The Majorana spinor representation of the Poincare group

Leonardo Pedro
Centro de Fisica Teorica de Particulas, Portugal
leonardo@cftp.ist.utl.pt

June 20, 2013

Abstract

The Majorana spinor is an element of a 4 dimensional real vector space. The Majorana spinor field is a space-time dependent Majorana spinor, solution of the free Dirac equation.

We show that the Majorana spinor field with finite mass is a real orthogonal irreducible representation of the Poincare group. The Majorana-Fourier and Majorana-Hankel transforms of Majorana spinor fields are defined and related to the linear and angular momentums of a spin one-half representation of the Poincare group.

Contents

1 Introduction

The Poincare group, also called inhomogeneous Lorentz group, has a real Lie algebra [1]. The irreducibility of a representation of a real Lie algebra may depend on whether the representation space is a real or complex Hilbert space. In a physicists language, the complex Hilbert spaces have twice the number of degrees of freedom of the real ones.

The Poincare group is the semi-direct product of the translations and Lorentz groups. Whether or not the Lorentz and Poincare groups include the parity and time reversal transformations depends on the context and authors. To be clear, we use the prefixes full/restricted when including/excluding parity and time reversal transformations. The fundamental representation of the Pin(3,1) group is a spin one-half representation of the full Lorentz group [2], while the fundamental representation of the SL(2,C) subgroup is a spin one-half representation of the restricted Lorentz subgroup.

The unitary projective representations of the Poincare group on complex Hilbert spaces were studied by many authors, including Wigner [3–7]. Since Quantum Mechanics is based on complex Hilbert spaces [8], these studies were very important in the evolution of the role of symmetry in the Quantum theory [9]. Although Quantum Theory in real Hilbert spaces was investigated before [10], to our knowledge, the orthogonal projective representations of the Poincare group on real Hilbert spaces were not studied.

The Dirac spinor is an element of a 4 dimensional complex vector space, while the Majorana spinor is an element of a 4 dimensional real vector space [11]. The Majorana spinor representation of both SL(2,C) and Pin(3,1) is irreducible [12]. The spinor fields, space-time dependent spinors, are solutions of the free Dirac equation [13–16]. The Hilbert space of Dirac spinor fields is complex, while the Hilbert space of Majorana spinor fields is real.

To study a system of many neutral particles with spin one-half, Majorana spinor fields are extended with second quantization operators and are called Majorana quantum fields or Majorana fermions [17–19]. There are important applications of the Majorana quantum field in theories trying to explain phenomena in neutrino physics, dark matter searches, the fractional quantum Hall effect and superconductivity [20]. Note that Majorana quantum fields are related to but are different from the Majorana spinor fields.

In the context of Clifford Algebras, there are studies on the geometric square roots of -1 [21–23] and on the generalizations of the Fourier transform [24], with applications to image processing [25].

Our goal is to study the spin one-half representation of the Poincare group on the real Hilbert space of Majorana spinor fields.

We will show that the Majorana spinor representations of the groups SU(2), SL(2,C) and Pin(3,1) are irreducible.

The Majorana-Fourier and Majorana-Hankel orthogonal transforms of Majorana spinor fields are defined and related to the linear and angular momentums of a spin one-half projective representation of the Poincare group.

Finally we show that the spin one-half representations of the restricted and full Poincare groups on the Majorana spinor field are orthogonal and irreducible.

In chapter 2 we define the Majorana matrices and spinors. In chapter 3 we study the Majorana spinor projective representation of the Lorentz group. In chapter 4 we relate the Majorana and Pauli spinor fields. In 5 and 6 we define the Majorana-Fourier and

Majorana-Hankel transforms of a Majorana spinor. In 7 we show that the projective Poincare group representation on the Majorana spinor field is orthogonal and irreducible. In 8, we extend the Majorana transforms to include the energy. In 9, by comparison with the Dirac spinor field solutions of the free Dirac equation, we show that the Majorana transforms are related with the linear and angular momentums of a free particle with spin one-half.

The Dirac equation [13], is a complex 4x4 matrix differential equation, whose solution is a Dirac spinor field, describing one particle with spin one-half in interaction with an electromagnetic potential. The Dirac spinor is a 4x1 complex column matrix that transforms in a precise way under the action of Lorentz transformations.

In a Majorana basis, the free (that is, when the electromagnetic potential is null) Dirac equation is a real 4x4 matrix differential equation, whose solution can be a Majorana spinor field. The Majorana spinor is a Dirac spinor which is real in a Majorana basis.

To study a system of many particles with spin one-half, Dirac or Majorana spinor fields are extended with second quantization operators and called Dirac or Majorana quantum fields. There are important applications of the Majorana quantum field in theories trying to explain phenomena in neutrino physics, dark matter searches, the fractional quantum Hall effect and superconductivity [20].

There are very good references on spinors [11, 18, 19] and on its relation with the Lorentz group [2]. It is known that the Majorana spinor is an irreducible representation of the double cover of the proper orthochronous Lorentz group [12]. Yet, we could not find a detailed study of the Majorana spinor field solutions of the free Dirac equation (without second quantization operators).

In the context of Clifford Algebras, the generalization of the Dirac matrices algebra to other dimensions and metrics, there are studies on the geometric square roots of -1 [22, 23] and on the generalizations of the Fourier transform [24], with applications to image processing.

Our goal is to show that, without second quantization operators, all the kinematic properties of a free spin 1/2 particle are present in the Majorana spinor field solutions of the free Dirac equation. In chapter 2 we define the Majorana matrices and spinors. In chapter 3 we relate the Majorana and Pauli spinor fields. In 4 and 5 we define the Majorana-Fourier and Majorana-Hankel transforms of a Majorana spinor. In 6, we extend the Majorana transforms to include the energy. In 7, by comparison with the Dirac spinor field solutions of the free Dirac equation, we show that the Majorana transforms are related with the linear and angular momentums of a free particle with spin one-half.

2 Majorana, Dirac and Pauli Matrices and Spinors

The Majorana matrices, $i\gamma^\mu$ with $\mu = 0, 1, 2, 3$, are the Dirac Gamma matrices, γ^μ, times the imaginary unit. The notation maintains explicit the relation between the Majorana and Dirac Gamma matrices.

