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Basic Multivector Calculus

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Abstract: We begin with introducing the generalization of real, complex, and quaternion numbers to hypercomplex numbers, also known as Clifford numbers, or *multivectors* of geometric algebra. Multivectors encode everything from vectors, rotations, scaling transformations, improper transformations (reflections, inversions), geometric objects (like lines and spheres), spinors, and tensors, and the like. Multivector calculus allows to define functions mapping multivectors to multivectors, differentiation, integration, function norms, multivector Fourier transformations and wavelet transformations, filtering, windowing, etc. We give a *basic introduction* into this general mathematical language, which has fascinating applications in physics, engineering, and computer science.

1. Introduction

"Now faith is being sure of what we hope for and certain of what we do not see. This is what the ancients were commended for. By faith we understand that the universe was formed at God's command, so that what is seen was not made out of what was visible." [7]

The German 19th century mathematician H. Grassmann had the clear vision, that his "extension theory (now developed to geometric calculus) … forms the keystone of the entire structure of mathematics."[6] The algebraic "grammar" of this universal form of calculus is *geometric algebra* (or Clifford algebra). That geometric calculus is a truly unifying approach to all of calculus will be demonstrated here by developing some basics of the *vector differential calculus* part of geometric calculus.

The basic geometric algebra[1,2] necessary for this is compiled in section 2. Then section 3 develops vector differential calculus with the help of few simple definitions. This approach is generically coordinate free, and fully shows both the concrete and abstract geometric and algebraic beauty of the "keystone" of mathematics. The full generalization to multivector calculus is shown in [2,10], applications in [11].

2. Basic Geometric Algebra

This section is a basic *summary* of important relationships in geometric algebra. This summary mainly serves as a reference section for the *vector differential calculus* to be developed in the following section. Most of the relationships listed here are to be found in the synopsis of geometric algebra and in chapters 1 and 2 of [1], as well as in chapter 1 of [2], together with relevant proofs. Beyond that [1] and [2] follow a much more didactic approach for newcomers to geometric algebra.

$\mathcal{G}(I)$ is the full *geometric algebra* over all vectors in the n-dimensional unit *pseudoscalar* $I = \vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n$. $\mathcal{A}_n \equiv \mathcal{G}^1(I)$ is the n-dimensional vector sub-space of grade-1 elements in $\mathcal{G}(I)$ spanned by $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$. For vectors $\vec{a}, \vec{b}, \vec{c} \in \mathcal{A}_n \equiv \mathcal{G}^1(I)$ and scalars $\alpha, \beta, \lambda, \tau$; $\mathcal{G}(I)$ has the fundamental properties of

- associativity

$$\vec{a}(\vec{b}\vec{c}) = (\vec{a}\vec{b})\vec{c}, \; \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}, \tag{1}$$

- commutativity $\alpha\vec{a} = \vec{a}\alpha, \;\; \vec{a} + \vec{b} = \vec{b} + \vec{a},$ (2)
- distributivity

$$\vec{a}(\vec{b} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}, \;\; (\vec{b} + \vec{c})\vec{a} = \vec{b}\vec{a} + \vec{c}\vec{a}, \tag{3}$$

- linearity $\alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b} = (\vec{a} + \vec{b})\alpha,$ (4)
- scalar square (vector length $|\vec{a}|$)

$$\vec{a}^2 = \vec{a}\vec{a} = \vec{a} \cdot \vec{a} = |\vec{a}|^2. \tag{5}$$

The *geometric product* $\vec{a}\vec{b}$ is related to the (scalar) *inner product* $\vec{a} \cdot \vec{b}$ and to the (bivector or 2-vector) *outer product* $\vec{a} \wedge \vec{b}$ by

$$\vec{a}\vec{b} = \vec{a}\cdot\vec{b} + \vec{a}\wedge\vec{b}\,, \qquad (6)$$

with

$$\vec{a}\cdot\vec{b} = \frac{1}{2}(\vec{a}\vec{b}+\vec{b}\vec{a}) = \vec{b}\cdot\vec{a} = \vec{a}\vec{b} - \vec{a}\wedge\vec{b} \overset{(13)}{=} \langle \vec{a}\vec{b}\rangle_0, \quad (7)$$

$$\vec{a}\wedge\vec{b} = \frac{1}{2}(\vec{a}\vec{b}-\vec{b}\vec{a}) = -\vec{b}\wedge\vec{a} = \vec{a}\vec{b} - \vec{a}\cdot\vec{b} \overset{(13)}{=} \langle \vec{a}\vec{b}\rangle_2 \quad (8)$$

