Foundations of Multidimensional Wavelet Theory:

The Quaternion Fourier Transform and its Generalizations

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1. Basic facts about Quaternions

Gauss, Rodrigues and Hamilton’s 4D quaternion algebra H over R:

\[ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1, \]

with isomorphisms \( H \approx \text{Cl}(0,2) \approx \text{Cl}^+(3,0) \).

\( \text{Cl}^+(3,0) \) is the even subalgebra of Clifford geometric algebra \( \text{Cl}(3,0) \), with basis \( \{1, e_{12} = e_1 e_2, \, e_{13} = e_1 e_3, \, e_{23} = e_2 e_3\} \) for an orthonormal basis \( \{e_1, \, e_2, \, e_3\} \) of \( \mathbb{R}^3 \). The quaternion

\[ q = q_r + q_i + q_j + q_k \in H, \quad q_r, \, q_i, \, q_j, \, q_k \in \mathbb{R} \]

has the quaternion conjugate (reversion in \( \text{Cl}(3,0) \))

\[ q^\ast = q_r - q_i - q_j - q_k, \]

This leads to the norm of \( q \in H \)

\[ ||q|| = \sqrt{(q^\ast q)} = \sqrt{(q_r^2 + q_i^2 + q_j^2 + q_k^2)}. \]

Quaternions (and quaternion valued functions) can be split in two ways:

\[ q = q_r + iq_i + jq_j + kq_k \quad \text{or} \quad q = q_r + q_i = (q+iqj)/2 + (q-iqj)/2. \]

The second split allows to write

\[ q_\pm = \{q_r \pm q_k + i(q_i - q_j)\} (1 \pm k)/2 = (1 \pm k)\{q_r \pm q_k + j(q_i - q_j)\}/2. \]

Applying (5) and (6) to the quaternionic kernel \( K = \text{exp}(-ixu) \text{exp}(-jyv) \) gives

\[ K_\pm = \text{exp}(-i(xu \mp yv)) (1 \pm k)/2 = (1 \pm k) \text{exp}(-j(yv \mp xu))/2. \]

For 2D quaternion valued functions \( f, g \) we can define the inner product (\( x = xe_1 + ye_2 \))

\[ (f, g) = \int f(x)g^\ast(x) \, dx dy, \]

with real scalar part

\[ \langle f, g \rangle = \int f(x)g^\ast(x) \, dx dy, \]

and norm

\[ ||f|| = \sqrt{(f,f)} = \sqrt{\langle f, f \rangle}. \]
2. Quaternion Fourier Transform (QFT)

Ell [1] defined the QFT for application to 2D linear time-invariant systems of PDEs. Later it was extensively applied to 2D image processing [2], including color. This spurred research into optimized numerical applications. The invertible QFT of a 2D quaternion valued signal \( f \) is defined as

\[
F\{f\} = \frac{1}{2\pi} \int f(x) e^{-j(xu+yv)} dx dy.
\]

The scalar product (9) gives the Plancherel theorem

\[
\langle f, g \rangle = \frac{1}{2\pi} \| F\{f\} \| / 2\pi.
\]

As corollary we get the Parseval (Rayleigh’s) theorem for signal energy preservation

\[
\| f \| = \| F\{f\} \| / 2\pi.
\]

Equations (5) and (15) reduce the computation of \( F\{f\} \) to the four QFTs of real functions \( f, f', f, f' \). And (15) shows that every theorem for the QFT of real 2D functions results in a theorem for quaternion-valued functions. For example a general linear non-singular transformation \( A \) of the QFT of 2D real signals can in this way be generalized to 2D quaternion-valued functions (for \( B \) compare [2])

\[
F\{f(Ax)\}(u) = \frac{1}{\det B} [ F\{f\}(B\cdot u) + F\{f\}(B\cdot u) + F\{f\}(B\cdot u) + F\{f\}(B\cdot u) ] j.
\]

Instead of (11) we can define the invertible right sided QFT (Clifford FT) as

\[
F\{f\}(u) = \int f(x) e^{-j(xu+yv)} dx dy,
\]

and obtain the Plancherel theorem

\[
\langle f, g \rangle = \langle F\{f\}, F\{g\} \rangle / (2\pi)^2.
\]

As corollary we again get a Parseval identity

\[
\| f \| = \| F\{f\} \| / 2\pi = \| F\{f\} \| / 2\pi.
\]

For \( F \), linearity and dilation properties hold, some other properties need commutation dependent modifications.