Definition 2.1. The Majorana matrices, $i\gamma^\mu$, are 4×4 unitary matrices with anti-commutator $\{i\gamma^\mu, i\gamma^\nu\}$:

$$(i\gamma^\mu)(i\gamma^\nu) + (i\gamma^\nu)(i\gamma^\mu) = -2g^{\mu\nu},\ \mu, \nu = 0, 1, 2, 3 \tag{2.1}$$

Where $g = diag(1,-1,-1,-1)$ is the Minkowski metric. The pseudo-scalar is $i\gamma^5 \equiv -\gamma^0\gamma^1\gamma^2\gamma^3$.

Remark 2.2. *Pauli's fundamental theorem implies that the Majorana matrices are unique up to an unitary similarity transformation.*

The product of 2 Dirac Gamma matrices is minus the product of 2 corresponding Majorana matrices: $\gamma^\mu\gamma^\nu = -i\gamma^\mu i\gamma^\nu$.

In a Majorana basis, the Majorana matrices are 4×4 real orthogonal matrices. An example of the Majorana matrices in a particular Majorana basis is:

$$\begin{aligned}
i\gamma^1 &= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} & i\gamma^2 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} & i\gamma^3 &= \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
i\gamma^0 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} & i\gamma^5 &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} & &= -\gamma^0\gamma^1\gamma^2\gamma^3
\end{aligned} \tag{2.2}$$

Definition 2.3. The Dirac spinor is a 4×1 complex column matrix, that transforms in a precise way under the action of Lorentz transformations.

The space of Dirac spinors is a 4 dimensional complex vector space.

Definition 2.4. Let S be a unitary matrix such that $Si\gamma^\mu S^\dagger$ is real, for $\mu = 0,1,2,3$.

The set of Majorana spinors, *Pinor*, is the subset of Dirac spinors u verifying the Majorana condition:

$$(Su)^* = (Su) \tag{2.3}$$

Where * denotes complex conjugation and † denotes hermitian conjugate.

Remark 2.5. *Let W be a subset of a vector space V over $\mathbb{C}$. W is a real vector space iff:*

1) $0 \in W$;

2) If $u, v \in W$, then $u + v \in W$;

3) If $u \in W$ and $c \in \mathbb{R}$, then $cu \in W$.

From the previous remark, the set of Majorana spinors is a 4 dimensional real vector space, while the set of Dirac spinors is a 8 dimensional real vector space. Note that the linear combinations of Majorana spinors with complex scalars do not verify the Majorana condition. The Majorana spinor, in a Majorana basis, is a 4×1 real column matrix.

Definition 2.6. The Pauli matrices σ^k, $k \in \{1,2,3\}$ are 2×2 hermitian, unitary, anti-commuting, complex matrices. The Pauli spinor is a 2×1 complex column matrix. The space of Pauli spinors is denoted by *Pauli*.

The space of Pauli spinors, *Pauli*, is a 2 dimensional complex vector space and a 4 dimensional real vector space.

Remark 2.7. *Pauli's fundamental theorem guarantees that the Pauli matrices are unique up to an unitary similarity transformation.*

3 Hilbert spaces of Majorana and Pauli spinor fields

Definition 3.1. The complex Hilbert space of Pauli spinors, *Pauli*, has the internal product:

$$< \phi, \psi >= \phi^\dagger \psi;\ \phi, \psi \in Pauli \tag{3.1}$$

Definition 3.2. The real Hilbert space of Majorana spinors, *Pinor*, has the internal product:

$$< \Phi, \Psi >= \Phi^\dagger \Psi;\ \Phi, \Psi \in Pinor \tag{3.2}$$

Definition 3.3. Consider that $\{M_+, M_-, i\gamma^0 M_+, i\gamma^0 M_-\}$ and $\{P_+, P_-, iP_+, iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces *Pinor* and *Pauli*, respectively, verifying:

$$\gamma^3\gamma^5 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm \tag{3.3}$$

Let H be a real Hilbert space. For all $h \in H$, the bijective linear map $\Theta_H : Pauli \otimes_{\mathbb{R}} H \to Pinor \otimes_{\mathbb{R}} H$ is defined by:

$$\Theta_H(h \otimes_{\mathbb{R}} P_+) = h \otimes_{\mathbb{R}} M_+,\ \Theta_H(h \otimes_{\mathbb{R}} iP_+) = h \otimes_{\mathbb{R}} i\gamma^0 M_+ \tag{3.4}$$

$$\Theta_H(h \otimes_{\mathbb{R}} P_-) = h \otimes_{\mathbb{R}} M_-,\ \Theta_H(h \otimes_{\mathbb{R}} iP_-) = h \otimes_{\mathbb{R}} i\gamma^0 M_- \tag{3.5}$$

Definition 3.4. Let H_n, with $n \in \{1, 2\}$, be two real Hilbert spaces and $U : Pauli \otimes_{\mathbb{R}} H_1 \to Pauli \otimes_{\mathbb{R}} H_2$ be an operator. The operator $U^\Theta : \Theta_{H_2} \circ U \circ \Theta_{H_1}^{-1} : Pinor \otimes_{\mathbb{R}} H_1 \to Pinor \otimes_{\mathbb{R}} H_2$ is defined as $U^\Theta \equiv \Theta_{H_2} \circ U \circ \Theta_{H_1}^{-1}$.

Remark 3.5. *Let H_n, with $n \in \{1, 2\}$, be two Hilbert spaces with internal products $<,>: H_n \times H_n \to \mathbb{F}$,($\mathbb{F} = \mathbb{R}, \mathbb{C}$). A linear operator $U : H_1 \to H_2$ is unitary iff:*
1) it is surjective;
2) for all $x \in H_1$, $< U(x), U(x) >=< x, x >$.