The inner and the outer product are both linear and distributive

$$\vec{a}\cdot(\alpha\vec{b}+\beta\vec{c}) = \alpha\vec{a}\cdot\vec{b} + \beta\vec{a}\cdot\vec{c}\,, \qquad (9)$$

$$\vec{a}\wedge(\alpha\vec{b}+\beta\vec{c}) = \alpha\vec{a}\wedge\vec{b} + \beta\vec{a}\wedge\vec{c}\,. \qquad (10)$$

A unit vector $\hat{a}$ in the direction of $\vec{a}$ is

$$\hat{a} \equiv \frac{\vec{a}}{|\vec{a}|}, \text{ with } \hat{a}^2 = \hat{a}\hat{a} = 1, \; \vec{a} = \hat{a}|\vec{a}|. \qquad (11)$$

The inverse of a vector is

$$\vec{a}^{-1} = \frac{1}{\vec{a}} \equiv \frac{\vec{a}}{\vec{a}^2} = \frac{\vec{a}}{|\vec{a}|^2} = \frac{\hat{a}}{|\vec{a}|}. \qquad (12)$$

A multivector A can be uniquely decomposed into its homogeneous grade k parts ($\langle\;\rangle_k$ grade k selector):

$$A = \underbrace{\langle A\rangle_0}_{scalar} + \underbrace{\langle A\rangle_1}_{vector} + \underbrace{\langle A\rangle_2}_{bivector} + \ldots + \underbrace{\langle A\rangle_k}_{k-vector} + \ldots + \underbrace{\langle A\rangle_n}_{\substack{pseudo\\ scalar}} \quad (13)$$

If A is homogeneous of grade k one often simply writes

$$A = \langle A\rangle_k = A_k\,. \qquad (14)$$

Grade selection is invariant under scalar multiplication

$$\lambda\langle A\rangle_k = \langle \lambda A\rangle_k\,. \qquad (15)$$

The consistent definition of inner and outer products of vectors $\vec{a}$ and r-vectors A_r is

$$\vec{a}\cdot A_r \equiv \langle \vec{a}A_r\rangle_{r-1} = \frac{1}{2}(\vec{a}A_r - (-1)^r A_r\vec{a}), \qquad (16)$$

$$\vec{a}\wedge A_r \equiv \langle \vec{a}A_r\rangle_{r+1} = \frac{1}{2}(\vec{a}A_r + (-1)^r A_r\vec{a}) \qquad (17)$$

By linearity the full geometric product of a vector and a multivector A is then

$$\vec{a}A = \vec{a}\cdot A + \vec{a}\wedge A\,. \qquad (18)$$

This extends to the distributive multiplication with arbitrary multivectors A, B

$$\vec{a}(A+B) = \vec{a}A + \vec{a}B\,. \qquad (19)$$

The inner and outer products of homogeneous multivectors A_r and B_s are defined ([2], p. 6, (1.21), (1.22)) as

$$A_r\cdot B_s \equiv \langle A_r B_s\rangle_{|r-s|} \text{ for } r,s>0\,, \qquad (20)$$

$$A_r\cdot B_s \equiv 0 \text{ for } r=0 \text{ or } s=0\,, \qquad (21)$$

$$A_r\wedge B_s \equiv \langle A_r B_s\rangle_{r+s}\,, \qquad (22)$$

$$A_r\wedge\lambda = \lambda\wedge A_r = \lambda A_r \text{ for scalar } \lambda\,. \qquad (23)$$

