NOT SO RATIONAL
A Cantorian argument on the rationality of the rational line

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Abstract. In the year 1874 Cantor proved the set of rational numbers is denumerable. An immediate consequence of this result is the impossibility of non-countable partitions of the real line, also proved by Cantor in 1885. Inspired by Cantor 1874 and 1885 proofs, the following argument defines a partition of an interval of positive rational numbers whose successive parts are defined a la Cantor by means of the successive elements of an w-ordered sequence of positive rational numbers that contains all positive rational numbers. It is then proved each part of the partition contains positive rational numbers that are not members of the defining sequence.

1. Introduction
1. For the purpose of the following discussion, a partition \( P_{ab} \) of a right closed interval \((a, b]\) of positive rational numbers will be any set of right-closed intervals of positive rational numbers which are also disjoint and successively adjacent, and whose union is the whole interval:

\[
P_{ab} = \{\ldots (q_i, q_{i+1}], \ (q_{i+1}, q_{i+2}], \ (q_{i+2}, q_{i+3}] \ldots \}
\]

\[
\bigcup_i (q_i, q_{i+1}] = (a, b]
\]

As a consequence of their adjacency, the right endpoint of each interval coincides with the left endpoint of the next one, provided that a next interval does exist. As a consequence of their disjointness, each endpoint can only belong to a different interval. Therefore, it holds:

\[
(q_i, q_{i+1}] \cup (q_{i+1}, q_{i+2}] = (q_i, q_{i+2}]
\]

\[
(q_i, q_{i+1}] \cap (q_{j+1}, q_{j+2}] = \emptyset, \ \forall i, \forall j
\]

\[
\ldots < q_i < q_{i+1} < q_{i+2} < \ldots
\]

where ’\(<\)’ stands for the natural order of precedence of the rational numbers.

2. As is well known, the set of rational numbers in their natural order of precedence is densely ordered. So, if \(a\) and \(b\) are any two different rational numbers, then the interval \((a, b]\) contains infinitely many different rational numbers, no matter how close \(a\) and \(b\) are. Or in other words (and contrary to what happens with any natural number in the sequence of natural numbers 1, 2, 3…), no rational number has an immediate successor in the natural order of precedence of rational numbers. This trivial property of rational numbers will be of capital importance in the following argument.
2. A partition a la Cantor

3. Let \( f \) be a one to one correspondence between the set \( \mathbb{N} \) of natural numbers and the denumerable set \( \mathbb{Q}^+ \) of positive rational numbers, and consider the sequence \( \langle q_n \rangle \) defined by

\[
\forall i \in \mathbb{N} : q_i = f(i)
\]

Since \( f \) is a one to one correspondence, it is quite clear \( \langle q_n \rangle \) contains all positive rational numbers. Obviously, \( f \) induces an \( \omega \)-order\(^1\) in \( \mathbb{Q}^+ \) that makes it possible to consider successively all of their elements: \( q_1, q_2, q_3 \ldots \), which in turn makes it possible the next definition.

4. Let \( (a, b) \) be any right closed interval of positive rational numbers. By following the same strategy as in Cantor’ 1874 argument \([1]\), we will now define a partition of \( (a, b) \) means of the successive elements \( q_1, q_2, q_3 \ldots \) of the sequence \( \langle q_n \rangle \) in the following way:

- Starting from \( q_1 \) and following the order \( q_1, q_2, q_3 \ldots \) find the first element \( q_e \) of \( \langle q_n \rangle \) in \( (a, b) \).
- Define the adjacent and disjoint intervals \( (a, q_e], (q_e, b] \). Obviously \( (a, q_e] \cup (q_e, b] = (a, b] \).
- Starting from \( q_{e+1} \) and following the order \( q_{e+1}, q_{e+2}, q_{e+3} \ldots \) find the first element \( q_j \) of \( \langle q_n \rangle \) in \( (a, b) \).
- Since the union of the adjacent and disjoint intervals previously defined is \( (a, b) \), \( q_j \) must belong to one of those intervals. Replace that interval with the intervals \( (x, q_j] \) and \( (q_j, y) \), where \( x \) and \( y \) are respectively the left and the right endpoint of the replaced interval. Obviously \( (x, q_j] \cup (q_j, y) = (x, y] \).
- Starting from \( q_{j+1} \) and following the order \( q_{j+1}, q_{j+2}, q_{j+3} \ldots \) find the first element \( q_m \) of \( \langle q_n \rangle \) in \( (a, b) \).
- Since the union of all the adjacent and disjoint intervals previously defined is \( (a, b) \), \( q_m \) must belong to one of those intervals. Replace that interval with the intervals \( (x, q_m] \) and \( (q_m, y] \), where \( x \) and \( y \) are respectively the left and the right endpoint of the replaced interval. Obviously \( (x, q_m] \cup (q_m, y] = (x, y] \).
- And so on.

5. Once all the successive elements \( q_1, q_2, q_3 \ldots \) of \( \langle q_n \rangle \) have been considered, we will have a partition \( P_{ab} \) of the rational interval \( (a, b) \). It is in fact a partition:

- The union of all intervals of the set \( P_{ab} \) is the interval \( (a, b] \) since in each replacement (including the first replacement of \( (a, b] \)) the replaced interval is the union of the replacing intervals.
- Each replaced interval \( (x, y] \) is always replaced by two adjacent and disjoint intervals \( (x, z] \) and \( (z, y] \), and so that their union \( (x, z] \cup (z, y] \) has the same endpoints \( x \) and \( y \) as the replaced interval \( (x, y] \). In consequence all replacements maintain the adjacency and disjointness of the replaced interval.
- The intersection of any two elements of \( P_{ab} \) is empty because in each replacement the intersection of the replacing intervals is always empty because the replacing intervals are always disjoint.