Remark 3.6. *Given two real Hilbert spaces H_1, H_2 and an unitary operator $U : H_1 \to H_2$, the inverse operator $U^{-1} : H_2 \to H_1$ is defined by:*

$$< x, U^{-1} y >=< Ux, y >,\ x \in H_1, y \in H_2 \tag{3.6}$$

Proposition 3.7. *Let H_n, with $n \in \{1, 2\}$, be two real Hilbert spaces. The following two statements are equivalent:*
1) The operator $U : Pauli \otimes_{\mathbb{R}} H_1 \to Pauli \otimes_{\mathbb{R}} H_2$ is unitary;
2) The operator $U^\Theta : Pinor \otimes_{\mathbb{R}} H_1 \to Pinor \otimes_{\mathbb{R}} H_2$ is orthogonal.

Proof. Because Θ_{H_n} is bijective, U is surjective iff $\Theta_{H_2} \circ U \circ \Theta_{H_1}^{-1}$ is surjective.

For all $g \in Pauli \otimes_{\mathbb{R}} H_1$, we have:

$$< g, g >=< \Theta_{H_1}(g), \Theta_{H_1}(g) > \tag{3.7}$$

$$< U(g), U(g) >=< \Theta_{H_2}(U(g)), \Theta_{H_2}(U(g)) > \tag{3.8}$$

Since Θ_{H_n} is bijective, we get that the following two statements are equivalent:
1) for all $g \in Pauli \otimes_{\mathbb{R}} H_1$, $< g, g >=< U(g), U(g) >$;
2) for all $g' \in Pinor \otimes_{\mathbb{R}} H_1$, $< g', g' >=< \Theta_{H_2}(U(\Theta_{H_1}^{-1}(g'))), \Theta_{H_2}(U(\Theta_{H_1}^{-1}(g'))) >$. $\square$

Definition 3.8. The space of Majorana spinor fields over a set S, $Pinor(S) \equiv Pinor \otimes_{\mathbb{R}} L^2(S)$, is the real Hilbert space of Majorana spinors whose entries, in a Majorana basis, are real Lebesgue square integrable functions of S.

Definition 3.9. The space of Pauli spinor fields over a set S, $Pauli(S) \equiv Pauli \otimes_{\mathbb{R}} L^2(S)$ is the complex Hilbert space of Pauli spinors whose components are complex Lebesgue square integrable functions of S.

4 Linear Momentum of Majorana spinor fields

Definition 4.1. $L^2(\mathbb{R}^n)$ is the real Hilbert space of real functions of n real variables whose square is Lebesgue integrable in $\mathbb{R}^n$. The internal product is:

$$< f, g > \equiv \int d^n x f(x) g(x),\ \ f, g \in L^2(\mathbb{R}^n) \tag{4.1}$$

Remark 4.2. *The Pauli-Fourier Transform $\mathcal{F}_P : Pauli(\mathbb{R}^n) \to Pauli(\mathbb{R}^n)$ is an unitary operator defined by:*

$$\mathcal{F}_P\{\psi\}(\vec{p}) \equiv \int d^n \vec{x} \frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^n}} \psi(\vec{x}),\ \ \psi \in Pauli(\mathbb{R}^n) \tag{4.2}$$

Where the domain of the integral is $\mathbb{R}^n$.

Definition 4.3. The Majorana-Fourier Transform $\mathcal{F}_M : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is an operator defined by:

$$\mathcal{F}_M\{\Psi\}(\vec{p}) \equiv \int d^3 \vec{x}\ \frac{e^{-i\gamma^0 \vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} \Psi(\vec{x}),\ \ \Psi \in Pinor(\mathbb{R}^3) \tag{4.3}$$

Where the domain of the integral is $\mathbb{R}^3$, $m \geq 0$, $E_p \equiv \sqrt{\vec{p}^2 + m^2}$ and $\not{p} = E_p \gamma^0 - \vec{p} \cdot \vec{\gamma}$.

Proposition 4.4. *The Majorana-Fourier Transform is an unitary operator.*

Proof. The Majorana-Fourier Transform can be written as:

$$\mathcal{F}_M\{\Psi\}(\vec{p}) \equiv \sqrt{\frac{E_p + m}{2E_p}} \Big(\int d^3 \vec{x}\ \frac{e^{-i\gamma^0 \vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} \Psi(\vec{x}) \Big) \tag{4.4}$$

$$- \sqrt{\frac{E_p - m}{2E_p}} \frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \Big(\int d^3 \vec{x}\ \frac{e^{+i\gamma^0 \vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} \Psi(\vec{x}) \Big) \tag{4.5}$$

So, one gets:

$$\mathcal{F}_M\{\Psi\} = S \circ \mathcal{F}_P^\Theta\{\Psi\} \tag{4.6}$$

Where $S : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} S\{\Psi\}(+\vec{p}) \\ S\{\Psi\}(-\vec{p}) \end{bmatrix} \equiv \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & -\sqrt{\frac{E_p-m}{2E_p}} \frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ \sqrt{\frac{E_p-m}{2E_p}} \frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} \begin{bmatrix} \Psi(+\vec{p}) \\ \Psi(-\vec{p}) \end{bmatrix} \tag{4.7}$$

We can check that the 2×2 matrix appearing in the equation above is orthogonal. Therefore S is an unitary operator. Since $\mathcal{F}_P^\Theta$ is also unitary, $\mathcal{F}_M$ is unitary. □

Proposition 4.5. *The inverse Majorana-Fourier Transform verifies:*

$$(\gamma^0\vec{\gamma}\cdot\vec{\partial}+i\gamma^0 m)\mathcal{F}_M^{-1}\{\Psi\}(\vec{x})=(\mathcal{F}_M^{-1}\circ R)\{\Psi\}(\vec{x}) \tag{4.8}$$

Where $\Psi\in Pinor(\mathbb{R}^3)$ *and* $R: Pinor(\mathbb{R}^3)\to Pinor(\mathbb{R}^3)$ *is a bijective linear map defined by* $R\{\Psi\}(\vec{p})=i\gamma^0 E_p\Psi(\vec{p})$*.*

