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Clifford Algebra $Cl_{3,0}$-valued Wavelets and Uncertainty Inequality for Clifford Gabor Wavelet Transformation

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1 Introduction

The purpose of this paper is to construct Clifford algebra $Cl_{3,0}$ -valued wavelets using the similitude group *SIM(3)* and then give a detailed explanation of their properties using the Clifford Fourier transform. Our approach can generalize complex Gabor wavelets to multivectors called Clifford Gabor wavelets. Finally, we describe some of their important properties which we use to establish a new uncertainty principle for the Clifford Gabor wavelet transform.

2 Preliminaries

Let us consider an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the real 3D Euclidean vector space $\mathbb{R}^3$. The geometric algebra over $\mathbb{R}^3$ denoted by $Cl_{3,0}$ then has the graded $2^3 = 8$-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \boldsymbol{e}_{123} = i_3\}. \quad (1)$$

By convention the geometric product obeys

$$\boldsymbol{e}_i\, \boldsymbol{e}_j + \boldsymbol{e}_j\, \boldsymbol{e}_i = 2\delta_{ij}, \quad i, j = 1, 2, 3.$$

We introduce an inner product of functions f, g defined on $\mathbb{R}^3$ with values in $Cl_{3,0}$

$$\begin{aligned} \langle f, g\rangle_{L^2(\mathbb{R}^3;Cl_{3,0})} &= \int_{\mathbb{R}^3} f(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\; d^3\boldsymbol{x} \\ &= \sum_{A,B} \boldsymbol{e}_A \widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^3} f_A(\boldsymbol{x}) g_B(\boldsymbol{x})\; d^3\boldsymbol{x}, \end{aligned} \quad (2)$$

where $\widetilde{g(\boldsymbol{x})} = \sum_B g_B(\boldsymbol{x})\widetilde{\boldsymbol{e}_B}$, and $\widetilde{\boldsymbol{e}_B}$ is reverse of $\boldsymbol{e}_B$ of (1) formed by writing all its vector factors in reverse order.

Definition 1 *The Clifford Fourier transform [1] of $f(\boldsymbol{x}) \in L^2(\mathbb{R}^3; Cl_{3,0})$, $\int_{\mathbb{R}^3} \sqrt{f * f} d^3\boldsymbol{x} < \infty$ is the function $\mathcal{F}\{f\}$: $\mathbb{R}^3 \to Cl_{3,0}$ given by*

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\boldsymbol{x})\, e^{-i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}, \quad (3)$$

where $\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3$, and $\boldsymbol{\omega} \cdot \boldsymbol{x} = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3$.

Theorem 1 *The Clifford Fourier transform $\mathcal{F}\{f\}$ of $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ is invertible and its inverse is calculated by*

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{\omega}. \quad (4)$$

3 Clifford algebra $Cl_{3,0}$-valued wavelet transform

Let us define the following unitary linear operator

$$\begin{aligned} U_{a,\boldsymbol{\theta},\boldsymbol{b}} : L^2(\mathbb{R}^3; Cl_{3,0}) &\longrightarrow L^2(\mathbb{R}^3; Cl_{3,0}) \\ \psi(\boldsymbol{x}) &\longrightarrow U_{a,\boldsymbol{\theta},\boldsymbol{b}}\psi(\boldsymbol{x}) \\ &= \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}), \end{aligned} \quad (5)$$

where

$$\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) = \frac{1}{a^{\frac{3}{2}}} \psi(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x} - \boldsymbol{b}}{a})). \quad (6)$$

We call $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ admissible wavelet (see [2, 5]) if

$$C_\psi = \int_{\mathbb{R}^+} \int_{S0(3)} a^3 \{\widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\, d\mu, \quad (7)$$

is an invertible multivector constant and finite at a.e. $\boldsymbol{\omega} \in \mathbb{R}^3$. In this case $d\mu(a, \boldsymbol{\theta}) = \frac{da d\boldsymbol{\theta}}{a^4}$, and $d\boldsymbol{\theta}$ is the left Haar measure on $SO(3)$.

