Rational Structure, General Solution and Naked Barred Galaxies

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**Abstract** Rational structure in two dimension means that not only there exists an orthogonal net of curves in the plane but also, for each curve, the stellar density on one side of the curve is in constant ratio to the density on the other side of the curve. Such a curve is called a proportion curve or a Darwin curve. Such a distribution of matter is called a rational structure. Spiral galaxies are blended with dust and gas. Their longer wavelength (e.g. infrared) images present mainly the stellar distribution, which is called the naked galaxies. Jin He found many evidences that galaxies are rational stellar distribution. We list a few examples. Firstly, galaxy components (disks and bars) can be fitted with rational structure. Secondly, spiral arms can be fitted with Darwin curves. Thirdly, rational structure dictates New Universal Gravity which explains constant rotation curves simply and elegantly. This article presents the systematic theory of rational structure, its general solution and geometric meaning. A preliminary application to spiral galaxies is also discussed.

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The standard equation system for rational structure is the following which is presented as the formula (78) in the following Section 6

$$\begin{cases} \varphi'_y = \psi'_x, \;\; \varphi'_x = -\psi'_y, \\ W(f(x,y), \sqrt{\varphi^2+\psi^2}) = 0, \\ (2\sqrt{\varphi^2+\psi^2}\mathrm{A} + \frac{W'_\sigma}{W'_\tau}\tilde{A})\varphi = (2\sqrt{\varphi^2+\psi^2}\mathrm{B} + \frac{W'_\sigma}{W'_\tau}\tilde{B})\psi \end{cases}$$

Now we derive it step by step.

# 1 Rational Structure

Rational structure in two dimension

$$\rho(x, y) \tag{1}$$

means that not only there exists an orthogonal net of curves in the plane

$$x = x(\lambda, \mu), \; y = y(\lambda, \mu) \tag{2}$$

but also, for each curve, the matter density on one side of the curve is in constant ratio to the density on the other side of the curve. Such a curve is called a proportion curve or a Darwin curve. Such a distribution of matter is called a rational structure.

Because the ratio of density $\rho(x,y)$ is proportional to the derivative to the logarithm of the density

$$f(x,y) = \ln \rho(x,y) \tag{3}$$

we from now on, are only concerned with the logarithmic density $f(x,y)$. We know that, given the two partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \tag{4}$$

the structure $f(x,y)$ is determined provided that the Green's theorem is satisfied

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{5}$$

Now we are interested in rational structure. Instead of calculating the partial derivatives (4), we calculate the directional derivatives along the tangent direction to the above curves (2)

$$\frac{\partial f}{\partial l_\lambda}, \quad \frac{\partial f}{\partial l_\mu} \tag{6}$$

where $l_\lambda$ is the linear length on the $(x,y)$ plane and is along the row curves whose parameter is $\lambda$ while $l_\mu$ is the linear length along the column curves whose parameter is $\mu$. Given the two partial derivatives (6), however, the structure $f(x,y)$ may not be determined. A similar Green's theorem must be satisfied

$$\frac{\partial}{\partial \mu}\left(P\,\frac{\partial f}{\partial l_\lambda}\right) - \frac{\partial}{\partial \lambda}\left(Q\,\frac{\partial f}{\partial l_\mu}\right) = 0 \tag{7}$$

where

$$P(\lambda,\mu) = \sqrt{x_\lambda'^2 + y_\lambda'^2}, \quad Q(\lambda,\mu) = \sqrt{x_\mu'^2 + y_\mu'^2} \tag{8}$$

are the lengths or magnitudes of the vectors $(x'_\lambda, y'_\lambda)$ and $(x'_\mu, y'_\mu)$ on the $(x,y)$ plane, respectively. Note that we have used the simple notation $x'_\lambda = \frac{\partial x}{\partial \lambda}$. From now on, we always use the similar simple notation. To simplify the expression of our equations, we introduce some more notations

$$u(\lambda,\mu) = \frac{\partial f}{\partial l_\lambda}, \quad v(\lambda,\mu) = \frac{\partial f}{\partial l_\mu} \tag{9}$$

The condition of rational structure is that $u$ depends only on $\lambda$ and $v$ depends only on $\mu$

$$u = u(\lambda), \;\; v = v(\mu) \tag{10}$$

Now we prove the condition. Assume you walk along a row curve. The logarithmic ratio of the density on your left side to the immediate density on your right side is approximately the directional derivative of $f(x,y)$ along the column direction. That is, the logarithmic ratio is approximately the directional derivative $v(\lambda,\mu)$. Because $v(\lambda,\mu)$ is constant along the row curve (rational), $v(\lambda,\mu)$ is independent of $\lambda$, $v = v(\mu)$. Similarly, we can prove that $u(\lambda,\mu) = u(\lambda)$.

## 2 Rational Structure Equation

In the case of rational structure, the directional derivatives, $u = \frac{\partial f}{\partial l_\lambda}$ and $v = \frac{\partial f}{\partial l_\mu}$, are the functions of the single variables $\lambda$ and $\mu$, respectively (see the formula (10)). Therefore, the Green's theorem (7) turns out to be much simpler that is called the rational structure equation [1,2]

$$u(\lambda)P'_\mu = v(\mu)Q'_\lambda \tag{11}$$

To transform the equation and find its geometric meaning, we calculate

$$\begin{aligned} P'_\mu &= (x'_\lambda x''_{\lambda\mu} + y'_\lambda y''_{\lambda\mu})/P, \\ Q'_\lambda &= (x'_\mu x''_{\lambda\mu} + y'_\mu y''_{\lambda\mu})/Q \end{aligned} \tag{12}$$

That is,

$$\begin{aligned} P'_\mu &= \hat{\mathbf{x}}'_\lambda \cdot \mathbf{x}''_{\lambda\mu}, \\ Q'_\lambda &= \hat{\mathbf{x}}'_\mu \cdot \mathbf{x}''_{\lambda\mu} \end{aligned} \tag{13}$$

where boldface letters are the notations of vectors

$$\mathbf{x} = (x, y), \quad \mathbf{x}'_\lambda = (x'_\lambda, y'_\lambda), \quad \mathbf{x}''_{\lambda\mu} = (x''_{\lambda\mu}, y''_{\lambda\mu}), \quad \text{etc.} \tag{14}$$

The hats above letters mean that the corresponding vectors are unit ones. The dot symbol is the inner product of vectors. The geometric meaning of the formula (13) is that $P'_\mu$ is the projection of the vector $\mathbf{x}''_{\lambda\mu}$ in the direction of the vector $\mathbf{x}'_\lambda$ and $Q'_\lambda$ is the projection of the same vector in the direction of the vector $\mathbf{x}'_\mu$. They are also the geometric meaning of the rational structure equation (16).