Proof. We have $\mathcal{F}_M^{-1}=(\mathcal{F}_P^{\Theta})^{-1}\circ S^{-1}$. Then:

$$(\gamma^0\vec{\gamma}\cdot\vec{\partial}+i\gamma^0 m)(\mathcal{F}_P^{\Theta})^{-1}\{\Psi\}(\vec{x})=((\mathcal{F}_P^{\Theta})^{-1}\circ Q)\{\Psi\}(\vec{x}) \tag{4.9}$$

Where $Q: Pinor(\mathbb{R}^3)\to Pinor(\mathbb{R}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} Q\{\Psi\}(+\vec{p}) \\ Q\{\Psi\}(-\vec{p}) \end{bmatrix}\equiv\begin{bmatrix} i\gamma^0 m & i\vec{p}\cdot\vec{\gamma} \\ -i\vec{p}\cdot\vec{\gamma} & i\gamma^0 m \end{bmatrix}\begin{bmatrix} \Psi(+\vec{p}) \\ \Psi(-\vec{p}) \end{bmatrix} \tag{4.10}$$

Now we show that $Q\circ S^{-1}=S^{-1}\circ R$:

$$\begin{bmatrix} i\gamma^0 m & i\vec{p}\cdot\vec{\gamma} \\ -i\vec{p}\cdot\vec{\gamma} & i\gamma^0 m \end{bmatrix}\begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ -\sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix}= \tag{4.11}$$

$$=\begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ -\sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix}\begin{bmatrix} i\gamma^0 E_p & 0 \\ 0 & i\gamma^0 E_p \end{bmatrix} \tag{4.12}$$

□

5 Angular momentum of Majorana spinor fields

Definition 5.1. Let $\vec{x}\in\mathbb{R}^3$. The spherical coordinates parametrization is:

$$\vec{x}=r(\sin(\theta)\sin(\varphi)\vec{e_1}+\sin(\theta)\sin(\varphi)\vec{e_2}+\cos(\theta)\vec{e_3}) \tag{5.1}$$

where $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ is a fixed orthonormal basis of $\mathbb{R}^3$ and $r\in[0,+\infty[$ $\theta\in[0,\pi]$, $\varphi\in[-\pi,\pi]$.

Definition 5.2. Let

$$\mathbb{S}^3\equiv\{(p,l,\mu): p\in\mathbb{R}_{\geq 0}; l,\mu\in\mathbb{Z}; l\geq 1; -l\leq\mu\leq l-1\} \tag{5.2}$$

The Hilbert space $L^2(\mathbb{S}^3)$ is the real Hilbert space of real Lebesgue square integrable functions of $\mathbb{S}^3$. The internal product is:

$$<f,g>=\sum_{l=1}^{+\infty}\sum_{\mu=-l}^{l-1}\int_0^{+\infty}dpf(p,l,\mu)g(p,l,\mu),\ \ f,g\in L^2(\mathbb{S}^3) \tag{5.3}$$

Definition 5.3. The Pauli-Hankel transform $\mathcal{H}_P: Pauli(\mathbb{R}^3)\to Pauli(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}_P\{\psi\}(p,l,\mu)\equiv\int r^2drd(\cos\theta)d\varphi\frac{2p}{\sqrt{2\pi}}\lambda_{l\mu}^{\dagger}(pr,\theta,\varphi)\psi(r,\theta,\varphi),\ \ \psi\in Pauli(\mathbb{R}^3) \tag{5.4}$$

The domain of the integral is $\mathbb{R}^3$. The matrices $\lambda_{l\mu}$, the spherical Bessel function of the first kind j_n [26], the Pauli spherical matrices $\omega_{l\mu}$ [27], the spherical harmonics $Y_{l\mu}$ and the associated Legendre functions of the first kind $P_{l\mu}$ are:

$$\lambda_{l\mu}(r,\theta,\varphi) \equiv \omega_{l\mu}(\theta,\varphi)\Big(j_l(r)\frac{1+\sigma^3}{2} + j_{l-1}(r)\frac{1-\sigma^3}{2}\Big) \tag{5.5}$$

$$j_l(r) \equiv r^l\Big(-\frac{1}{r}\frac{d}{dr}\Big)^l\frac{\sin r}{r} \tag{5.6}$$

$$\omega_{l\mu}(\theta,\varphi) \equiv \Big(-\sqrt{\frac{l-\mu}{2l+1}}Y_{l,\mu}(\theta,\varphi) + \sqrt{\frac{l+\mu+1}{2l+1}}Y_{l,\mu+1}(\theta,\varphi)\sigma^1\Big)\frac{1+\sigma^3}{2} \tag{5.7}$$

$$+\Big(\sqrt{\frac{l+\mu}{2l-1}}Y_{l-1,\mu}(\theta,\varphi)\sigma^1 + \sqrt{\frac{l-\mu-1}{2l-1}}Y_{l-1,\mu+1}(\theta,\varphi)\Big)\frac{1-\sigma^3}{2} \tag{5.8}$$

$$Y_{l\mu}(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^\mu(\cos\theta)e^{i\mu\varphi} \tag{5.9}$$

$$P_l^\mu(\xi) \equiv \frac{(-1)^\mu}{2^l l!}(1-\xi^2)^{\mu/2}\frac{\mathrm{d}^{l+\mu}}{\mathrm{d}\xi^{l+\mu}}(\xi^2-1)^l \tag{5.10}$$

Remark 5.4. *Due to the properties of spherical harmonics and Bessel functions, the Pauli-Hankel transform is an unitary operator. The inverse Pauli-Hankel Transform verifies:*

$$\vec{\sigma}\cdot\vec{\partial}\ \mathcal{H}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_P^{-1}\circ R')\{\psi\}(\vec{x}) \tag{5.11}$$