The inner (and outer) product is again linear and distributive

$$(\lambda A_r)\cdot B_s = A_r\cdot(\lambda B_s) = \lambda(A_r\cdot B_s) = \lambda A_r\cdot B_s \qquad (24)$$

$$A_r\cdot(B_s + C_t) = A_r\cdot B_s + A_r\cdot C_t\,, \qquad (25)$$

$$\lambda(B_s + C_t) = \lambda B_s + \lambda C_t\,. \qquad (26)$$

The *reverse* of a multivector is

$$\tilde{A} = \sum_{k=1}^{n}(-1)^{k(k-1)/2}\langle A\rangle_k. \qquad (27)$$

[2] uses a *dagger* instead of the *tilde*.
Special examples are

$$\tilde{\lambda} = \lambda\,,\; \tilde{\vec{a}} = \vec{a}\,,\; (\vec{a}\wedge\vec{b})^{\sim} = \vec{b}\wedge\vec{a} = -\vec{a}\wedge\vec{b}\,, \ldots \quad (28)$$

The scalar *magnitude* $|A|$ of a multivector A is

$$|A|^2 \equiv \underbrace{\tilde{A}*A}_{\text{scalar product}} \equiv \langle A\rangle_0^{\,2} + \sum_{r=1}^{n}\langle\tilde{A}\rangle_r\cdot\langle A\rangle_r\,, \qquad (29)$$

where the separate term $\langle A\rangle_0^{\,2}$ is in particular due to the definition of the inner product in [2], p. 6, (1.21). The magnitude allows to define the inverse for simple k-blade vectors

$$A^{-1} \equiv \frac{\tilde{A}}{|A|^2}\,, \text{ with } A^{-1}A = AA^{-1} = 1. \qquad (30)$$

Alternative ways to express $\vec{a}\in\mathcal{A}_n \equiv \mathcal{G}^1(I)$ are

$$I\wedge\vec{a} = 0 \text{ or } I\vec{a} = I\cdot\vec{a}\,. \qquad (31)$$

The projection of $\vec{a}$ into $\mathcal{A}_n \equiv \mathcal{G}^1(I)$ is

$$P_I(\vec{a}) = P(\vec{a}) \equiv \sum_{k=1}^{n} \vec{a}^k \vec{a}_k \cdot \vec{a} = \sum_{k=1}^{n} \vec{a}_k \vec{a}^k \cdot \vec{a}, \quad (32)$$

where $\vec{a}^k$ is the *reciprocal frame* defined by

$$\vec{a}^k \cdot \vec{a}_j = \delta_j^k = \text{Kronecker delta} = \begin{cases} 1 \text{ if } & j = k \\ 0 \text{ if } & j \neq k \end{cases}. \quad (33)$$

A general convention is that inner products $\vec{a} \cdot \vec{b}$ and outer products $\vec{a} \wedge \vec{b}$ have priority over geometric products $\vec{a}\vec{b}$, e.g.

$$\vec{a} \cdot \vec{b}\vec{c} \wedge \vec{d}\vec{e} = (\vec{a} \cdot \vec{b})(\vec{c} \wedge \vec{d})\vec{e}\,. \quad (34)$$

The projection of a multivector B on a subspace described by a simple m-vector (m-blade) $A_m = \vec{a}_1 \wedge \vec{a}_2 \wedge \ldots \wedge \vec{a}_m, m \leq n$ is

$$P_A(B) \equiv \underbrace{(B \cdot A) \cdot A^{-1}}_{general} \overset{s \leq m}{=} \underbrace{A^{-1} \cdot (A \cdot B_{(s)})}_{\text{degree dependent}}, \quad (35)$$

$$P_A(\langle B \rangle_0) \equiv \langle B \rangle_0, \quad P_A(\langle B \rangle_n) \equiv \langle B \rangle_n \cdot AA^{-1}, \quad (36)$$

the exceptions for scalars $\langle B \rangle_0$ and pseudoscalars $\langle B \rangle_n$ being again due to the definition of the inner product in [2], p. 6, (1.21). A projection of one factor of an inner product has the effect

$$\vec{a} \cdot P(\vec{b}) = P(\vec{a}) \cdot P(\vec{b}) = P(\vec{a}) \cdot \vec{b}. \quad (37)$$