Finally, our rational structure equation becomes

$$u(\lambda)\hat{\mathbf{x}}'_\lambda \cdot \hat{\mathbf{x}}''_{\lambda\mu} = v(\mu)\hat{\mathbf{x}}'_\mu \cdot \hat{\mathbf{x}}''_{\lambda\mu} \tag{15}$$

However, the solution $f(x, y)$ of the equation may not be rational structure because the net of curves may not be orthogonal. The solution of the following equation system must be rational

$$\begin{cases} u(\lambda)\hat{\mathbf{x}}'_\lambda \cdot \hat{\mathbf{x}}''_{\lambda\mu} = v(\mu)\hat{\mathbf{x}}'_\mu \cdot \hat{\mathbf{x}}''_{\lambda\mu}, \\ \mathbf{x}'_\lambda \cdot \mathbf{x}'_\mu = 0 \end{cases} \tag{16}$$

where the second equation is the orthogonal condition.

## 3 Rational Structure Equations without $u(\lambda)$ and $v(\lambda)$

Now we want to change the rational structure equation into the one without the apparent quantities $u(\lambda)$ and $v(\lambda)$. We know that $f(x, y)$ is the function of Cartesian coordinates $x$ and $y$. However, the equation system (2) suggests that it is also the composite function of the parameters $\lambda, \mu$. We use the same symbol $f$ to denote the composite function. For example, we denote its partial derivatives by $f'_\lambda, f'_\mu$. It is straightforward to show that the partial derivatives are

$$\begin{aligned} f'_\lambda &= u(\lambda)P(\lambda, \mu), \\ f'_\mu &= v(\mu)Q(\lambda, \mu) \end{aligned} \tag{17}$$

That is,

$$\begin{aligned} u(\lambda) &= f'_\lambda / P(\lambda,\mu), \\ v(\mu) &= f'_\mu / Q(\lambda,\mu) \end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned} (u(\lambda))'_\mu &= (f'_\lambda / P(\lambda,\mu))'_\mu = 0, \\ (v(\mu))'_\lambda &= \left(f'_\mu / Q(\lambda,\mu)\right)'_\lambda = 0 \end{aligned} \tag{19}$$

That is,

$$\begin{aligned} f''_{\lambda\mu} &= \tfrac{f'_\lambda}{P} P'_\mu, \\ f''_{\mu\lambda} &= \tfrac{f'_\mu}{Q} Q'_\lambda \end{aligned} \tag{20}$$

Finally we have our rational structure equation system without the apparent quantities $u(\lambda)$ and $v(\lambda)$

$$\begin{cases} \tfrac{f'_\lambda}{P} P'_\mu = \tfrac{f'_\mu}{Q} Q'_\lambda, \\ f''_{\lambda\mu} = \tfrac{f'_\mu}{Q} Q'_\lambda, \\ \mathbf{x}'_\lambda \cdot \mathbf{x}'_\mu = 0 \end{cases} \tag{21}$$

which is the necessary and sufficient condition for rational structure. That is, the equation system is the iff (if and only if) condition for rational structure. Noticing the formulas (18), we find out that the first equation of the equation system is nothing but an alternative form of the rational structure equation (11).

Using the derivative rule of composite functions for $f$, we have different form of the equation system

$$\begin{cases} \frac{(f'_x x'_\lambda + f'_y y'_\lambda)(x'_\lambda x''_{\lambda\mu} + y'_\lambda y''_{\lambda\mu})}{x'^2_\lambda + y'^2_\lambda} = \frac{(f'_x x'_\mu + f'_y y'_\mu)(x'_\mu x''_{\lambda\mu} + y'_\mu y''_{\lambda\mu})}{x'^2_\mu + y'^2_\mu}, \\ f''_{xx} x'_\lambda x'_\mu + f''_{xy}(x'_\lambda y'_\mu + x'_\mu y'_\lambda) + f''_{yy} y'_\lambda y'_\mu + f'_x x''_{\lambda\mu} + f'_y y''_{\lambda\mu} = \frac{(f'_x x'_\mu + f'_y y'_\mu)(x'_\mu x''_{\lambda\mu} + y'_\mu y''_{\lambda\mu})}{x'^2_\mu + y'^2_\mu}, \\ \mathbf{x}'_\lambda \cdot \mathbf{x}'_\mu = 0 \end{cases} \tag{22}$$

Using the same vector notation in the last Section, we have the other form of the equation system

$$\begin{cases} \nabla f \cdot \hat{\mathbf{x}}'_\lambda \, \hat{\mathbf{x}}'_\lambda \cdot \mathbf{x}''_{\lambda\mu} = \nabla f \cdot \hat{\mathbf{x}}'_\mu \, \hat{\mathbf{x}}'_\mu \cdot \mathbf{x}''_{\lambda\mu}, \\ (x'_\lambda \; y'_\lambda) \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} \begin{pmatrix} x'_\mu \\ y'_\mu \end{pmatrix} + (f'_x \; f'_y) \begin{pmatrix} x''_{\lambda\mu} \\ y''_{\lambda\mu} \end{pmatrix} = \nabla f \cdot \hat{\mathbf{x}}'_\mu \, \hat{\mathbf{x}}'_\mu \cdot \mathbf{x}''_{\lambda\mu}, \\ \mathbf{x}'_\lambda \cdot \mathbf{x}'_\mu = 0 \end{cases} \tag{23}$$

The geometric meaning of the first equation in the equation system is that the two vectors

$$\begin{aligned} \mathbf{C} &= \begin{pmatrix} \hat{\mathbf{x}}'_\lambda \cdot \mathbf{x}''_{\lambda\mu} \\ \hat{\mathbf{x}}'_\mu \cdot \mathbf{x}''_{\lambda\mu} \end{pmatrix} \\ \mathbf{D} &= \begin{pmatrix} \nabla f \cdot \hat{\mathbf{x}}'_\mu \\ \nabla f \cdot \hat{\mathbf{x}}'_\lambda \end{pmatrix} \end{aligned} \tag{24}$$

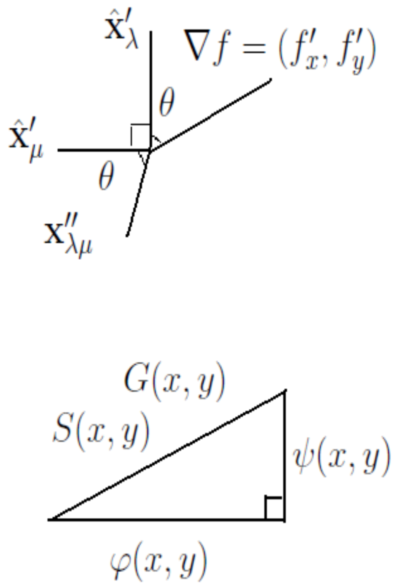


Figure 1: Up panel: The directions of the two vectors $\mathbf{x}''_{\lambda\mu}$ and $\nabla f = (f'_x, f'_y)$ are symmetric with respect to the bisector line between the two vectors $\mathbf{x}'_\lambda$ and $\mathbf{x}'_\mu$.
Down panel: The quantities $G \equiv S$, $|\varphi|$ and $|\psi|$ are the lengths of the hypotenuse and the two sides of a right triangle, respectively (see the formulas (69) and (86)).