Where $\psi \in Pauli(\mathbb{S}^3)$ *and* $R' : Pauli(\mathbb{S}^3) \to Pauli(\mathbb{S}^3)$ *is a bijective linear map defined by:*

$$R'\{\psi\}(p,l,\mu) \equiv p\sigma^1\sigma^3\psi(p,l,\mu) \tag{5.12}$$

Definition 5.5. The Majorana-Hankel transform $\mathcal{H}_M : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}_M\{\Psi\}(p,l,\mu) \equiv \int r^2 dr d(\cos\theta)d\varphi\frac{2p}{\sqrt{2\pi}}\Delta^\dagger(p,l,\mu,r,\theta,\varphi)\Psi(r,\theta,\varphi),\ \Psi \in Pinor(\mathbb{R}^3) \tag{5.13}$$

$$\Delta(p,l,\mu,r,\theta,\varphi) \equiv \sqrt{\frac{E_p+m}{2E_p}}\Lambda_{l\mu}(pr,\theta,\varphi) + \sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu\Lambda_{l,-\mu-1}(pr,\theta,\varphi)i\gamma^3 \tag{5.14}$$

Where the matrices $\Lambda_{l\mu}(r,\theta,\varphi) \equiv \Theta\circ\lambda_{l\mu}(r,\theta,\varphi)\circ\Theta^{-1}$ are obtained from the Pauli matrices $\lambda_{l\mu}$ replacing (i,σ^1,σ^3) by $(i\gamma^0,\gamma^1\gamma^5,\gamma^3\gamma^5)$.

Proposition 5.6. *The Majorana-Hankel transform is an unitary operator.*

Proof. The Majorana-Hankel transform can be written as:

$$\mathcal{H}_M = S\circ\mathcal{H}_P^\Theta \tag{5.15}$$

Where $S : Pinor(\mathbb{S}^3) \to Pinor(\mathbb{S}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} S\{\Psi\}(p,l,\mu) \\ S\{\Psi\}(p,l,-\mu-1) \end{bmatrix} \equiv \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu i\gamma^3 \\ -\sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu i\gamma^3 & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} \begin{bmatrix} \Psi(p,l,\mu) \\ \Psi(p,l,-\mu-1) \end{bmatrix} \tag{5.16}$$

We can check that the 2×2 matrix appearing in the equation above is orthogonal. Therefore S is an unitary operator. Since $\mathcal{H}_P^\ominus$ is also unitary, $\mathcal{H}_M$ is unitary. □

Proposition 5.7. *The inverse Majorana-Hankel Transform verifies:*

$$(\gamma^0\vec{\gamma}\cdot\vec{\partial} + i\gamma^0 m)\mathcal{H}_M^{-1}\{\Psi\}(\vec{x}) = (\mathcal{H}_M^{-1}\circ R)\{\Psi\}(\vec{x}) \tag{5.17}$$

Where $\Psi \in Pinor(\mathbb{S}^3)$ and $R : Pinor(\mathbb{S}^3) \to Pinor(\mathbb{S}^3)$ is a bijective linear map defined by:

$$R\{\Psi\}(p,l,\mu) \equiv i\gamma^0 E_p \Psi(p,l,\mu) \tag{5.18}$$

Proof. We have $\mathcal{H}_M^{-1} = (\mathcal{H}_P^\ominus)^{-1}\circ S^{-1}$. Then we can check that $i\gamma^5\Lambda_{l\mu}(pr,\theta,\varphi) = -(-1)^\mu\Lambda_{l,-\mu-1}(pr,\theta,\varphi)i\gamma^1$.

Therefore, the inverse Pauli-Hankel Transform verifies:

$$(\gamma^0\vec{\gamma}\cdot\vec{\partial} + i\gamma^0 m)\ (\mathcal{H}_P^\ominus)^{-1}\{\Psi\}(\vec{x}) = ((\mathcal{H}_P^\ominus)^{-1}\circ Q)\{\psi\}(\vec{x}) \tag{5.19}$$

Where $\Psi \in Pinor(\mathbb{S}^3)$ and $Q : Pinor(\mathbb{S}^3) \to Pinor(\mathbb{S}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} Q\{\Psi\}(p,l,\mu) \\ Q\{\Psi\}(p,l,-\mu-1) \end{bmatrix} \equiv \begin{bmatrix} i\gamma^0 m & (-1)^\mu\gamma^0\gamma^3 p \\ -(-1)^\mu\gamma^0\gamma^3 p & i\gamma^0 m \end{bmatrix} \begin{bmatrix} \Psi(p,l,\mu) \\ \Psi(p,l,-\mu-1) \end{bmatrix} \tag{5.20}$$

Now we show that $Q\circ S^{-1} = S^{-1}\circ R$:

$$\begin{bmatrix} i\gamma^0 m & (-1)^\mu\gamma^0\gamma^3 p \\ -(-1)^\mu\gamma^0\gamma^3 p & i\gamma^0 m \end{bmatrix} \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu i\gamma^3 \\ -\sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu i\gamma^3 & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} = \tag{5.21}$$

$$= \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu i\gamma^3 \\ -\sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu i\gamma^3 & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} \begin{bmatrix} i\gamma^0 E_p & 0 \\ 0 & i\gamma^0 E_p \end{bmatrix} \tag{5.22}$$

□

6 Majorana spinor representation of the Lorentz group

Remark 6.1. *The Lorentz group, $O(1,3) \equiv \{\lambda \in \mathbb{R}^{4\times 4} : \lambda^T\eta\lambda = \eta\}$, is the set of real matrices that leave the metric, $\eta = diag(1,-1,-1,-1)$, invariant.*

The proper orthochronous Lorentz subgroup is defined by $SO^+(1,3) \equiv \{\lambda \in O(1,3) : det(\lambda) = 1, \lambda^0{}_0 > 0\}$. It is a normal subgroup. The discrete Lorentz subgroup of parity and time-reversal is $\Delta \equiv \{1, \eta, -\eta, -1\}$.

The Lorentz group is the semi-direct product of the previous subgroups, $O(1,3) = \Delta \ltimes SO^+(1,3)$.

Remark 6.2. *$Pin(3,1)$ [2] is the group of endomorphisms of Majorana spinors that leave the space of linear combinations of the Majorana matrices invariant, that is:*

$$Pin(3,1) \equiv \Big\{ S \in End(Pinor):\ detS = 1,\ S^{-1}(i\gamma^\mu)S = \Lambda^\mu{}_\nu i\gamma^\nu,\ \Lambda \in O(1,3) \Big\} \quad (6.1)$$

The map $\Lambda : Pin(3,1) \to O(1,3)$ defined by:

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu \equiv S^{-1}(i\gamma^\mu)S \quad (6.2)$$

is two-to-one and surjective. It defines a group homomorphism.