For a multivector $B \in \mathcal{G}(A_m)$, with $A = A_m$ we have

$$(\vec{a} \wedge B) \cdot A = (\vec{a} \wedge B)A = \vec{a} \cdot (BA) \quad \text{if} \quad \vec{a} \wedge A = 0\,. \quad (38)$$

Reordering rules for products of homogeneous multivector are

$$A_r \cdot B_s = (-1)^{r(s-r)} B_s \cdot A_r \quad \text{for} \quad r \leq s\,, \quad (39)$$

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r\,. \quad (40)$$

Elementary combinations that occur often are

$$\vec{a} \cdot (\vec{b} \wedge \vec{c}) = (\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\vec{b} = \vec{a} \cdot \vec{b}\vec{c} - \vec{a} \cdot \vec{c}\vec{b}, \quad (41)$$

$$\begin{aligned} &(\vec{a} \wedge \vec{b}) \cdot (\vec{c} \wedge \vec{d}) = \vec{a} \cdot (\vec{b} \cdot (\vec{c} \wedge \vec{d})) = \\ &(\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) - (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}), \end{aligned} \quad (42)$$

$$\begin{aligned} &(\vec{a} \wedge \vec{b})^2 = (\vec{a} \wedge \vec{b}) \cdot (\vec{a} \wedge \vec{b}) = (\vec{a} \cdot \vec{b})^2 - \vec{a}^2\vec{b}^2 = \\ &-(\vec{b} \wedge \vec{a}) \cdot (\vec{a} \wedge \vec{b}) = -\left|\vec{a} \wedge \vec{b}\right|^2, \end{aligned} \quad (43)$$

and the *Jacobi identity*

$$\vec{a} \cdot (\vec{b} \wedge \vec{c}) + \vec{b} \cdot (\vec{c} \wedge \vec{a}) + \vec{c} \cdot (\vec{a} \wedge \vec{b}) = 0. \quad (44)$$

The *commutator product* of multivectors A,B is

$$A \times B \equiv \frac{1}{2}(AB - BA). \quad (45)$$

One useful identity using it is

$$(\vec{a} \wedge \vec{b}) \times A = \vec{a}\vec{b} \cdot A - A \cdot \vec{a}\vec{b} = \vec{a}\vec{b} \wedge A - A \wedge \vec{a}\vec{b}. \quad (46)$$

The commutator product is to be distinguished from the *cross product*, which is strictly limited to the three-dimensional Euclidean case with unit pseudoscalar I_3:

$$\vec{a} \times \vec{b} \equiv (\vec{b} \wedge \vec{a})I_3 = -(\vec{a} \wedge \vec{b})I_3. \quad (47)$$

For more on basic geometric algebra I refer to [1,2,9] and to section 3 of [3].

3. Vector Differential Calculus

This section shows how to differentiate functions on linear subspaces of the universal geometric algebra $\mathcal{G}$ by vectors. It has wide applications particularly to mechanics and physics in general [1]. Separate concepts of gradient, divergence and curl merge into a single concept of vector derivative, united by the geometric product.

The relationship of differential and derivative is clarified. The *Taylor expansion* (P. 12) is applied to important examples, yielding e.g. the *Legendre polynomials* (P. 36). The *integrability* (P. 42, etc.) of multivector functions are defined and discussed. Throughout this section a number of basic differentials and derivations are performed explicitly illustrating ease and power of the calculus. ([1]-[5]).

As for the notation: P. 7 refers to proposition 7 of this section. Def. 13 refers to definition 13 of this section. (6) refers to equation number (6) in the previous section on basic geometric algebra.

Standard definitions of continuity and scalar differentiability apply to multivector-valued functions, because the scalar product determines a unique "distance" $|A-B|$ between two elements $A,B \in \mathcal{G}(I)$.

