

The left-hand side of (32) and (34), for the variables (y, A, V) , has of the form:

$$\frac{\gamma}{u} = \left(\frac{pV}{y}\right)\cdot\left(\frac{y}{A}\right)^{p}, \quad p \geq 3. \tag{35}$$

For integers (y, A, V) , the left-hand side of (32) is a rational number, and the right-hand side is a radical degree $2k$ of rational numbers D_k , $k \geq 1$. However, for equality (32), a radical degree $2k$ must be a rational number for all values of $k \geq 1$. This requirement is actually not feasible for an infinite sequence $k = 1, 2, 3, \ldots$. In other words, the radical in the right-hand side of (32), at least for one of $k \geq 1$, is an irrational quantity. So, for integers (y, A, V) , equality (32) is impossible: the left-hand side is a rational number, and the right-hand is an irrational quantity.

In this case, according to (35), at least one of the variables (y, V, A) is an irrational quantity. Since, by hypothesis, (V, A) - integers, then the variable y is an irrational number.

So, equation (1) has no solution in integers (y, A, V) . Thus, equation (1.1) also has no solution in integers (x, y, z) . If in equation (1.1) x is integer, then at least one of the variables (y, z) is an irrational quantity. Therefore, Fermat's Last Theorem is true.

3. AUXILIARY EQUATION OF THE THIRD DEGREE

1. Consider the equality (2.34):

$$\frac{\gamma}{u} = \left(\frac{3p \; - \; (3p-1)\cdot u}{3p \; - \; 2v} \right)^{\frac{1}{2}}, \quad p \geq 3, \qquad (1)$$

$$0 < v < u < 1.$$

The variables (v, u) are:

$$v = \frac{pV}{y}, \qquad u = \left(\frac{A}{y} \right)^{p}, \quad p \geq 3 \qquad (2)$$

Taking into account (2), we represent equation (1) in the form:

$$3p\cdot\left(\frac{pV}{y}\right)^{2} - 2\left(\frac{pV}{y}\right)^{3} = 3p\cdot\left(\frac{A}{y}\right)^{2p} - 2(3p-1)\cdot\left(\frac{A}{y}\right)^{3p}. \qquad (3)$$

We write equation (3) with respect to the variable y :

$$3p^{3}\cdot V^{2}\cdot y^{3p-2} - 2p^{3}\cdot V^{3}\cdot y^{3p-3} - 3p\cdot A^{2p}\cdot y^{p} + (3p-1)\cdot A^{3p} = 0. \quad (4)$$

Equation (4) is a transcendental equation of degree $(3p-2)$ with respect to the change y .

According to (4), the equation for the number of A is:

$$(3p-1)\cdot A^{3p} - 3p\cdot y^{p}\cdot A^{2p} + 3p^{3}\cdot y^{3p-2}\cdot V^{2}\cdot\left(1-\frac{2V}{3y}\right) = 0. \quad (5)$$

By (1.7), from (5) we obtain the equation for the number of U :

$$(3p-1)\cdot V\cdot U^{3} - 3p\cdot y^{p}\cdot U^{2} + 3p^{3}\cdot y^{3p-2}\cdot\left(1-\frac{2V}{3y}\right) = 0. \qquad (6)$$

Equation (6) is a cubic equation for the number of U .

2. Consider the equation (6).

Equation (6), we will present in the form of a cubic equation:

$$t^3 + 3h\cdot t + 2q = 0\,, \tag{7}$$

$$U = t + \sigma\,, \qquad \sigma = \frac{p\cdot y^p}{(3p-1)\cdot V}\,, \quad \sigma > 1 \tag{8}$$

$$h = -\sigma^2\,, \qquad q = -(1-\lambda)\cdot\sigma^3\,, \tag{9}$$

$$\lambda = \frac{3}{2}\left(1 - \frac{2V}{3y}\right)\cdot\left(\frac{(3p-1)\cdot V}{y}\right)^2. \tag{10}$$

For the solution of (7), we apply the method of auxiliary variables $(\rho\,,\varphi)$, (Chapter 2.4., [3]).

The numbers of real roots of equation (7) depends on the sign of the discriminate D :

$$D = q^2 + h^3 = -\lambda\cdot(2-\lambda)\cdot\sigma^6\,. \tag{11}$$

Auxiliary value ρ is equal to:

$$\rho = \pm\sqrt{|h|} = \pm\sigma\,. \tag{12}$$

Sign ρ must coincide with the sign of q . Auxiliary value φ is determined depending on the signs of $(h\,,D)$. According to (9), we have $h<0$.