Definition 2 *We define the Clifford wavelet transform with respect to the mother wavelet $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ as follows*

$$\begin{aligned} T_\psi : L^2(\mathbb{R}^3; Cl_{3,0}) &\rightarrow L^2(\mathbb{R}^3; Cl_{3,0}) \\ f &\rightarrow T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{x}) \widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})}\, d^3\boldsymbol{x} \\ &= \langle f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} \rangle_{L^2(\mathbb{R}^3; Cl_{3,0})} \quad (8) \end{aligned}$$

Theorem 2 *Let $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be an admissible Clifford mother wavelet and $f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$ arbitrary. Then we have*

$$\langle T_\psi f, T_\psi g \rangle_{L^2(\mathcal{G}; Cl_{3,0})} = \langle f C_\psi, g \rangle_{L^2(\mathbb{R}^3; Cl_{3,0})}. \quad (9)$$

Theorem 3 *(Inverse Clifford $Cl_{3,0}$ wavelet transform) Let $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford mother wavelet that satisfies the admissibility condition (7). Then any $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ can be decomposed as*

$$f(\boldsymbol{x}) = \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{b}, \boldsymbol{\theta})\, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, C_\psi^{-1}\, d\mu d^3\boldsymbol{b}. \quad (10)$$

4 Clifford Gabor wavelets

Complex Gabor wavelets can be extended to multivectors. They take the form

$$\begin{aligned} \psi^c(\boldsymbol{x}) &= g(\boldsymbol{x}; \sigma_1, \sigma_2, \sigma_3) \times \quad (11) \\ &\quad \left(e^{i_3 \boldsymbol{\omega}_0 \cdot \boldsymbol{x}} - e^{-\frac{1}{2}(\sigma_1^2 u_0^2 + \sigma_2^2 u_0^2 + \sigma_3^2 w_0^2)} \right) \\ &= g(\boldsymbol{x}; \sigma_1, \sigma_2, \sigma_3)\, e^{i_3 \boldsymbol{\omega}_0 \cdot \boldsymbol{x}} - \eta(\boldsymbol{x}), \quad (12) \end{aligned}$$

where the correction term η [3] is defined by

$$\eta(\boldsymbol{x}) = g(\boldsymbol{x}; \sigma_1, \sigma_2, \sigma_3) e^{-\frac{1}{2}(\sigma_1^2 u_0^2 + \sigma_2^2 u_0^2 + \sigma_3^2 w_0^2)},$$

and $g(\boldsymbol{x}; \sigma_1, \sigma_2, \sigma_3)$ is a 3D Gaussian function.

The admissibility constant (7) will be real-valued, i.e.

$$C_{\psi^c} = \int_{\mathbb{R}^+} \int_{SO(3)} a^3 |\mathcal{F}\{\psi^c\}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})|^2 d\mu < \infty, \quad (13)$$

is a real constant and finite at a.e. $\boldsymbol{\omega} \in \mathbb{R}^3$.

5 Uncertainty inequality for Clifford Gabor wavelets

It is known that the uncertainty principle plays an important role in the development and understanding of quantum physics. In quantum physics it states e.g. that particle momentum and position cannot be simultaneously measured with arbitrary precision. In classical harmonic analysis the uncertainty principle establishes for a function and its Fourier transform a minimum of the products of the variances. Here we will see how the Clifford Gabor wavelet transform and the Clifford Fourier transform of a multivector function are related.

Theorem 4 *(Uncertainty principle for Clifford Gabor wavelet) Let ψ^c be an admissible Clifford Gabor wavelet. Assume $\|f\|^2_{L^2(\mathbb{R}^3; Cl_{3,0})} = F < \infty$ for every $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, then the following inequality holds*

$$\|\boldsymbol{b} T_{\psi^c} f(a, \boldsymbol{\theta}, \boldsymbol{b})\|^2_{L^2(\mathcal{G}; Cl_{3,0})} \times \\ \|\boldsymbol{\omega} \hat{f}\|^2_{L^2(\mathbb{R}^3; Cl_{3,0})} \geq 3 C_{\psi^c} \frac{(2\pi)^3}{4} F^2.$$

References

[1] B. Mawardi and E. Hitzer, *Clifford Fourier Transformation and Uncertainty Principle for the Clifford Geometric Algebra $Cl_{3,0}$*, Adv. App. Cliff. Alg., Vol. 16, No. 1, 2006, pp. 41-61.

[2] F. Brackx and F. Sommen, *Benchmarking of Three-Dimensional Clifford Wavelet Functions*, Complex Variables, Vol. 47, No. 7, 2002, pp. 577-588.

[3] S. T. Ali, J. P. Antoine, and J. P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, Springer Verlag, 2000.

[4] Marius Mitrea, *Clifford Wavelets, Singular Integrals and Hardy Spaces,* Lectures Notes in Mathematics 1575, Springer Verlag, 1994.

[5] J. Zhao, *Clifford algebra-valued Admissible Wavelets Associated with Admissible Group,* Acta Scientiarium Naturalium Universitatis Pekinensis, Vol. 41, No. 5, 2005.

[6] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Vol. 76 of Research Notes in Mathematics, Pitman Advanced Publishing Program, Boston, 1982.

[7] T. Bülow, *Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images,* PhD thesis, University of Kiel, 1999.

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