Few types of chains of primes arising in the study of pseudoprimes

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Abstract. While studying Fermat pseudoprimes I met few interesting generic forms of numbers that have the property to generate chains of primes and pseudoprimes. I list in this paper few such types of chains.

I. Recurrent chains

I.1.
Chains of primes of the form \( P_0, P_1 = P_0\cdot n - n + 1, P_2 = P_0\cdot n - n + 1, \ldots, P_k = P_{k-1}\cdot n - n + 1 \), where \( P_0, P_1, \ldots, P_k \) are primes and \( n \) is a positive integer, \( n > 1 \).

Note: For \( n = 2 \), we obtain the Cunningham chain of the second kind, i.e. \( P_{i+1} = 2\cdot P_i - 1 \).

For instance:

A chain of primes of length 6, for \( n = 3 \), should have the form: \( p, 3p - 2, 9p - 8, 27p - 26, 81p - 80, 243p - 242 \). It can be seen that \( p \) must be of the form \( 10k + 1 \) for a chain of length bigger than 3. Such a chain, of length 5, is: 61, 181, 541, 1621, 4861.

A chain of primes of length 5, for \( n = 4 \), should have the form: \( p, 4p - 3, 16p - 15, 64p - 63, 256p - 255 \). It can be seen that \( p \) can’t be of the form \( 10k + 7 \). Such a chain, of length 4, is: 23, 89, 353, 1409.

Notes:
The formula \( P_{i+1} = P_i\cdot n - n + 1 \) can lead to the formation of chains of Fermat pseudoprimes, for instance for Carmichael numbers \([P_0, P_1] = [1729, 46657]\) and \( n = 27 \).

The formula \( P_{i+1} = P_i\cdot (P_i\cdot n - n + 1) \) can also lead to the formation of chains of Fermat pseudoprimes; for instance, for \( n = 2 \), \( P_{i+1} = P_i\cdot (2\cdot P_i - 1) \) leads to the formation of the following Poulet numbers \([P_0, P_1]: [645, 831405], [1729, 5977153] \text{ etc.}; \) for \( n = 3 \), \( P_{i+1} = P_i\cdot (3\cdot P_i - 2) \) leads
to the formation of the following Poulet numbers \([P_0, P_1]: [341, 348161], [645, 1246785]\) etc. (see the sequence A215343 in OEIS).

I.2.
Chains of primes of the form \(P_0, P_1 = P_0*n - n - 1, P_2 = P_1*n - n - 1, \ldots, P_k = P_{k-1}*n - n - 1\), where \(P_0, P_1, \ldots, P_k\) are primes and \(n\) is a positive integer, \(n > 1\).

For instance:
A chain of primes of length 6, for \(n = 2\), should have the form: \(p, 2p - 3, 4p - 9, 8p - 21, 16p - 45, 32p - 93\). It can be seen that \(p\) must be of the form \(10k + 3\) for a chain of length bigger than 3. Such a chain, of length 4, is: 113, 223, 443, 883.

A chain of primes of length 5, for \(n = 3\), should have the form: \(p, 3p - 4, 9p - 16, 27p - 52, 81p - 160\). It can be seen that \(p\) must be of the form \(10k + 7\) for a chain of length bigger than 3. Such a chain, of length 4, is: 7, 17, 47, 137.

Note: I met this type of numbers in the study of Fermat pseudoprimes to base 2 with three prime factors (see the sequence A215672 in OEIS). Most of them can be written as \(p*(p*n - n + 1)*(p*m - m + 1)\) or as \(p*(p*n - n - 1)*(p*m - m - 1)\).

I.3.
Chains of primes of the form \(P_0, P_1 = P_0*n + n + 1, P_2 = P_1*n + n + 1, \ldots, P_k = P_{k-1}*n + n + 1\), where \(P_0, P_1, \ldots, P_k\) are primes and \(n\) is a positive integer, \(n > 1\).

Note: For \(n = 2\), we obtain the Cunningham chain of the first kind, i.e. \(P_{i+1} = 2*P_i + 1\).

For instance:
A chain of primes of length 6, for \(n = 3\), should have the form: \(p, 3p + 2, 9p + 8, 27p + 26, 81p + 80, 243p + 242\). It can be seen that \(p\) must be of the form \(10k + 9\) for a chain of length bigger than 3. Such a chain, of length 4, is: 29, 89, 269, 809.