Provided that $D\le 0$, the value of φ and the solutions of equation (7) are:

$$\cos\varphi = \beta\,, \qquad \beta = \frac{q}{\rho^3}\,, \tag{13}$$

$$t_1 = -\,2\rho\cdot\cos\left(\frac{\varphi}{3}\right), \tag{14a}$$

$$t_i = +\,2\rho\cdot\cos\left(\frac{\pi}{3} \mp \frac{\varphi}{3}\right), \quad i = 2, 3\,. \tag{14b}$$

Under the condition $D > 0$, we have the expressions:

$$ch\,\varphi = \beta, \qquad \beta = \frac{q}{\rho^3}, \tag{15}$$

$$t_1 = -2\rho \cdot ch\left(\frac{\varphi}{3}\right). \tag{16}$$

The solutions (t_2, t_3) are imaginary.

A. Provided that $\lambda > 2$, we have the following values:

$$q > 0, \quad D > 0, \quad \rho = +\sigma, \quad \beta = \lambda - 1 .$$

According to (15), the value φ is:

$$\varphi = \ln\left(\lambda - 1 + \sqrt{\lambda \cdot (\lambda - 2)}\right), \quad \lambda > 2, \quad \varphi > 0. \tag{17}$$

According to (8), the real solution of equation (6) has the form:

$$U = -\sigma \cdot \left(2ch\frac{\varphi}{3} - 1\right), \quad \sigma > 0, \quad \varphi > 0. \tag{18}$$

So, provided $\lambda > 2$, the equation (6) has no positive solutions.
According to (10), condition $\lambda > 2$ can be written as:

$$\frac{2\sqrt{3}}{9} < \frac{pV}{y} < 1 .$$

B. Provided that $\lambda = 2$, we have the following values:

$$q = \sigma^3, \quad D = 0, \quad \rho = +\sigma, \quad \beta = 1, \quad \varphi = 0 .$$

In this case, the solutions (14a) - (14b) are:

$$t_1 = -2\sigma, \quad t_2 = t_3 = \sigma .$$

In view of (8), the solutions of equation (6) are:

$$U_1 = -\sigma , \quad U_2 = U_3 = 2\sigma .$$

Thus, equation (6) has one positive solution:

$$U = 2\sigma . \qquad (19)$$

With (8), the solution (19), we will present in the form of:

$$U = \frac{2p \cdot y^p}{(3p-1) \cdot V} . \qquad (20)$$

Using (1.7) and (20), we obtain an expression for A^p :

$$A^p = \frac{2p \cdot y^p}{(3p-1)} , \quad p \geq 3 . \qquad (21)$$

We will present (21) in the form:

$$\frac{A}{y} = \left(\frac{2p}{3p-1} \right)^{\frac{1}{p}} , \quad p \geq 3 . \qquad (22)$$

The right-hand side of (22) is an irrational quantity, by definition. Consequently, at least one of the variables (y, A) is an irrational number.

So, provided $\lambda = 2$, the equation (5) has no solution in integers (y, A) .

According to (10), condition $\lambda = 2$ can be written as:

$$\frac{3}{4} \cdot \left(\frac{3p-1}{p} \right)^2 \cdot \left(\frac{pV}{y} \right)^2 \cdot \left(1 - \frac{2}{3p} \cdot \frac{pV}{y} \right) = 1 .$$

C. Provided that $\lambda = 1$, we have the following values:

$$q = 0 , \; D < 0 , \; \rho = +\sigma , \quad \beta = 0 , \; \varphi = \frac{\pi}{2} .$$

In this case, the solutions (14a) - (14b) are:

$$t_1 = -\sqrt{3}\cdot\sigma, \quad t_2 = +\sqrt{3}\cdot\sigma, \quad t_3 = 0. \qquad (23)$$

In view of (8) and (23), the solutions of equation (6) are:

$$U_1 = -\sigma\cdot\left(\sqrt{3} - 1\right),$$

$$U_2 = +\sigma\cdot\left(\sqrt{3} + 1\right), \qquad (24)$$

$$U_3 = \sigma.$$

According to (24), equation (6) has only one rational solution:

$$U = \sigma . \qquad (25)$$

Using (1.7) and (8), we will present the solution (25) in the form:

$$\frac{A}{y} = \left(\frac{p}{3p-1}\right)^{\frac{1}{p}}, \quad p \geq 3 . \qquad (26)$$

The right-hand side of (26) is an irrational quantity, by definition. Consequently, at least one of the variables (y, A) is an irrational number.

So, provided $\lambda = 1$, the equation (5) has no solution in integers (y, A).