Definition 1 (*directional derivative*)

$F = F(\vec{x})$ multivector-valued function of a vector variable $\vec{x}$ defined on an *n*-dimensional vector space $\mathcal{A}_n = \mathcal{G}^1(I)$, *I* unit pseudoscalar. $\vec{a} \in \mathcal{A}_n$.

$$\vec{a}\cdot\vec{\partial}F \equiv \frac{dF(\vec{x}+\vec{a}\,\tau)}{d\tau} = \lim_{\tau\to 0}\frac{dF(\vec{x}+\vec{a}\,\tau)-dF(\vec{x})}{\tau}$$

Nomenclature: *derivative* of *F* in the *direction* $\vec{a}$, $\vec{a}$ *-derivative* of *F*. ([1] uses $\nabla \equiv \vec{\partial}$, [2] uses $\partial \equiv \vec{\partial}$.)

Proposition 2 (*distributivity w.r.t. vector argument*)

$$(\vec{a}+\vec{b})\cdot\vec{\partial}F = \vec{a}\cdot\vec{\partial}F + \vec{b}\cdot\vec{\partial}F,\ \ \vec{a},\vec{b} \in \mathcal{A}_n$$

Proposition 3 For scalar λ

$$(\lambda\vec{a})\cdot\vec{\partial}F = \lambda(\vec{a}\cdot\vec{\partial}F)$$

Proposition 4 (*distributivity w.r.t. multivector-valued function*)

$$\vec{a}\cdot\vec{\partial}(F+G) = \vec{a}\cdot\vec{\partial}F + \vec{a}\cdot\vec{\partial}G$$

$F = F(\vec{x})$, $G = G(\vec{x})$ multivector-valued functions of a vector variable $\vec{x}$. In the notation of Def. 13:

$$\underline{F+G} = \underline{F} + \underline{G}.$$

Proposition 5 (*product rule*)

$$\vec{a}\cdot\vec{\partial}(FG) = (\vec{a}\cdot\vec{\partial}F)G + F(\vec{a}\cdot\vec{\partial}G)$$

In the notation of Def. 13: $\underline{FG} = \underline{F}G + F\underline{G}$.

Proposition 6 (*grade invariance*)

$$\vec{a}\cdot\vec{\partial}\langle F\rangle_k = \langle \vec{a}\cdot\vec{\partial}F\rangle_k$$

$\vec{a}\cdot\vec{\partial}$ is therefore said to be a *scalar differential operator.*

Proposition 7 (*scalar chain rule*)

$$\vec{a}\cdot\vec{\partial}F = (\vec{a}\cdot\vec{\partial}\lambda)\frac{dF}{d\lambda}$$

$F = F(\lambda(\vec{x}))$, $\lambda = \lambda(\vec{x})$ scalar valued function.

Proposition 8 (*identity*) $\vec{a}\cdot\vec{\partial}\vec{x} = \vec{a}$.

Proposition 9 (*constant function*) *A* independent of $\vec{x}$:

$$\vec{a}\cdot\vec{\partial}A = 0.$$

Proposition 10 (*vector length*)

$$\vec{a}\cdot\vec{\partial}|\vec{x}| = \frac{\vec{a}\cdot\vec{x}}{|\vec{x}|} = \vec{a}\cdot\hat{x}.$$

$\hat{x} \equiv \frac{\vec{x}}{|\vec{x}|}$ unit vector in the direction of $\vec{x}$ (11).

Proposition 11 (*direction function*)

$$\vec{a}\cdot\vec{\partial}\hat{x} = \frac{\vec{a} - \vec{a}\cdot\hat{x}\hat{x}}{|\vec{x}|} = \frac{\hat{x}\hat{x}\wedge\vec{a}}{|\vec{x}|}.$$

Proposition 12 (*Taylor expansion*)

$$F(\vec{x}+\vec{a}) = \exp(\vec{a}\cdot\vec{\partial})F(\vec{x}) = \sum_{k=0}^{\infty}\frac{(\vec{a}\cdot\vec{\partial})^k}{k!}F(\vec{x}).$$

Definition 13 (*continuously differentiable, differential*)

F is *continuously differentiable* at $\vec{x}$ if for each fixed $\vec{a}$ $\vec{a}\cdot\vec{\partial}F(\vec{y})$ exists and is a continuous function of $\vec{y}$ for each $\vec{y}$ in a neighborhood of $\vec{x}$.