are parallel to each other. Therefore, we have the another form of the equation system

$$\begin{cases} \mathbf{C} = h(\lambda, \mu)\mathbf{D}, \\ (x'_\lambda \; y'_\lambda) \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} \begin{pmatrix} x'_\mu \\ y'_\mu \end{pmatrix} + (f'_x \; f'_y) \begin{pmatrix} x''_{\lambda\mu} \\ y''_{\lambda\mu} \end{pmatrix} = \nabla f \cdot \hat{\mathbf{x}}'_\mu \, \hat{\mathbf{x}}'_\mu \cdot \mathbf{x}''_{\lambda\mu}, \\ \mathbf{x}'_\lambda \cdot \mathbf{x}'_\mu = 0 \end{cases} \tag{25}$$

where $h$ is an unknown function. The geometric meaning of the formula is shown in the up panel of Figure 1. It is that the directions of the two vectors $\mathbf{x}''_{\lambda\mu}$ and $\nabla f = (f'_x, f'_y)$ are symmetric with respect to the bisector line between the two vectors $\mathbf{x}'_\lambda$ and $\mathbf{x}'_\mu$.

# 4 Rational Equation System without Composite Functions

The above equation systems may not be easy for their solutions because they are involved with composite functions

$$f \circ \mathbf{x} \tag{26}$$

The chain rule for its first partial derivatives is well known

$$f'_\lambda = \nabla f \cdot \mathbf{x}'_\lambda \tag{27}$$

However, the rule for its higher-order derivatives may not be familiar,

$$(f \circ \mathbf{x})''_{\lambda\mu} = \sum_i f'_{x_i} x''_{i\lambda\mu} + \sum_{ij} f''_{x_i x_j} x'_{i\lambda} x'_{j\mu} \tag{28}$$

where $x_i$ or $x_j$ is $x$ and $y$.

Therefore, we want to find rational structure equation system without composite functions. To do so, we need the inverse functions of the functions (2)

$$\lambda = \lambda(x, y),\ \mu = \mu(x, y) \tag{29}$$

The inverse functions satisfy an orthogonal condition

$$\lambda'_x \mu'_x + \lambda'_y \mu'_y = 0 \tag{30}$$

which is resulted from the original orthogonal condition, i. e., the second equation of the equation system (16). However, to derive the equation system from the old one which is shown in the above Section 3, we need the chain rules for composite functions (formulas (27) and (28)).

The first chain rule is

$$\begin{pmatrix} x'_\lambda & x'_\mu \\ y'_\lambda & y'_\mu \end{pmatrix} \begin{pmatrix} \lambda'_x & \lambda'_y \\ \mu'_x & \mu'_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{31}$$

With the introduction of a new symbol

$$\delta = \lambda'_x \mu'_y - \lambda'_y \mu'_x \tag{32}$$

we have

$$\begin{array}{ll} x'_\lambda = \mu'_y/\delta, & x'_\mu = -\lambda'_y/\delta, \\ y'_\lambda = -\mu'_x/\delta, & y'_\mu = \lambda'_x/\delta \end{array} \tag{33}$$

Now we consider the second chain rule (28) for the composite relation $\mathbf{x} \to (\lambda, \mu) \to \mathbf{x}$. The corresponding result for the second derivative $''_{xx}$ is

$$0 = x'_\lambda \lambda''_{xx} + x'_\mu \mu''_{xx} + x''_{\lambda\lambda} \lambda'_x \lambda'_x + 2x''_{\lambda\mu} \lambda'_x \mu'_x + x''_{\mu\mu} \mu'_x \mu'_x \tag{34}$$

That is,

$$\begin{aligned} x''_{\lambda\lambda} \lambda'_x \lambda'_x + x''_{\lambda\mu} 2\lambda'_x \mu'_x + x''_{\mu\mu} \mu'_x \mu'_x &= -x'_\lambda \lambda''_{xx} - x'_\mu \mu''_{xx} \\ &= \mu'_y \tfrac{-\lambda''_{xx}}{\delta} + (-\lambda'_y) \tfrac{-\mu''_{xx}}{\delta} \end{aligned} \tag{35}$$

Other second derivatives are

$$\begin{aligned} x''_{\lambda\lambda} \lambda'_x \lambda'_y + x''_{\lambda\mu} (\lambda'_x \mu'_y + \lambda'_y \mu'_x) + x''_{\mu\mu} \mu'_x \mu'_y &= -x'_\lambda \lambda''_{xy} - x'_\mu \mu''_{xy}, \\ x''_{\lambda\lambda} \lambda'_y \lambda'_y + x''_{\lambda\mu} 2\lambda'_y \mu'_y + x''_{\mu\mu} \mu'_y \mu'_y &= -x'_\lambda \lambda''_{yy} - x'_\mu \mu''_{yy}, \\ y''_{\lambda\lambda} \lambda'_x \lambda'_x + y''_{\lambda\mu} 2\lambda'_x \mu'_x + y''_{\mu\mu} \mu'_x \mu'_x &= -y'_\lambda \lambda''_{xx} - y'_\mu \mu''_{xx}, \\ y''_{\lambda\lambda} \lambda'_x \lambda'_y + y''_{\lambda\mu} (\lambda'_x \mu'_y + \lambda'_y \mu'_x) + y''_{\mu\mu} \mu'_x \mu'_y &= -y'_\lambda \lambda''_{xy} - y'_\mu \mu''_{xy}, \\ y''_{\lambda\lambda} \lambda'_y \lambda'_y + y''_{\lambda\mu} 2\lambda'_y \mu'_y + y''_{\mu\mu} \mu'_y \mu'_y &= -y'_\lambda \lambda''_{yy} - y'_\mu \mu''_{yy} \end{aligned} \tag{36}$$

These formulas (35) and (36) are the linear algebraic equation system for the six variables $x''_{\lambda\lambda}, \cdots, y''_{\mu\mu}$. However, the righthand sides have the derivatives $'_\lambda$ and $'_\mu$. What we want is the derivative $'_x$ or $'_y$. Fortunately, we have the formulas (33) to replace the derivatives $'_\lambda$ and $'_\mu$. We have already done this in the formula (35). Because of linearity, the solution of our algebraic equation system depends solely on the two other equation systems whose constant terms (i. e., righthand-side terms) are $\frac{-\lambda''_{xx}}{\delta}, \frac{-\lambda''_{xy}}{\delta}, \cdots$ and $\frac{-\mu''_{xx}}{\delta}, \frac{-\mu''_{xy}}{\delta}, \cdots$, respectively (please take a look at the righthand side of the equation (35)). We use the symbols $\Lambda$ and $\Omega$ to denote the corresponding solutions, respectively. For example,

$$\Lambda_{01} = \frac{\begin{vmatrix} \lambda'_x\lambda'_x & -\lambda''_{xx}/\delta & \mu'_x\mu'_x \\ \lambda'_x\lambda'_y & -\lambda''_{xy}/\delta & \mu'_x\mu'_y \\ \lambda'_y\lambda'_y & -\lambda''_{yy}/\delta & \mu'_y\mu'_y \end{vmatrix}}{\Phi} \tag{37}$$