According to (10), condition $\lambda = 1$ can be written as:

$$\frac{3}{2}\cdot\left(\frac{3p-1}{p}\right)^2\cdot\left(\frac{pV}{y}\right)^2\cdot\left(1 - \frac{2}{3p}\cdot\frac{pV}{y}\right) = 1 .$$

D. Provided that $1 < \lambda < 2$, we have the following values:

$$q > 0, \; D < 0, \quad \rho = +\sigma, \quad \beta = \lambda - 1 ,$$

$$\varphi = arc\cos\left(\lambda - 1\right), \quad 0 < \lambda - 1 < 1. \qquad (27)$$

Solutions (14a) – (14b) take the form:

$$t_1 \;=\; -\;2\sigma\cdot\cos\left(\frac{\varphi}{3}\right), \qquad \text{(28a)}$$

$$t_i \;=\; +\;\sigma\cdot\left(\cos\frac{\varphi}{3} \;\pm\; \sqrt{3}\cdot\sin\frac{\varphi}{3}\right), \quad i=2,3. \qquad \text{(28b)}$$

In view of (27), for the parameter of φ , we have the inequality:

$$0 \;<\; \frac{\varphi}{3} \;<\; \frac{\pi}{6},$$

$$\frac{\sqrt{3}}{2} \;<\; \cos\frac{\varphi}{3} \;<\; 1, \qquad 0 \;<\; \sqrt{3}\cdot\sin\frac{\varphi}{3} \;<\; \frac{\sqrt{3}}{2}. \qquad \text{(29)}$$

Using (8) and (28), we will present the solutions of equation (6) in the form:

$$U \;=\; \sigma\cdot\left(1 \;+\; \frac{t_i}{\sigma}\right), \quad i=1,2,3 \;. \qquad \text{(30)}$$

By (29), the functions $\left(\cos\frac{\varphi}{3},\; \sin\frac{\varphi}{3}\right)$, under the condition $\left(0 \;<\; \frac{\varphi}{3} \;<\; \frac{\pi}{6}\right)$, take only the irrational values. Since σ is a rational number, then the solutions (t_1, t_2, t_3) are the irrational values. Hence, according to (30), U is also an irrational number.

In view of (1.7), (8) and (30), we obtain the following expression:

$$\frac{A}{y} \;=\; \left(\frac{p}{3p-1}\right)^{\frac{1}{p}}\cdot\left(1 \;+\; \frac{t_i}{\sigma}\right)^{\frac{1}{p}}, \quad p\geq 3, \quad i=1,2,3 \;. \qquad \text{(31)}$$

The right-hand side of (31) is an irrational quantity, by definition. Consequently, at least one of the variables (y, A) is an irrational number.

So, provided $1<\lambda<2$, the equation (5) has no solution in integers (y, A) .

According to (10), condition $1<\lambda<2$ can be written as:

$$\frac{\sqrt{6}}{9} \;<\; \frac{pV}{y} \;<\; \frac{2\sqrt{3}}{9} \quad .$$

E. Provided that $0<\lambda<1$, we have the following values:

$$q\ <\ 0\,,\ D\ <\ 0\,,\quad \rho=-\,\sigma\,,\qquad \beta\ =1\ -\ \lambda\quad ,$$

$$\varphi\ =\ arc\cos\left(1\ -\ \lambda\ \right),\quad 0<\ 1\ -\ \lambda\ <1\,. \qquad (32)$$

Solutions (14a) – (14b) take the form:

$$t_1\ =\ +\ 2\sigma\cdot\cos\left(\frac{\varphi}{3}\right), \qquad (33a)$$

$$t_i\ =\ -\ \sigma\cdot\left(\ \cos\frac{\varphi}{3}\ \pm\ \sqrt{3}\cdot\sin\frac{\varphi}{3}\ \right),\quad i=2,3\,. \qquad (33b)$$

In view of (32), for the parameter of φ , we have the inequalities (29).
In view of (8) and (33), solutions of equation (6) have the form (30).
The parameter $\left(\frac{A}{y}\right)$ is defined by expression (31).
So, provided $0<\lambda<1$, the equation (5) also has no solution in integers $(y\,,A)$.
According to (10), condition $0<\lambda<1$ can be written as:

$$0\ <\ \frac{pV}{y}\ <\ \frac{\sqrt{6}}{9}\,,\quad p\geq 3\,.$$

3. According to (2), the left-hand side of equation (1) there is a proper fraction (2.35):

$$\frac{\gamma}{u}\ =\ \left(\frac{pV}{y}\right)\cdot\left(\frac{y}{A}\right)^p,\quad p\geq 3\,,\quad 0<\gamma<u<1\,.$$

Since equation (5) has no solution in integers $(y\,,A)$, then for integers $(\ y\,,A\,,V\ )$ the equality (1) is not feasible, the left-hand side there is a proper fraction, while the right-hand side is an irrational quantity, by definition. Consequently, for integers $(\ y\,,A\,,V\ )$ and under the condition $k\geq 1$, the equality (2.32) also is impossible: the left-hand side there is a rational number, and the right side is an irrational quantity.

So, equation (1.12) has no integer solutions $(\ y\,,A\,,V\ )$. Thus, equation (1.1) also has no solution in integers $(\,x\,,\,y\,,\,z\,)$. If in the equation (1.1) x is integer, then at least one of the variables $(\,y\,,z\,)$ is an irrational quantity. Therefore, Fermat's Last Theorem is true.

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June 2011. Tartu

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