If *F* is defined and continuously differentiable at $\vec{x}$, then, for fixed $\vec{x}$, $\vec{a}\cdot\vec{\partial}F(\vec{x})$ is a linear function of $\vec{a}$, the (*first*) *differential* of *F*.

$$\underline{F}(\vec{a},\vec{x}) = F_{\vec{a}}(\vec{x}) \equiv \vec{a}\cdot\vec{\partial}F(\vec{x}).$$

([1], p. 107 uses $F' \equiv \underline{F}$.)

Suppressing $\vec{x}$, or for fixed $\vec{x}$:

$$\underline{F} = \underline{F}(\vec{a}) = F_{\vec{a}} \equiv \vec{a}\cdot\vec{\partial}F.$$

Proposition 14 (*linearity*)

$$\underline{F}(\vec{a}+\vec{b}) = \underline{F}(\vec{a}) + \underline{F}(\vec{b})$$

λ scalar: $\underline{F}(\lambda\vec{a}) = \lambda\underline{F}(\vec{a})$

Proposition 15 (*linear approximation*)

For $|\vec{r}| = |\vec{x} - \vec{x}_0|$ sufficiently small:

$$F(\vec{x}) - F(\vec{x}_0) \approx \underline{F}(\vec{x} - \vec{x}_0) = \underline{F}(\vec{x}) - \underline{F}(\vec{x}_0).$$

Proposition 16 (*chain rule*)

$$\frac{dF}{dt}\big(\vec{x}(t)\big) = \left(\frac{d}{dt}\vec{x}(t)\right) \cdot \vec{\partial} F(\vec{x})\Big|_{\vec{x}=\vec{x}(t)}$$

Definition 17 (*vector derivative*)

Differentiation of *F* by its argument $\vec{x}$: $\vec{\partial}_{\vec{x}} F(\vec{x}) = \vec{\partial} F$,

with the differential operator $\vec{\partial}_{\vec{x}}$, assumed to

(i) have the algebraic properties of a vector in $\mathcal{A}_n \equiv \mathcal{G}^1(I)$, *I* unit pseudoscalar; and

(ii) that $\vec{a} \cdot \vec{\partial}_{\vec{x}}$ with $\vec{a} \in \mathcal{A}_n$ is $\vec{a} \cdot \vec{\partial}_{\vec{x}} F$ as in Def. 1.

Proposition 18 (*algebraic properties of* $\vec{\partial}_{\vec{x}}$)

$$I \wedge \vec{\partial}_{\vec{x}} \overset{(31)}{=} 0, \qquad I\vec{\partial}_{\vec{x}} \overset{(31)}{=} I \cdot \vec{\partial}_{\vec{x}}$$

$$\vec{\partial}_{\vec{x}} = P_I(\vec{\partial}_{\vec{x}}) \overset{(32)}{=} \sum_{k=1}^{n} \vec{a}^k \vec{a}_k \cdot \vec{\partial}_{\vec{x}},$$

where the $\vec{a}^k$ express the algebraic vector properties and the $\vec{a}_k \cdot \vec{\partial}_{\vec{x}}$ the scalar differential properties.

Definition 19 (*gradient*)

The vector field $\vec{f} = \vec{f}(\vec{x}) \equiv \vec{\partial}_{\vec{x}} \Phi(\vec{x}) = \vec{\partial}\Phi$ for a scalar function $\Phi = \Phi(\vec{x})$ is called the gradient of Φ.

Propostion 20 (*3-dimensional cross product*)

For $\vec{b}$ independent of $\vec{x} \in \mathcal{A}_3 \equiv \mathcal{G}^1(I_3)$:

$$\vec{a} \cdot \vec{\partial}(\vec{x} \times \vec{b}) = \vec{a} \times \vec{b}.$$

Proposition 21

$$\vec{a} \cdot \vec{\partial}\big(\vec{x} \cdot \langle A \rangle_r\big) = \vec{a} \cdot \langle A \rangle_r,$$

A independent of $\vec{x}$.