$$\Omega_{01} = \frac{\begin{vmatrix} \lambda'_x\lambda'_x & -\mu''_{xx}/\delta & \mu'_x\mu'_x \\ \lambda'_x\lambda'_y & -\mu''_{xy}/\delta & \mu'_x\mu'_y \\ \lambda'_y\lambda'_y & -\mu''_{yy}/\delta & \mu'_y\mu'_y \end{vmatrix}}{\Phi} \tag{38}$$

where

$$\Phi = \begin{vmatrix} \lambda'_x\lambda'_x & 2\lambda'_x\mu'_x & \mu'_x\mu'_x \\ \lambda'_x\lambda'_y & \lambda'_x\mu'_y + \lambda'_y\mu'_x & \mu'_x\mu'_y \\ \lambda'_y\lambda'_y & 2\lambda'_y\mu'_y & \mu'_y\mu'_y \end{vmatrix} \tag{39}$$

It is straightforward to show that

$$\begin{aligned} \Lambda_{01} &= \tfrac{1}{\Phi}(\lambda'_y\mu'_y\lambda''_{xx} - (\lambda'_x\mu'_y + \lambda'_y\mu'_x)\lambda''_{xy} + \lambda'_x\mu'_x\lambda''_{yy}) \\ &= \tfrac{1}{\Phi}(\lambda'_x\mu'_x(\lambda''_{yy} - \lambda''_{xx}) - (\lambda'_x\mu'_y + \lambda'_y\mu'_x)\lambda''_{xy}), \\ \\ \Omega_{01} &= \tfrac{1}{\Phi}(\lambda'_y\mu'_y\mu''_{xx} - (\lambda'_x\mu'_y + \lambda'_y\mu'_x)\mu''_{xy} + \lambda'_x\mu'_x\mu''_{yy}) \\ &= \tfrac{1}{\Phi}(\lambda'_x\mu'_x(\mu''_{yy} - \mu''_{xx}) - (\lambda'_x\mu'_y + \lambda'_y\mu'_x)\mu''_{xy}), \\ \\ \Phi &= (\lambda'_x\mu'_y - \lambda'_y\mu'_x)^3 = \delta^3 \end{aligned} \tag{40}$$

where we have used the orthogonal condition (30).

Finally we have our solution to the algebraic equation

$$\begin{array}{lll} x''_{\lambda\lambda} = \mu'_y\Lambda_{00} - \lambda'_y\Omega_{00}, & x''_{\lambda\mu} = \mu'_y\Lambda_{01} - \lambda'_y\Omega_{01}, & x''_{\mu\mu} = \mu'_y\Lambda_{11} - \lambda'_y\Omega_{11}, \\ y''_{\lambda\lambda} = -\mu'_x\Lambda_{00} + \lambda'_x\Omega_{00}, & y''_{\lambda\mu} = -\mu'_x\Lambda_{01} + \lambda'_x\Omega_{01}, & y''_{\mu\mu} = -\mu'_x\Lambda_{11} + \lambda'_x\Omega_{11} \end{array} \tag{41}$$

The formulas (33) and (41) are all we need to be substituted into the equation system (21) or (22) to achieve a different form without composite functions. Let us do it step by step

$$\begin{aligned} \tfrac{f'_\lambda}{P}P'_\mu &= \tfrac{f'_x x'_\lambda + f'_y y'_\lambda}{x'^2_\lambda + y'^2_\lambda}(x'_\lambda x''_{\lambda\mu} + y'_\lambda y''_{\lambda\mu}) \\ &= \tfrac{f'_x\mu'_y - f'_y\mu'_x}{\mu'^2_x + \mu'^2_y}(\mu'_y(\mu'_y\Lambda_{01} - \lambda'_y\Omega_{01}) - \mu'_x(-\mu'_x\Lambda_{01} + \lambda'_x\Omega_{01})) \\ &= (f'_x\mu'_y - f'_y\mu'_x)\Lambda_{01} \end{aligned} \tag{42}$$

where we have used the orthogonal condition (30). Similarly we have

$$\begin{aligned} \tfrac{f'_\mu}{Q}Q'_\lambda &= \tfrac{(f'_x x'_\mu + f'_y y'_\mu)(x'_\mu x''_{\lambda\mu} + y'_\mu y''_{\lambda\mu})}{x'^2_\mu + y'^2_\mu} \\ &= (-f'_x\lambda'_y + f'_y\lambda'_x)\Omega_{01} \end{aligned} \tag{43}$$

Then we have

$$\begin{array}{ll} f''_{\lambda\mu} & = \frac{1}{\delta^2}(-f''_{xx}\lambda'_y\mu'_y + f''_{xy}(\lambda'_x\mu'_y + \lambda'_y\mu'_x) - f''_{yy}\lambda'_x\mu'_x) \\ & +f'_x(\mu'_y\Lambda_{01} - \lambda'_y\Omega_{01}) + f'_y(-\mu'_x\Lambda_{01} + \lambda'_x\Omega_{01}) \end{array} \tag{44}$$

Finally, the equation system (21) or (22) is transformed into the one without composite functions

$$\begin{cases} (f'_x\mu'_y - f'_y\mu'_x)\Lambda_{01} = (-f'_x\lambda'_y + f'_y\lambda'_x)\Omega_{01}, \\ \\ \frac{1}{\delta^2}(-f''_{xx}\lambda'_y\mu'_y + f''_{xy}(\lambda'_x\mu'_y + \lambda'_y\mu'_x) - f''_{yy}\lambda'_x\mu'_x) \\ +f'_x(\mu'_y\Lambda_{01} - \lambda'_y\Omega_{01}) + f'_y(-\mu'_x\Lambda_{01} + \lambda'_x\Omega_{01}) = (f'_x\mu'_y - f'_y\mu'_x)\Lambda_{01}, \\ \\ \lambda'_x\mu'_x + \lambda'_y\mu'_y = 0 \end{cases} \tag{45}$$