$Pin(3,1)$ is the semi-direct product of the groups $Spin^+(3,1) \equiv \{e^{\theta^j i\gamma^5\gamma^0\gamma^j + b^j\gamma^0\gamma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$ and $\Omega \equiv \{\pm 1, \pm i\gamma^0, \pm\gamma^0\gamma^5, \pm i\gamma^5\}$. The group homomorphisms $\Lambda : Spin^+(3,1) \to SO^+(1,3)$ and $\Lambda : \Omega \to \Delta$ are two-to-one and surjective. $Spin^+(3,1)$ is isomorphic to $SL(2,\mathbb{C})$, while the unitary subgroup $Spin^+(3,1) \cap SU(4) = \{e^{\theta^j i\gamma^5\gamma^0\gamma^j} : \theta^j \in \mathbb{R},\ j \in \{1,2,3\}\}$ is isomorphic to $SU(2)$.

Definition 6.3. The Majorana spinor representation of $Pin(3,1)$ and subgroups is defined by the action of $S \in Pin(3,1)$ in the space of Majorana spinors.

Remark 6.4. *A unitary matrix representation of a group is irreducible iff there is no basis where all the matrices of the representation can be block diagonalized (in a non-trivial way).*

Proposition 6.5. *The Majorana spinor representation of $Spin^+(1,3) \cap SU(4)$ (isomorphic to $SU(2)$), is irreducible.*

Proof. In a Majorana basis, the automorphisms of Majorana spinors are 4×4 non-singular real matrices. We can check that $i\gamma^5\gamma^0\gamma^j \in Spin^+(1,3) \cap SU(4),\ j \in \{1,2,3\}$. These matrices square to -1 and anti-commute. If there is a basis where they are all block diagonal, then the blocks also square to -1 and anti-commute. But there is only one (linear independent) 2×2 real matrix that squares to -1 and no 1×1 real matrix that squares to -1. Therefore, the representation is irreducible. □

7 Majorana spinor field representation of the Poincare group

Consider a Majorana spinor field $\Psi \in Pinor(\mathbb{R}^3)$. Let the Dirac Hamiltonian, H, be defined in the configuration space by:

$$iH\{\Psi\}(\vec{x}) \equiv (\gamma^0\vec{\gamma}\cdot\vec{\partial} + i\gamma^0 m)\Psi(\vec{x}),\ m \geq 0 \quad (7.1)$$

In the momentum space:

$$iH\{\Psi\}(\vec{p}) \equiv i\gamma^0 E_p \Psi(\vec{p}) \quad (7.2)$$

The free Dirac equation is verified by:

$$(\partial_0 + iH)e^{-iHx^0}\{\Psi\} = 0 \quad (7.3)$$

Definition 7.1. Given a Majorana spinor field $\Psi \in Pinor(\mathbb{R}^3)$, we define $\Psi(x) \equiv e^{-iHx^0}\{\Psi\}(\vec{x})$. The Majorana spinor field representation of the Poincare group is defined by:

$$P(S,b)\{\Psi\}(x) \equiv S\Psi(\Lambda_S^{-1}x + b) \tag{7.4}$$

Where $S \in Pin(3,1)$, $\Lambda_S \in O(1,3)$ is such that $\Lambda_S{}^{\mu}{}_{\nu}\gamma^{\nu} = S\gamma^{\mu}S^{-1}$ and $b \in \mathbb{R}^4$.

Proposition 7.2. *The Majorana spinor field representation of the Poincare group is unitary.*

Proof. 1) The representation of the Poincare group is surjective. That is, for all $\Psi \in Pinor(\mathbb{R}^3)$, there is a $\Phi(x) = S^{-1}\Psi(\Lambda_S^{-1}(x-a))$ such that:

$$\Psi(x) = S\Phi(\Lambda_S x + a) \tag{7.5}$$

2) The only part of the Poincare representation that is not easy to see that is unitary are the boosts. Let S be a boost transformation:

$$< \mathcal{F}_M \circ S\{\Psi\}, \mathcal{F}_M \circ S\{\Psi\} >= \int d^3\vec{x}d^3\vec{y}\Psi^{\dagger}(\vec{y})F(\vec{y},\vec{x},S)\Psi(\vec{x}) \tag{7.6}$$

Then:

$$\begin{aligned}
F(\vec{y},\vec{x},S) &= \int \frac{d^3\vec{p}}{(2\pi)^3} S^{\dagger} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{\Lambda}(p)\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} S && (7.7)\\
&= \int \frac{d^3\vec{p}}{(2\pi)^3} S^{\dagger} \frac{\cos(\vec{\Lambda}(p)\cdot(\vec{y}-\vec{x}))\not{p}\gamma^0 + \sin(\vec{\Lambda}(p)\cdot(\vec{y}-\vec{x}))i\gamma^0 m}{E_p} S && (7.8)\\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 E_p} \{\cos(\vec{\Lambda}(p)\cdot(\vec{y}-\vec{x}))\not{\Lambda}(p)\gamma^0 + \sin(\vec{\Lambda}(p)\cdot(\vec{y}-\vec{x}))i\gamma^0 m\} && (7.9)\\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 E_p} \{\cos(\vec{p}\cdot(\vec{y}-\vec{x}))\not{p}\gamma^0 + \sin(\vec{p}\cdot(\vec{y}-\vec{x}))i\gamma^0 m\} && (7.10)\\
&= F(\vec{y},\vec{x},1) && (7.11)
\end{aligned}$$

Therefore:

$$< \mathcal{F}_M \circ S\{\Psi\}, \mathcal{F}_M \circ S\{\Psi\} >=< \mathcal{F}_M\{\Psi\}, \mathcal{F}_M\{\Psi\} > \tag{7.12}$$

Using the fact that the Majorana-Fourier transform is unitary, we conclude that:

$$< S\{\Psi\}, S\{\Psi\} >=< \Psi, \Psi > \tag{7.13}$$

So, the representation of the Poincare group is unitary. □

Proposition 7.3. *The Majorana spinor field representation of the inhomogeneous restricted Lorentz group, for a finite mass, is irreducible.*