Proposition 22

$$\vec{a} \cdot \vec{\partial}[\vec{x} \cdot (\vec{x} \wedge \vec{b})] = \vec{a} \cdot (\vec{x} \wedge \vec{b}) + \vec{x} \cdot (\vec{a} \wedge \vec{b})$$

Proposition 23

For $\vec{x}'$ independent of $\vec{x}$ and $r \equiv |\vec{r}| = |\vec{x} - \vec{x}'|$:

$$\vec{a} \cdot \vec{\partial} r = \vec{a} \cdot \frac{\vec{r}}{r} = \vec{a} \cdot \hat{r}, \qquad \text{where } \hat{r} = \frac{\vec{r}}{r}.$$

Proposition 24 $\quad \vec{a} \cdot \vec{\partial}\hat{r} = \dfrac{\hat{r}\hat{r} \wedge \vec{a}}{r}.$

Proposition 25 $\quad \vec{a} \cdot \vec{\partial}(\hat{r} \cdot \vec{a}) = \dfrac{|\hat{r} \wedge \vec{a}|^2}{r}.$

Proposition 26 $\quad \vec{a} \cdot \vec{\partial}(\hat{r} \wedge \vec{a}) = \dfrac{\hat{r} \cdot \vec{a}\vec{a} \wedge \hat{r}}{r}.$

Proposition 27 $\quad \vec{a} \cdot \vec{\partial}|\hat{r} \wedge \vec{a}| = -\dfrac{\hat{r} \cdot \vec{a}|\hat{r} \wedge \vec{a}|}{r}.$

Proposition 28 $\quad \vec{a} \cdot \vec{\partial}\dfrac{1}{\vec{r}} = -\dfrac{1}{\vec{r}}\vec{a}\dfrac{1}{\vec{r}}.$

Proposition 29 $\quad \vec{a} \cdot \vec{\partial}\dfrac{1}{r^2} = -2\dfrac{\vec{a} \cdot \hat{r}}{r^3}.$

Proposition 30 $\quad \dfrac{1}{2}(\vec{a} \cdot \vec{\partial})^2 \dfrac{1}{r^2} = \dfrac{3(\vec{a} \cdot \hat{r})^2 - |\hat{r} \wedge \vec{a}|^2}{r^4}.$

Proposition 31

$$\frac{1}{6}(\vec{a} \cdot \vec{\partial})^3 \frac{1}{r^2} = \frac{-4(\vec{a} \cdot \hat{r})^3 + 4|\hat{r} \wedge \vec{a}|^2 \vec{a} \cdot \hat{r}}{r^5}.$$

Proposition 32 $\quad \vec{a} \cdot \vec{\partial} \log r = \dfrac{\vec{a} \cdot \vec{r}}{r^2}.$

Proposition 33

For integer *k* and $\vec{r} \neq 0$ if *k*<0 :

$$\vec{a} \cdot \vec{\partial}\vec{r}^{2k} = 2k\vec{a} \cdot \vec{r}\vec{r}^{2(k-1)}.$$

Proposition 34

For integer *k* and $\vec{r} \neq 0$ if 2*k*+1<0 :

$$\vec{a} \cdot \vec{\partial}\vec{r}^{2k+1} = \vec{r}^{2k}(\vec{a} + 2k\vec{a} \cdot \hat{r}\vec{r}).$$

Proposition 35 (*Taylor expansion of* $\dfrac{1}{\vec{x} - \vec{a}}$)

$$\frac{1}{\vec{x}-\vec{a}} = \frac{1}{\vec{x}} + \frac{1}{\vec{x}}\vec{a}\frac{1}{\vec{x}} + \frac{1}{\vec{x}}\vec{a}\frac{1}{\vec{x}}\vec{a}\frac{1}{\vec{x}} + \ldots \quad .$$

Proposition 36 (*Legendre Polynomials*)
The Legendre Polynomials P_n are defined by:

$$\frac{1}{|\vec{x}-\vec{a}|} \equiv \sum_{n=0}^{\infty} \frac{P_n(\hat{x}\vec{a})}{|\vec{x}|^{n+1}} = \sum_{n=0}^{\infty} \frac{P_n(\vec{x}\vec{a})}{|\vec{x}|^{2n+1}}.$$