That is,

$$\begin{cases} (f'_x\mu'_y - f'_y\mu'_x)\Lambda_{01} = (-f'_x\lambda'_y + f'_y\lambda'_x)\Omega_{01}, \\ \frac{1}{\delta^2}(f''_{xx}\lambda'_y\mu'_y - f''_{xy}(\lambda'_x\mu'_y + \lambda'_y\mu'_x) + f''_{yy}\lambda'_x\mu'_x) = (-f'_x\lambda'_y + f'_y\lambda'_x)\Omega_{01}, \\ \lambda'_x\mu'_x + \lambda'_y\mu'_y = 0 \end{cases} \tag{46}$$

or

$$\begin{cases} (f'_x\mu'_y - f'_y\mu'_x)(\lambda'_x\mu'_x(\lambda''_{yy} - \lambda''_{xx}) - (\lambda'_x\mu'_y + \lambda'_y\mu'_x)\lambda''_{xy}) \\ = (-f'_x\lambda'_y + f'_y\lambda'_x)(\lambda'_x\mu'_x(\mu''_{yy} - \mu''_{xx}) - (\lambda'_x\mu'_y + \lambda'_y\mu'_x)\mu''_{xy}), \\ \\ (\lambda'_x\mu'_y - \lambda'_y\mu'_x)(\lambda'_x\mu'_x(f''_{yy} - f''_{xx}) - (\lambda'_x\mu'_y + \lambda'_y\mu'_x)f''_{xy}) \\ = (\lambda'_xf'_y - \lambda'_yf'_x)(\lambda'_x\mu'_x(\mu''_{yy} - \mu''_{xx}) - (\lambda'_x\mu'_y + \lambda'_y\mu'_x)\mu''_{xy}), \\ \\ \lambda'_x\mu'_x + \lambda'_y\mu'_y = 0 \end{cases} \tag{47}$$

where we have used the orthogonal condition (30). The equation system involves three unknowns $f(x, y), \lambda(x, y)$ and $\mu(x, y)$.

If we express the first equation in the equation system (47) into a form of array multiplication and further require that the two functions (29) satisfy Cauchy-Riemann equations,

$$\begin{array}{l} \lambda'_x = \mu'_y, \\ \lambda'_y = -\mu'_x \end{array} \tag{48}$$

then the equation reduces to

$$f'_x(\lambda'_y\lambda''_{xx} - \lambda'_x\lambda''_{xy}) + f'_y(\lambda'_x\lambda''_{xx} + \lambda'_y\lambda''_{xy}) = 0 \tag{49}$$

The rational structure equation system (47) becomes

$$\begin{cases} f'_x(\lambda'_y\lambda''_{xx} - \lambda'_x\lambda''_{xy}) + f'_y(\lambda'_x\lambda''_{xx} + \lambda'_y\lambda''_{xy}) = 0, \\ \\ (\lambda'^2_x + \lambda'^2_y)(-\lambda'_x\lambda'_y(f''_{yy} - f''_{xx}) + (\lambda'^2_y - \lambda'^2_x)f''_{xy}) \\ = (\lambda'_xf'_y - \lambda'_yf'_x)(-2\lambda'_x\lambda'_y\lambda''_{xy} + (\lambda'^2_y - \lambda'^2_x)\lambda''_{xx}), \\ \\ \lambda''_{xx} + \lambda''_{yy} = 0 \end{cases} \tag{50}$$

which is our main result, i. e. the necessary and sufficient condition for rational structure. The equation system involves two unknowns $f(x, y)$ and $\lambda(x, y)$. We will know that the choice (48) leads our equation system to being very simple and separates Green's theorem (11) from the requirement of orthogonality of the proportion curves. But the solution of rational structure equation system becomes the search for the correct complex analytic functions.

# 5 Rational Standard Equation

Now we want to change the above equation system (50) into the so-called standard equation system which looks simpler and more elegant, and is easier for its general solution. To do so, we need introduce new unknowns

$$\begin{cases} \alpha(x, y) = \lambda'_x, \\ \beta(x, y) = \lambda'_y \end{cases} \tag{51}$$

which describes the proportion curves of rational structure. The introduction means the following identity

$$\alpha'_y = \beta'_x \tag{52}$$

In the meantime, the third equation of the equation system (50) becomes

$$\alpha'_x = -\beta'_y \tag{53}$$

They together mean that the complex function

$$\beta(x, y) + \mathrm{i}\, \alpha(x, y) \tag{54}$$

is an analytic function, and the formulas (52) and (53) are its Cauchy-Riemann equations.

Now we turn to the first equation of the rational equation system (50). With the new variables, it is

$$f'_x(\beta\alpha'_x - \alpha\beta'_x) + f'_y(\alpha\alpha'_x + \beta\beta'_x) = 0 \tag{55}$$

Using the Cauchy-Riemann equations (52) and (53), we have

$$f'_x((\alpha^2)'_y + (\beta^2)'_y) = f'_y((\alpha^2)'_x + (\beta^2)'_x) \tag{56}$$

Now we introduce another quantity

$$G(x, y) = \alpha^2 + \beta^2, \tag{57}$$

and the first equation of the equation system (50) becomes finally

$$f'_x G'_y = f'_y G'_x \tag{58}$$

This is called the stretch equation for the given vector field $(f'_x, f'_y)$. It is the simplest first order partial differential equation because it is linear and homogeneous. Its solution is completely determined by the function $H(x, y)$

$$\begin{cases} G'_x = H(x, y) f'_x, \\ G'_y = H(x, y) f'_y \end{cases} \tag{59}$$

We call $H(x,y)$ a stretch of the stretch equation (58). It is straightforward to show that $H(x,y)$ itself must be another solution of the equation and must be determined by another stretch $I(x,y)$. $I(x,y)$ must be some other solution of the stretch equation and must be determined by some other stretch $J(x,y)$, and so forth. Therefore, the general solution of the stretch equation (58) corresponds to all stretches. However, our solution $G(x,y)$ must be simultaneously the squared modulus of some analytic complex function (54).

Now we turn to the second equation of the equation system (50). With the new variables, it is

$$\begin{aligned}&(\alpha^2+\beta^2)(-\alpha\beta(f''_{yy}-f''_{xx})+(\beta^2-\alpha^2)f''_{xy})\\&=(\alpha f'_y-\beta f'_x)(-2\alpha\beta\alpha'_y+(\beta^2-\alpha^2)\alpha'_x)\end{aligned}\tag{60}$$

That is,

$$\begin{aligned}&G(\alpha\beta(f''_{yy}-f''_{xx})+(\alpha^2-\beta^2)f''_{xy})\\&+(\beta f'_x-\alpha f'_y)(2\alpha\beta\alpha'_y+(\alpha^2-\beta^2)\alpha'_x)=0\end{aligned}\tag{61}$$

The final factor of the above equation can be further transformed with the Cauchy-Riemann equations (52) and (53),

$$\begin{aligned}&2\alpha\beta\alpha'_y+(\alpha^2-\beta^2)\alpha'_x\\&=\alpha\beta\alpha'_y+\alpha\beta\alpha'_y+\tfrac{1}{2}\alpha(\alpha^2)'_x+\beta^2\beta'_y\\&=\tfrac{1}{2}\beta(\alpha^2)'_y+\alpha\beta\beta'_x+\tfrac{1}{2}\alpha(\alpha^2)'_x+\tfrac{1}{2}\beta(\beta^2)'_y\\&=\tfrac{1}{2}\beta(\alpha^2)'_y+\tfrac{1}{2}\alpha(\beta^2)'_x+\tfrac{1}{2}\alpha(\alpha^2)'_x+\tfrac{1}{2}\beta(\beta^2)'_y\\&=\tfrac{1}{2}\beta(\alpha^2+\beta^2)'_y+\tfrac{1}{2}\alpha(\alpha^2+\beta^2)'_x\\&=\tfrac{1}{2}\alpha G'_x+\tfrac{1}{2}\beta G'_y\\&=\tfrac{1}{2}\alpha Hf'_x+\tfrac{1}{2}\beta Hf'_y\end{aligned}\tag{62}$$