6. The partition \( P_{ab} \) will necessarily contain an interval whose left endpoint is \( a \), because all replacements (including the first replacement of \( (a, b] \)) maintain the endpoints of the

\(^1\)The ordering of a well ordered set whose ordinal is \( \omega \), as the set \( \mathbb{N} \) of natural numbers in their natural order of precedence.
replaced interval in the new replacing intervals. Let \((a, q_s]\) be that interval. Since all rational intervals are densely ordered, between \(a\) and \(q_s\) infinitely many different rationals do exist. Let \(q \neq q_s\) be any element of \((a, q_s]\). Obviously \(q\) can only belong to \((a, q_s]\) because all \(P_{ab}\) intervals are disjoint to each other. It is also clear \(q\) is a positive rational number. But it cannot be an element \(q_v\) of the sequence \(\langle q_n \rangle\) because if that were the case \(q_v\) would be an element of \((a, q_s]\) and only of \((a, q_s]\), and then the interval \((a, q_s]\) would have been replaced with the intervals \((a, q_v], (q_v, q_s]\) when considering \(q_v\) in Definition 4. The same argument can be applied to any other interval of the partition \(P_{ab}\). This proves the sequence \(\langle q_n \rangle\), that contains all positive rational numbers, does not contain all positive rational numbers.

7. Notice, on the other hand, that all rational numbers in \((a, b]\) are elements of \(\langle q_n \rangle\), therefore all of them will have been considered by Definition 4 to define \(P_{ab}\), which means every element in \((a, b]\) is the endpoint of two adjacent and disjoint intervals of \(P_{ab}\). In consequence, all those intervals are empty intervals whose endpoints are two different rational numbers, which is incompatible with the dense ordering of \((a, b]\). This contradiction suggests the impossibility for a denumerable set (i.e. one that can be \(\omega\)-ordered) to be densely ordered.

8. Unnecessary as it may seem, let me recall at this point the following words by Wilfrid Hodges [9, p. 4]:

How does anybody get into a state of mind where they persuade themselves that you can criticize an argument by suggesting a different argument which doesn’t reach the same conclusion?

In effect, if two different arguments are formally correct but they reach contradictory conclusions they are proving a contradiction, surely derived from an inconsistent common assumption. So, the only way to reject an argument is to indicate where and why that argument fails.

3. DISCUSSION

9. Cantor’s Beiträge (‘Contributions’)\(^2\), published in 1895 (Part I, [5]) and 1897 (Part II, [6]) contain the fundaments of the theory of transfinite cardinals and ordinals. Epigraph 6 of the first article begins by assuming the existence of the set of all finite cardinals as a complete totality (although rather than an assumption it is introduced as an example of ‘transfinite aggregate’ whose existence as a complete totality Cantor took for granted). This implicit assumption (equivalent to our modern Axiom of Infinity) is the only assumption in Cantor’s theory of transfinite numbers. From it, Cantor successfully derived the existence of increasing transfinite ordinals (Theorems §15 A-K) and cardinals (Theorems §16 D-F). The consistency of Cantor theory rests, therefore, on the consistency of that unique foundational assumption.

10. In 1874 Cantor proved for the first time the set of real numbers is not denumerable [1], [2], [8]. Two of the three final alternatives of Cantor’s proof could also be applied to the set of rational numbers [10, pp. 37-44]. In consequence, it is necessary to prove the conflicting alternatives are never satisfied in the case of the set of rational numbers. Otherwise that set would and would not be denumerable. Until now, and as far as I know, this problem has not even been raised.

\(^2\)English translation [7].
In 1891 Cantor proved for the second time the set of real numbers is not denumerable, now by his celebrated diagonal method, an impeccable Modus Tollens [4]. Cantor antidiagonal is a real number in the real interval \((0, 1)\), and being real it will be either rational or irrational. If it were rational we would have the same problem as with Cantor’s 1874 argument. So, it should be formally proved that no permutation of the \(\aleph_0\) rows of Cantor’s table yields a rational diagonal (rational antidiagonals are immediately derived from rational diagonals).

The above referred Cantor’s 1874 argument begins by proving the set of algebraic numbers (and then the set of rational numbers) is denumerable. Some years after, in 1885, Cantor published an immediate consequence of this result: non-denumerable partitions of the real line are impossible, for the sole reason that if they were possible the set of rational numbers would be non-denumerable [3]. So, as in the cases of Cantor’s 1874 and 1891 arguments and for the same reasons, we should prove the impossibility of non-countable partitions of the real line by means of an argument independent of Cantor immediate proof.

In conclusion, and in order to ensure set theory is free of inconsistencies related to the cardinality of the set of rational numbers, Cantor’s 1784, 1885 and 1891 arguments should be completed in the sense indicated in 10-12.

On the other hand, the above argument 3-6 proves the sequence \(\langle q_n \rangle\) contains and does not contain all positive rational numbers. Which can only mean the set of rational numbers is and is not denumerable. If that were the case, and in agreement with 9, the assumed existence of the infinite sets as complete totalities would be inconsistent, simply because that assumption is the only assumption of the theory of transfinite numbers.

**References**