I.4.
Chains of primes of the form \(P_0, P_1 = P_0*n + n + 1, P_2 = P_1*n + n + 1, \ldots, P_k = P_{k-1}*n + n + 1\), where \(P_0, P_1, \ldots, P_k\) are primes and \(n\) is a positive integer, \(n > 1\).
For instance:
A chain of primes of length 6, for n = 2, should have the form: p, 2p + 3, 4p + 9, 8p + 21, 16p + 45, 32p + 93. It can be seen that p must be of the form 10k + 7 for a chain of length bigger than 3. Such a chain, of length 6, is: 47, 97, 197, 397, 797, 1597.

I.5.
Chains of primes of the form $P_0, P_1 = P_0n - d*n + d, P_2 = P_1*n - d*n + d, \ldots, P_k = P_{k-1}*n - d*n + d$, where $P_0, P_1, \ldots, P_k$ are primes, d is also a prime number and n is a positive integer, n > 1.

For instance:
A chain of this type of primes of length 6, for n = 2 and d = 7, should have the form: p, 2p - 7, 4p - 21, 8p - 49, 16p - 105, 32p - 217. It can be seen that p must be of the form 30k + 7 for a chain of length bigger than 3.

A chain of this type of primes of length 6, for n = 2 and d = 13, should have the form: p, 2p - 13, 4p - 39, 8p - 91, 16p - 195, 32p - 403. It can be seen that p must be of the form 30k + 13 for a chain of length bigger than 3. Such a chain, of length 4, is 163, 313, 613, 1213.

Note: I met this type of numbers in the study of Fermat pseudoprimes to base 2 with two prime factors (see the sequence A214305 in OEIS); for instance, for n = 3 and d = 73, $P_{i+1} = 3*P_i - 2*73$ leads to the formation of the following Poulet numbers $[P_0,P_1]$: [2701,7957] etc.; for n = 4 and d = 73, $P_{i+1} = 4*P_i - 3*73$ leads to the formation of the following Poulet numbers $[P_0,P_1]$: [2701,10585] etc.

II. Non-recurrent chains

II.1.
Chains of primes of the form $30*a*n - (a*p + a - 1)$, where p and $a*p + a - 1$ are primes and n has successive values of integers.

For instance:
For $p = 11$, $a = 2$, n from -1 to 3 we have, in absolute value, the following chain of primes of length 5: 83, 23, 37, 97, 157.
For \( p = 23, a = 2, \) \( n \) from -3 to 2 we have, in absolute value, the following chain of primes of length 6: 227, 167, 107, 47, 13, 73.

For \( p = 7, a = 3, \) \( n \) from -1 to 2 we have, in absolute value, the following chain of primes of length 4: 113, 23, 67, 157.

**Note:** I met this type of numbers in the study of Carmichael numbers of the form \( C = ((30a*n - (a*p + a - 1))((30b*n - (b*p + b - 1))((30c*n - (c*p + c - 1))), \)
where \( p, a*p + a - 1, b*p + b - 1 \) and \( c*p + c - 1 \) are all primes. Many Carmichael numbers can be written in this form (see the sequence A182416 in OEIS).

**II.2.**
Chains of primes of the form \( 30a*n + (a*p + a - 1), \)
where \( p \) and \( a*p + a - 1 \) are primes and \( n \) has successive values of integers.

For instance:

For \( p = 19, a = 3, \) \( n \) from -1 to 2 we have, in absolute value, the following chain of primes of length 4: 31, 59, 149, 239.

**Note:** I met this type of numbers in the study of Carmichael numbers of the form \( C = ((30a*n + (a*p - a + 1))((30b*n + (b*p - b + 1))((30c*n + (c*p - c + 1))), \)
where \( p, a*p + a - 1, b*p + b - 1 \) and \( c*p + c - 1 \) are all primes. Many Carmichael numbers can be written in this form (see the sequence A182416 in OEIS).

**II.3.**
Chains of primes of the form \( 2*p*n - 2*n + p, \) where \( p \) and \( 2p - 1 \) are primes and \( n \) has successive values of integers.

For instance:

For \( p = 7, n \) from -5 to 3 we have, in absolute value, the following chain of primes of length 9: 53, 41, 29, 17, 5, 7, 19, 31, 43.

**Note:** I met this type of numbers in the study of Carmichael numbers of the form \( C = p*(2*p - 1)*(2*p*n - 2*n + p). \) I conjecture that all Carmichael numbers divisible with \( p \) and \( 2p - 1, \) where \( p \) and \( 2p - 1 \) are primes, can be written in this form (see the sequence A182207 in OEIS).