The explicit first four polynomials are:

$P_0(\vec{x}\vec{a}) = 1$

$P_1(\vec{x}\vec{a}) = \vec{x}\cdot\vec{a}$

$P_2(\vec{x}\vec{a}) = \frac{1}{2}\left[3(\vec{x}\cdot\vec{a})^2 - \vec{a}^2\vec{x}^2\right]$

$= (\vec{x}\cdot\vec{a})^2 + \frac{1}{2}(\vec{x}\wedge\vec{a})^2$

$P_3(\vec{x}\vec{a}) = \frac{1}{2}\left[5(\vec{x}\cdot\vec{a})^3 - 3\vec{a}^2\vec{x}^2\vec{x}\cdot\vec{a}\right] =$

$(\vec{x}\cdot\vec{a})^3 + \frac{3}{2}\vec{x}\cdot\vec{a}(\vec{x}\wedge\vec{a})^2$

$P_n(\vec{x}\vec{a}) = |\vec{x}|^n P_n(\hat{x}\vec{a}) = |\vec{x}|^n|\vec{a}|^n P_n(\hat{x}\hat{a}).$

Definition 37 (*redefinition of differential, over-dots*)

$$\underline{F}(\vec{a}) = \vec{a}\cdot\vec{\partial}F \overset{(7)}{=} \frac{1}{2}\left(\vec{a}\vec{\partial}F + \dot{\vec{\partial}}\vec{a}\dot{F}\right),$$

where the *over-dots* indicate, that only F is to be differentiated and not $\vec{a}$.

Proposition 38 For $\vec{a} \notin \mathcal{A}_n \equiv \mathcal{G}^1(I)$, $P = P_I$:

$$\vec{a}\cdot\vec{\partial}_{\vec{x}} = \vec{a}\cdot P(\vec{\partial}_{\vec{x}}) = P(\vec{a})\cdot\vec{\partial}_{\vec{x}}\,, \quad P(\vec{a}) \in \mathcal{A}_n\,.$$

Proposition 39

$$\underline{F}(\vec{a}) = \underline{F}(P(\vec{a})) = P(\vec{a})\cdot\vec{\partial}F\ .$$

$$\underline{F}(\vec{a}) = 0\text{, if } P(\vec{a}) = 0\,.$$

Proposition 40 (*differential of composite functions*)

For $F(\vec{x}) = G(f(\vec{x}))$ and

$$f : \vec{x} \in \mathcal{A}_n = \mathcal{G}^1(I) \to f(\vec{x}) \in \mathcal{A}'_n = \mathcal{G}^1(I')$$

$$\vec{a}\cdot\vec{\partial}F = \underline{f}(\vec{a})\cdot\vec{\partial}G$$

$$\underline{F}(\vec{a}) = \underline{G}(\underline{f}(\vec{a})) \quad \text{(Def. 13)}$$

The differential of composite functions is the composite of differentials.

$$\underline{F}(\vec{x},\vec{a}) = \underline{G}(f(\vec{x}), \underline{f}(\vec{x},\vec{a})) \quad \text{(explicit)}$$

Definition 41 (*second differential*)

$$F_{\vec{a}\vec{b}}(\vec{x}) \equiv \vec{b}\cdot\dot{\vec{\partial}}\vec{a}\cdot\vec{\partial}\dot{F}(\vec{x})\,.$$

Suppressing $\vec{x}$: $F_{\vec{a}\vec{b}} \equiv \vec{b}\cdot\dot{\vec{\partial}}\vec{a}\cdot\vec{\partial}\dot{F}$.

Proposition 42 (*integrability condition*) $F_{\vec{a}\vec{b}} = F_{\vec{b}\vec{a}}$.

The second differential is a *symmetric bilinear* function of its differential arguments $\vec{a},\ \vec{b}.$

Remark: Please see [2,9,10] for more results and proofs.

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