Substituting the result (62) into the equation (61), we transform the second equation of the equation system (50) into

$$\begin{aligned}&G(\alpha\beta(f''_{yy}-f''_{xx})-(\beta^2-\alpha^2)f''_{xy})\\&+\tfrac{1}{2}H(-\alpha\beta(f'^2_y-f'^2_x)+(\beta^2-\alpha^2)f'_xf'_y)=0\end{aligned}\tag{63}$$

Although the two independent unknowns $\alpha$ and $\beta$ greatly reduce the form of our equation system, we are not satisfied with them. We introduce two new independent unknowns

$$\varphi(x,y)=2\alpha\beta,\quad \psi(x,y)=\beta^2-\alpha^2,\tag{64}$$

and the equation (63) becomes

$$2G(\varphi(f''_{yy}-f''_{xx})-\psi\,2f''_{xy})=H(\varphi(f'^2_y-f'^2_x)-\psi\,2f'_xf'_y)\tag{65}$$

which is called the rational standard equation.

We see the highly symmetric expressions of the derivatives to $f(x,y)$. They appeared also in the formulas (6) in the reference [3] which goes a different way to approaching rational structure (the main unknown in the reference is the tangent angle of the proportion curves). Therefore, we introduce some symbols to denote the symmetric expressions

$$\begin{aligned}&\mathrm{A}=f''_{yy}-f''_{xx},\\&\mathrm{B}=2f''_{xy},\\&\tilde{A}=f'^2_y-f'^2_x,\\&\tilde{B}=2f'_xf'_y\end{aligned}\tag{66}$$

We know that $\alpha$ and $\beta$ satisfy Cauchy-Riemann equations. Accordingly, their transformations $\varphi$ and $\psi$ satisfy the same Cauchy-Riemann equations

$$\varphi'_y = \psi'_x, \ \ \varphi'_x = -\psi'_y \tag{67}$$

and the complex function

$$\Psi(x, y) = \psi(x, y) + \mathrm{i}\, \varphi(x, y) \tag{68}$$

is analytic. It is straightforward to show that

$$G^2 = \varphi^2 + \psi^2 \tag{69}$$

Finally the quantities $\alpha$ and $\beta$ are no longer needed, and the equation system (50) can be transformed into

$$\begin{cases} \varphi'_y = \psi'_x, \ \ \varphi'_x = -\psi'_y, \\ G^2(x, y) = \varphi^2 + \psi^2, \\ G'_x = H(x, y) f'_x, \ \ G'_y = H(x, y) f'_y, \\ (2G\mathrm{A} - H\tilde{A})\varphi = (2G\mathrm{B} - H\tilde{B})\psi \end{cases} \tag{70}$$

which involves five unknowns, $f(x, y), \varphi(x, y), \psi(x, y), G(x, y)$ and $H(x, y)$. The equation system is the necessary and sufficient condition for rational structure. If we substitute the formulas (51) into the above equation system (70) then it must return to the original equation system (50).

Two important results come from the rational standard equation system (70) immediately.

**Theorem 1** (geometric meaning): Corresponding to any point $(x, y)$ in the structure plane, the quantities $G, |\varphi|$ and $|\psi|$ are the lengths of the hypotenuse and the two sides of a right triangle, respectively (see Figure 1).

**Theorem 2** (separation of Green's theorem from the orthogonality of proportion curves): If we ignore the first two equations in (70) then for any arbitrary function $f(x, y)$ the remaining equation system has infinite solutions of $G(x, y)$ and accordingly infinite solutions of the pair of $\varphi(x, y)$ and $\psi$.

Theorem 2 results from two facts. The first fact is that our stretch equation is linear and homogeneous and has infinite solutions (see the next Section). The second fact is that the triangle solution in Figure 1 always exists. Therefore, the choice (48) leads our equation system to being very simple and separates Green's theorem (11) from the requirement of orthogonality of the proportion curves. But the solution of rational structure becomes the search for the correct complex analytic functions.

# 6 General Solution

In fact, the stretch equation (58) of the unknown $G(x, y)$ is a linear and homogeneous differential equation which has the standard general solution. The solution results from the related ordinary differential equation (the characteristic equation)

$$\frac{dx}{f'_y} = \frac{dy}{-f'_x} \tag{71}$$

That is,

$$0 = f'_x dx + f'_y dy = df(x, y) \tag{72}$$

The general solution to the above equation is

$$\sigma = f(x, y) = c \tag{73}$$

where $c$ is the arbitrary constant. The general solution to the stretch equation (58) is determined indirectly by the following equation

$$W(\sigma, \tau) = 0 \tag{74}$$

where $W(\sigma, \tau)$ is an arbitrary function of two variables $\sigma$ and $\tau$, and

$$\sigma = f(x, y), \quad \tau = G \tag{75}$$

That is,

$$W(f(x, y), G) = 0 \tag{76}$$

is the general solution to the stretch equation (58). The corresponding stretch is

$$H(x, y) = \frac{-W'_\sigma}{W'_\tau} = \frac{-W'_\sigma(f(x, y), G)}{W'_\tau(f(x, y), G)} \tag{77}$$

Finally, the equation system (70) is transformed into

$$\left\{ \begin{array}{l} \varphi'_y = \psi'_x, \quad \varphi'_x = -\psi'_y, \\ W(f(x, y), \sqrt{\varphi^2 + \psi^2}) = 0, \\ (2\sqrt{\varphi^2 + \psi^2}\mathrm{A} + \frac{W'_\sigma}{W'_\tau}\tilde{A})\varphi = (2\sqrt{\varphi^2 + \psi^2}\mathrm{B} + \frac{W'_\sigma}{W'_\tau}\tilde{B})\psi \end{array} \right. \tag{78}$$

where

$$\begin{array}{l} W'_\sigma = W'_\sigma(f(x, y), \sqrt{\varphi^2 + \psi^2}), \\ W'_\tau = W'_\tau(f(x, y), \sqrt{\varphi^2 + \psi^2}) \end{array} \tag{79}$$

The equation system involves four unknowns, $f(x, y), \varphi(x, y), \psi(x, y)$ and $W(\sigma, \tau)$.