3. \( GL(R^2) \) Transformation Properties

We observe that the split (7) results in two complex kernels \( K_{\pm} \) with complex units \( i \) (or \( j \)) apart from \((1 \pm k)/2\). We therefore analyze the transformation properties of \( F\{f\} \) in terms of \( F\{f_{\pm}\} \). We can prove that

\[
F\{f_{\pm}\}(u) = \int f_{\pm} e^{-j(yv+\mp xu)} dx dy = \int e^{\mp i(xu+yv)} f_{\pm} dx dy.
\]

Every \( A \in GL(R^2) \) can be decomposed to \( A=TR=RS \), with \( R \) a rotation, \( T \) and \( S \) symmetric with positive and negative eigenvalues (ev.). Positive (negative) ev. correspond to stretches (reflections and stretches perpendicular to line of reflection). Rotations can be composed by two reflections \( R_{u_m} U_{m} \). Elementary transformations are hence reflections (Cartan) and stretches. In Clifford geometric algebra \( U_{m} \) is given by the vector \( n \) normal to the line of reflection \( U_{m} x = -n^x n x \). Using \( xu+yv = x \cdot u - xu+yv = x \cdot (U_{m} u) \) we get

\[
F\{f\}(u) = \int f \cdot e^{-j \cdot x \cdot u} dx dy, \quad F\{f\}(u) = \int f \cdot e^{-j \cdot x \cdot (U_{m} u)} dx dy.
\]

We therefore get for automorphisms \( A \in GL(R^2) \), \( A^{-1} \) the adjoint inverse transformation of \( A \)
\[ F\{f(Ax)(u)\} = |\text{det}A|^{-1} F\{f; (A^{-1}u) \}, \quad F\{f(Ax)(u)\} = |\text{det}A|^{-1} F\{f; (U_\varepsilon A^{-1}U_\varepsilon u) \}. \] (22)

The combination of (22) gives therefore
\[ F\{f(Ax)(u)\} = |\text{det}A|^{-1} \left[ F\{f; (A^{-1}u)\} + F\{f; (U_\varepsilon A^{-1}U_\varepsilon u) \} \right]. \] (23)

For axial stretches we get (ab ≠ 0, a,b ∈ R)
\[ F\{f(As x)(u)\} = F\{f; (ue_1/a+ve_2/b)/|ab| \}. \] (24)

For reflections we get (a’ = U_\varepsilon a)
\[ F\{f(Ua x)(u)\} = F\{f; (Ua u)\} + F\{f; (Ua' u)\}. \] (25)

For rotations we get
\[ F\{f(Rx)(u)\} = F\{f; (Ru)\} + F\{f; (R' u)\}. \] (26)

4. Generalization to spatio-temporal signals

Quaternion isomorphisms and $GL(\mathbb{R}^{n,m})$ transformation laws allow generalization to higher dimensions. As an example we take an isomorphism to a subalgebra of the spacetime algebra $\text{Cl}(3,1)$ with time vector $e_0$, 3D volume $I_3 = e_1 e_2 e_3$, and spacetime volume $I_4 = e_0 e_1 e_2 e_3$, all three with negative square. \{e_0, I_3, I_4\} generate an algebra isomorphic to quaternions. This leads to an invertible spacetime FT for 4D multivector valued $\text{Cl}(3,1)$ functions $f$
\[ F_s\{f\} (u) = \int \exp(-e_0 ts) f(x) \exp(-I_3 x' \cdot u') d^4x, \] (27)

With $d^4x = dt dx dy dz$, \( x = te_0 + x', \ x' = xe_1 + ye_2 + ze_3, \ u = se_0 + u', u' = ue_1 + ve_2 + we_3 \). The space time split
\[ f_z = (f_{\pm} e_0/\sqrt{1})/2 \] (28)
yields therefore the transformation formulas (comp. [4,5])
\[ F_u\{f_z\} (u) = \int f_z (x) \exp(-I_3 (x' \cdot u' - ts)) d^4x = \int \exp(-e_0 ts x' \cdot u') f_z (x) d^4x. \] (29)

Our new results will serve for the further development of discrete and continuous multivector wavelets.

References
[1] T. A. Ell, in Proc. Of the 32 Conf. on Decision and Control, IEEE, 1993, pp. 1830-1841
[5] For further literature see mathematics publication section of: http://sinai.mech.fukui-u.ac.jp/