Proof. Suppose that we have for some Φ and Ψ, that for all $a \in \mathbb{R}^4$:

$$< \Phi, P(1,a)\{\Psi\} >= 0 \tag{7.14}$$

Doing a Fourier transform, the above equation can be written as:

$$\int \frac{d^3\vec{p}}{(2\pi)^3}\ \Phi^\dagger(\vec{p})e^{-i\gamma^0 p\cdot a}\Psi(\vec{p}) = 0 \tag{7.15}$$

$$\int \frac{d^3\vec{p}}{(2\pi)^3}\ \Phi^\dagger(\vec{p})\Big(\frac{1+\gamma^0}{2}e^{-ip\cdot a} + \frac{1-\gamma^0}{2}e^{ip\cdot a}\Big)\Psi(\vec{p}) = 0 \tag{7.16}$$

Now we multiply it by $e^{-i\vec{q}\cdot\vec{a}}$, with $\vec{q}$ arbitrary. Integrating in $\vec{a}$, we get:

$$\Phi^\dagger(\vec{q})\frac{1+\gamma^0}{2}\Psi(\vec{q})e^{-iE_q a^0} + \Phi^\dagger(-\vec{q})\frac{1-\gamma^0}{2}\Psi(-\vec{q})e^{iE_q a^0} = 0 \tag{7.17}$$

If we multiply the equation above by $e^{iE_q a^0}$ and we integrate a^0 from 0 to $2\pi/E_q$, we get $\Phi^\dagger(\vec{q})\frac{1+\gamma^0}{2}\Psi(\vec{q}) = 0$. Considering real and imaginary parts in separate, we obtain $\Phi^\dagger(\vec{q})\Psi(\vec{q}) = 0$ and $\Phi^\dagger(\vec{q})i\gamma^0\Psi(\vec{q}) = 0$.

Suppose $S \in Pin(1,3)$ verifies $S\not{q} = \not{p}S$. Then it can be written as: $S = B_p R B_q^{-1}$, where $R\not{l} = \not{l}R$ and B_p is any Lorentz transform verifying $B_p\not{l} = \not{p}B_p$. Now suppose $i\not{l} = i\gamma^0 m$, with $m > 0$. Then $B_p \equiv \frac{\not{p}\gamma^0+m}{\sqrt{E_p+m}\sqrt{2m}}$, where $p^0 = E_p$, satisfies $B_p\not{l} = \not{p}B_p$. Then R is a representation of $SU(2)$ and:

$$S\{\Psi\}(x) = \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} S\frac{\not{p}\gamma^0+m}{\sqrt{E_p+m}\sqrt{2E_p}}e^{-i\gamma^0\Lambda(p)\cdot x}\Psi(\vec{p}) \tag{7.18}$$

$$= \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} \frac{\not{\Lambda}(p)\gamma^0+m}{\sqrt{\Lambda^0(p)+m}\sqrt{2\Lambda^0(p)}}e^{-i\gamma^0\Lambda(p)\cdot x}R\sqrt{\frac{\Lambda^0(p)}{E_p}}\Psi(\vec{p}) \tag{7.19}$$

$$= \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} \frac{(\Lambda^{-1})^0(p)}{E_p}\frac{\not{p}\gamma^0+m}{\sqrt{E_p+m}\sqrt{2E_p}}e^{-i\gamma^0 p\cdot x}R\sqrt{\frac{E_p}{(\Lambda^{-1})^0(p)}}\Psi(\vec{\Lambda}^{-1}(p)) \tag{7.20}$$

Then:

$$\mathcal{F}_M \circ S\{\Psi\}(x^0,\vec{p}) = e^{-i\gamma^0 E_p x^0}R\sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}}\Psi(\vec{\Lambda}^{-1}(p)) \tag{7.21}$$

Since $m > 0$, for all $\vec{q}$ and $\vec{p}$, we can always find Λ such that $\vec{q} = \vec{\Lambda}(p)$. If the Poincare representation is reducible, since it is unitary, there are 2 states Ψ, Φ verifying for all $g \in SL(2,\mathbb{C})$ and $a \in \mathbb{R}^4$:

$$< \Phi, S_g \circ T(a)\{\Psi\} >= 0 \tag{7.22}$$

This implies that for all $\vec{p}$ and $\vec{q}$:

$$\frac{m}{E_p}\Phi^\dagger(\vec{q})R\Psi(\vec{p}) = 0 \tag{7.23}$$

R is a Majorana representation of $SU(2)$, which is irreducible, so the equation above is not true. Therefore the Poincare representation is irreducible. □

8 Energy of Majorana spinor fields

Definition 8.1. The Energy Transform $\mathcal{E} : Pinor(\mathbb{R}) \to Pinor(\mathbb{R})$ is an operator defined by:

$$\mathcal{E}\{\Psi\}(p^0) \equiv \int dx^0 \, \frac{e^{i\gamma^0 p^0 x^0}}{\sqrt{2\pi}} \Psi(x^0), \; \Psi \in Pinor(\mathbb{R}) \tag{8.1}$$

Where the domain of the integral is $\mathbb{R}$, $m \geq 0$.

Proposition 8.2. *The Energy transform is an unitary operator.*

Proof. The Energy transform can be written as:

$$\mathcal{E}\{\Psi\}(p^0) = \Theta_{L^2} \circ \mathcal{F}_P(-p^0) \circ \Theta_{L^2}^{-1}\{\Psi\} \tag{8.2}$$

Where $\mathcal{F}_P(-p^0)$ is a Pauli-Fourier transform over $\mathbb{R}$. Since the Pauli-Fourier transform is unitary, so is the Energy transform. $\square$

The energy transform can be applied in the time coordinate of a Majorana spinor field, x^0, after a (linear or spherical) momentum transform on the space coordinates, $\vec{x}$, to define an unitary energy-momentum transform:
- for the linear case $\mathcal{E} \circ \mathcal{F}_M : Pinor(\mathbb{R}^4) \to Pinor(\mathbb{R}^4)$;
- for the spherical case $\mathcal{E} \circ \mathcal{H}_M : Pinor(\mathbb{R}^4) \to Pinor(\mathbb{R} \times \mathbb{S}^3)$.