The general solution to the rational structure equation system (78) is all real function $f(x, y)$, all analytic complex function

$$\Psi(x, y) = \psi(x, y) + \mathrm{i}\, \varphi(x, y) \tag{80}$$

and all function

$$W(\sigma, \tau) \tag{81}$$

which satisfy the following two equations

$$\left\{ \begin{array}{l} W(f(x, y), \sqrt{\varphi^2 + \psi^2}) = 0, \\ (2\sqrt{\varphi^2 + \psi^2}\mathrm{A} + \frac{W'_\sigma}{W'_\tau}\tilde{A})\varphi = (2\sqrt{\varphi^2 + \psi^2}\mathrm{B} + \frac{W'_\sigma}{W'_\tau}\tilde{B})\psi \end{array} \right. \tag{82}$$

# 7 Sufficient Condition for Rational Structure

We may select a limited form of the function (81) and find a sufficient condition for rational structure. We will know that galaxy disks and double breasts have the following limited form

$$W(\sigma, \tau) = S(\sigma) - \tau \tag{83}$$

where

$$S(\sigma) \tag{84}$$

is an unknown function of single variable $\sigma$. Then the final equation of the equation system (78) becomes

$$(2S(f)\mathrm{A} - S'_f \tilde{A})\varphi = (2S(f)\mathrm{B} - S'_f \tilde{B})\psi \tag{85}$$

and the equation system (78) itself is

$$\left\{ \begin{array}{l} \varphi'_y = \psi'_x, \;\; \varphi'_x = -\psi'_y, \\ S^2(f(x,y)) = \varphi^2 + \psi^2, \\ (2\mathrm{A} - (\ln S)'_f \tilde{A})\varphi = (2\mathrm{B} - (\ln S)'_f \tilde{B})\psi \end{array} \right. \tag{86}$$

whose geometric meaning is that the quantities $S, |\varphi|$ and $|\psi|$ are the lengths of the hypotenuse and the two sides of a right triangle, respectively (see Figure 1).

# 8 Exponential Disk

Exponential disk is rational structure. Its logarithmic density is

$$f(x,y) = d_1 r \tag{87}$$

where $d_1$ is a constant and $r = \sqrt{x^2 + y^2}$. Its orthogonal net of curves is

$$\left\{ \begin{array}{l} x = e^{\lambda}\cos(\mu), \;\; y = e^{\lambda}\sin(\mu) \\ \lambda > -\infty \end{array} \right. \tag{88}$$

Its inverse is

$$\left\{ \begin{array}{l} \lambda = (1/2)\ln(x^2 + y^2), \\ \mu = \tan^{-1}\frac{y}{x} \end{array} \right. \tag{89}$$

We look for the corresponding $W(\sigma, \tau)$. We have $\sigma = f = d_1 r$ and $\tau = G = \sqrt{\lambda_x'^2 + \lambda_y'^2} = 1/r^2$. Therefore,

$$W(\sigma, \tau) = \frac{d_1^2}{\sigma^2} - \tau = \frac{d_1^2}{f^2} - G \equiv 0 \tag{90}$$

# 9 Heaven Breasts Structure

Heaven breasts structure is rational [4,5]. From now on, we change our convention so that the double breasts are paired horizontally. Previously we always present the bars of barred spiral galaxy pattern vertically, and the double breasts are demonstrated vertically accordingly. To be consistent with the Cauchy-Riemann equations of complex analytic functions (formulas (48)), we from now on, present the bars of barred spiral galaxy pattern horizontally. Accordingly, the double breasts are demonstrated horizontally and its logarithmic density is

$$f(x,y) = (b_2/3)\left((r^2 - b_1^2)^2 + 4b_1^2y^2\right)^{3/4} \tag{91}$$

where $b_1, b_2$ are constants. Its orthogonal net of curves is

$$\begin{cases} x = b_1\cosh(\lambda)\cos(\mu),\ y = b_1\sinh(\lambda)\sin(\mu), \\ \lambda \geq 0 \end{cases} \tag{92}$$

Its inverse is

$$\begin{cases} \lambda = \sinh^{-1}(p(x,y)/b_1), \\ \mu = \sin^{-1}(y/p(x,y)) \end{cases} \tag{93}$$

where

$$p(x,y) = \sqrt{\left(r^2 - b_1^2 + \sqrt{(r^2 - b_1^2)^2 + 4b_1^2y^2}\right)/2} \tag{94}$$

We introduce one more quantity

$$q(x,y) = \sqrt{(r^2 - b_1^2)^2 + 4b_1^2y^2} \tag{95}$$

It is straightforward to show that

$$\begin{aligned} &\tau = G = \sqrt{\lambda_x'^2 + \lambda_y'^2} \\ &= \tfrac{x^2p^2}{q^2(b_1^2+p^2)} + \tfrac{y^2(b_1^2+p^2)}{q^2p^2} \\ &= \tfrac{(-4b_1^2x^2)p^2}{-4b_1^2q^2(b_1^2+p^2)} + \tfrac{(4b_1^2y^2)(b_1^2+p^2)}{4b_1^2q^2p^2} \\ &= \tfrac{(q^2-(r^2+b_1^2)^2)((r^2-b_1^2+q)/2)}{-4b_1^2q^2(b_1^2+(r^2-b_1^2+q)/2)} + \tfrac{(q^2-(r^2-b_1^2)^2)(b_1^2+(r^2-b_1^2+q)/2)}{4b_1^2q^2(r^2-b_1^2+q)/2} \\ &= 1/q \end{aligned} \tag{96}$$

Therefore,

$$W(\sigma,\tau) = \frac{(b_2/3)^{2/3}}{\sigma^{2/3}} - \tau = \frac{(b_2/3)^{2/3}}{f^{2/3}} - G \equiv 0 \tag{97}$$

# 10 Application to Naked Galaxy Structure

If we ignore those strongly interacting galaxies or some extremely small galaxies, there exist only two types of galaxies: spiral galaxies and elliptical galaxies. There are two kinds of spiral galaxies. A spiral galaxy with a bar is called a barred spiral, and a spiral galaxy without a bar is called an ordinary spiral. Astronomers found out that the stellar density

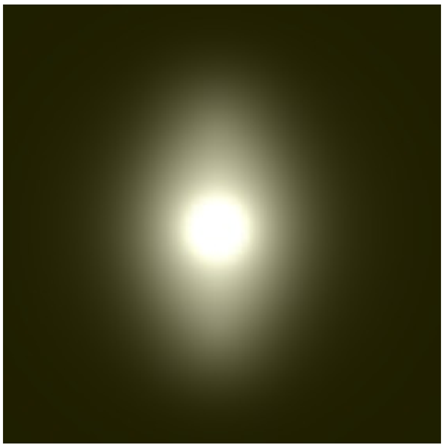

Figure 2: The simulated galaxy NGC 3275, i. e., the superimposed structure of double-breasts and exponential disk which is called the naked galaxy structure.

distribution of ordinary spiral galaxies, i. e., the naked ordinary spiral galaxies without much gas or dust, is basically an axi-symmetric disk described by the exponential disk (see the Section 8)

$$\rho_0(x, y) = d_0 \exp(d_1 r) \tag{98}$$

where $d_0$ is a constant. Its logarithmic density is

$$f_0(x, y) = d_1 r \tag{99}$$

Therefore, ordinary spiral galaxies are rational structure.

Astronomers have found out that the main structure of barred spiral galaxies is also the exponential disk. Therefore, we subtract the fitted exponential disk from a barred spiral galaxy image. What is left over? Jin He discovered that the left-over resembles human breasts [4,5]

$$\rho_i(x, y) = b_{i0} \exp\left((b_{i2}/3)\,\left((r^2 - b_{i1}^2)^2 + 4b_{i1}^2 y^2\right)^{3/4}\right) \tag{100}$$

where $b_{i0}, b_{i1}, b_{i2}$ are constants. Its logarithmic density is

$$f_i(x, y) = (b_{i2}/3)\,\left((r^2 - b_{i1}^2)^2 + 4b_{i1}^2 y^2\right)^{3/4} \tag{101}$$

Jin He calls it Heaven Breasts structure (see the above Section 9). Barred spiral galaxies, however, generally have more than a pair of breasts. The bar of barred spiral galaxies is composed of two or three pairs of breasts which are usually aligned. The addition of the two or three pairs of breasts to the major structure of exponential disk becomes a bar-shaped pattern which crosses galaxy center (see the simulated naked galaxy Figure 2).

Therefore, our model of barred spiral galaxy structure is

$$\rho(x, y) = \rho_0(x, y) + \rho_1(x, y) + \rho_2(x, y) \tag{102}$$

The logarithmic density of barred spiral galaxies is

$$f(x,y) = \ln\rho(x,y) = \ln(\rho_0(x,y) + \rho_1(x,y) + \rho_2(x,y)) \tag{103}$$

It is straightforward to show that

$$\nabla f = (\rho_0 \nabla f_0 + \rho_1 \nabla f_1 + \rho_2 \nabla f_2)/(\rho_0 + \rho_1 + \rho_2) \tag{104}$$

That is, the gradient of sum is the sum of weighted gradients. It is some kind of averaging which is usually denoted by a bar above the corresponding symbol of the variable. We follow the notation. Here is an example

$$f'_x = (\rho_0 f'_{0x} + \rho_1 f'_{1x} + \rho_2 f'_{2x})/(\rho_0 + \rho_1 + \rho_2) = \overline{f'_{ix}} \tag{105}$$

Here comes the big question. Is the sum of rational structures also a rational structure (see the formula (102))? Numerical calculation suggests that it is approximately true for the fitted galaxy values of the parameters $d_0, d_1, b_{i0}, b_{i1}, \cdots$ (see the paper [5]). However, we need mathematical justification. It is left for future work.

In case you need them, we present some useful formulas in the Appendix.

# References


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# 11 Appendix

In the following formulas, $\alpha, \beta, \gamma, \delta$ may be the Cartesian coordinates $x$ or $y$, or may be the parameters $\lambda$ or $\mu$.

$$f'_\alpha = \overline{f'_{i\alpha}}, \tag{106}$$

$$f''_{\alpha\beta} = \overline{f'_{i\alpha} f'_{i\beta}} + \overline{f''_{i\alpha\beta}} - \overline{f'_{i\alpha}}\ \overline{f'_{i\beta}}, \tag{107}$$

$$\begin{aligned} f'''_{\alpha\beta\gamma} &= \overline{f'_{i\alpha} f'_{i\beta} f'_{i\gamma}} + \overline{f''_{i\alpha\beta} f'_{i\gamma}} + \overline{f''_{i\beta\gamma} f'_{i\alpha}} + \overline{f''_{i\alpha\gamma} f'_{i\beta}} + \overline{f'''_{i\alpha\beta\gamma}} \\ &- \overline{f'_{i\alpha}}\ \overline{f'_{i\beta}}\ \overline{f'_{i\gamma}} - \overline{f''_{i\alpha\beta}}\ \overline{f'_{i\gamma}} - \overline{f''_{i\beta\gamma}}\ \overline{f'_{i\alpha}} - \overline{f''_{i\alpha\gamma}}\ \overline{f'_{i\beta}}, \end{aligned} \tag{108}$$

$$
\begin{aligned}
&f''''_{\alpha\beta\gamma\delta}\\
&= \overline{f'_{i\alpha} f'_{i\beta} f'_{i\gamma} f'_{i\delta}}\\
&+ \overline{f''_{i\alpha\beta} f'_{i\gamma} f'_{i\delta}} + \overline{f''_{i\alpha\gamma} f'_{i\beta} f'_{i\delta}} + \overline{f''_{i\alpha\delta} f'_{i\gamma} f'_{i\beta}} + \overline{f''_{i\beta\gamma} f'_{i\alpha} f'_{i\delta}} + \overline{f''_{i\beta\delta} f'_{i\gamma} f'_{i\alpha}} + \overline{f''_{i\gamma\delta} f'_{i\alpha} f'_{i\beta}}\\
&+ \overline{f''_{i\alpha\beta} f''_{i\gamma\delta}} + \overline{f''_{i\alpha\gamma} f''_{i\beta\delta}} + \overline{f''_{i\alpha\delta} f''_{i\beta\gamma}}\\
&+ \overline{f'''_{i\alpha\beta\gamma} f'_{i\delta}} + \overline{f'''_{i\alpha\beta\delta} f'_{i\gamma}} + \overline{f'''_{i\alpha\gamma\delta} f'_{i\beta}} + \overline{f'''_{i\beta\gamma\delta} f'_{i\alpha}} + \overline{f''''_{i\alpha\beta\gamma\delta}}\\
&- \overline{f'_{i\alpha}}\;\overline{f'_{i\beta}}\;\overline{f'_{i\gamma}}\;\overline{f'_{i\delta}}\\
&- \overline{f''_{i\alpha\beta}}\;\overline{f'_{i\gamma}}\;\overline{f'_{i\delta}} - \overline{f''_{i\alpha\gamma}}\;\overline{f'_{i\beta}}\;\overline{f'_{i\delta}} - \overline{f''_{i\alpha\delta}}\;\overline{f'_{i\gamma}}\;\overline{f'_{i\beta}} - \overline{f''_{i\beta\gamma}}\;\overline{f'_{i\alpha}}\;\overline{f'_{i\delta}} - \overline{f''_{i\beta\delta}}\;\overline{f'_{i\gamma}}\;\overline{f'_{i\alpha}} - \overline{f''_{i\gamma\delta}}\;\overline{f'_{i\alpha}}\;\overline{f'_{i\beta}}\\
&- \overline{f''_{i\alpha\beta}}\;\overline{f''_{i\gamma\delta}} - \overline{f''_{i\alpha\gamma}}\;\overline{f''_{i\beta\delta}} - \overline{f''_{i\alpha\delta}}\;\overline{f''_{i\beta\gamma}}\\
&- \overline{f'''_{i\alpha\beta\gamma}}\;\overline{f'_{i\delta}} - \overline{f'''_{i\alpha\beta\delta}}\;\overline{f'_{i\gamma}} - \overline{f'''_{i\alpha\gamma\delta}}\;\overline{f'_{i\beta}} - \overline{f'''_{i\beta\gamma\delta}}\;\overline{f'_{i\alpha}}
\end{aligned}
\tag{109}
$$