9 Conclusion

We fulfilled our goal to show that (without second quantization operators) all the kinematic properties of a free spin 1/2 particle with mass are present in the real solutions of the real free Dirac equation.

Acknowledgements

The work of Leonardo Pedro was supported by FCT under contract SFRH/BD/70688/2010.

References

[1] B. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate texts in mathematics. Springer-Verlag, 2003. `http://books.google.pt/books?id=j3T4mwEACAAJ`.

[2] M. Berg, C. De Witt-Morette, S. Gwo, and E. Kramer, "The Pin Groups in Physics," *Reviews in Mathematical Physics* **13** (2001) 953–1034, `arXiv:math-ph/0012006`.

[3] E. Wigner, "On Unitary Representations of the Inhomogeneous Lorentz Group," *The Annals of Mathematics* **40** no. 1, (Jan., 1939) 149+. `http://dx.doi.org/10.2307/1968551`.

[4] G. Mackey, *Unitary group representations in physics, probability, and number theory*. Advanced book classics. Addison-Wesley Pub. Co., 1989. http://books.google.pt/books?id=u2kPAQAAMAAJ.

[5] Y. Ohnuki, *Unitary Representations of the Poincaré Group and Relativistic Wave Equations*. World Scientific Publishing Company Incorporated, 1988. http://books.google.pt/books?id=RDLOOPIXZg0C.

[6] N. Straumann, "Unitary Representations of the inhomogeneous Lorentz Group and their Significance in Quantum Physics," *ArXiv e-prints* (Sept., 2008) , arXiv:0809.4942 [math-ph].

[7] S. Weinberg, *The Quantum Theory of Fields: Modern Applications*. The Quantum Theory of Fields. Cambridge University Press, 1995. http://books.google.pt/books?id=3ws6RJzqisQC.

[8] C. N. Yang. Cambridge University Press, 1987. http://dx.doi.org/10.1017/CBO9780511564253.006.

[9] D. Gross, "Symmetry in physics: Wigner's legacy," *Phys.Today* **48N12** (1995) 46–50.

[10] J. Myrheim, "Quantum mechanics on a real hilbert space," arXiv:quant-ph/9905037 [quant-ph].

[11] I. Todorov, "Clifford Algebras and Spinors," *ArXiv e-prints* (June, 2011) , arXiv:1106.3197 [math-ph].

[12] A. Aste, "A direct road to Majorana fields," *Symmetry* **2** (2010) 1776–1809, arXiv:0806.1690 [hep-th]. See section 5 on the Majorana spinor irrep of SL(2,C).

[13] P. A. M. Dirac, "The quantum theory of the electron," *Proc. R. Soc. Lond. A* **117** no. 778, (1928) 610–624. http://rspa.royalsocietypublishing.org/content/117/778/610.

[14] T. D. Newton and E. P. Wigner, "Localized states for elementary systems," *Rev. Mod. Phys.* **21** (Jul, 1949) 400–406. http://link.aps.org/doi/10.1103/RevModPhys.21.400.

[15] L. L. Foldy and S. A. Wouthuysen, "On the dirac theory of spin 1/2 particles and its non-relativistic limit," *Phys. Rev.* **78** (Apr, 1950) 29–36. http://link.aps.org/doi/10.1103/PhysRev.78.29.

[16] J. P. Costella and B. H. McKellar, "The Foldy-Wouthuysen transformation," *Am.J.Phys.* **63** (1995) 1119, arXiv:hep-ph/9503416 [hep-ph].

[17] E. Majorana, "A symmetric theory of electrons and positrons," *Nuovo Cim.* **14** (1937) 171–184. http://www2.phys.canterbury.ac.nz/editorial/Majorana1937-Maiani2.pdf.

[18] P. B. Pal, "Dirac, Majorana, and Weyl fermions," *American Journal of Physics* **79** (May, 2011) 485–498, `arXiv:1006.1718 [hep-ph]`.

[19] H. K. Dreiner, H. E. Haber, and S. P. Martin, "Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry," *Phys.Rept.* **494** (2010) 1–196, `arXiv:0812.1594 [hep-ph]`.

[20] J. Alicea, "New directions in the pursuit of Majorana fermions in solid state systems," *Reports on Progress in Physics* **75** no. 7, (July, 2012) 076501, `arXiv:1202.1293 [cond-mat.supr-con]`.

[21] D. Hestenes, "Real Spinor Fields," *J. Math. Phys.* **8** no. 4, (1967) 798808. `http://www.intalek.com/Index/Project/Research/RealSpinorFields.pdf`.

[22] D. Hestenes, "Gauge Gravity and Electroweak Theory," `arXiv:0807.0060 [gr-qc]`.

[23] E. Hitzer, J. Helmstetter, and R. Ablamowicz, "Square Roots of -1 in Real Clifford Algebras," *ArXiv e-prints* (Apr., 2012) , `arXiv:1204.4576 [math.RA]`.

[24] H. De Bie, "Clifford algebras, Fourier transforms and quantum mechanics," *ArXiv e-prints* (Sept., 2012) , `arXiv:1209.6434 [math-ph]`.

[25] T. Batard, M. Berthier, and C. Saint-Jean, "Cliffordfourier transform for color image processing," in *Geometric Algebra Computing*, E. Bayro-Corrochano and G. Scheuermann, eds., pp. 135–162. Springer London, 2010. `http://dx.doi.org/10.1007/978-1-84996-108-0_8`.

[26] G. S. Adkins, "Three-dimensional Fourier transforms, integrals of spherical Bessel functions, and novel delta function identities," *ArXiv e-prints* (Feb., 2013) , `arXiv:1302.1830 [math-ph]`.

[27] R. Szmytkowski, "Recurrence and differential relations for spherical spinors," *ArXiv e-prints* (Nov., 2010) , `arXiv:1011.3433 [math-ph]`.