The new accelerated coordinate system by the tetrad

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ABSTRACT
In the general relativity theory, find the the Rindler theory’s mistake, find the new accelerated theory that used the tetrad on the new method. And discover the new inverse-coordinate transformation of the new accelerated theory and expand to be the new accelerated theory of the accelerated observer that have the initial velocity.

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1. Introduction

This theory is to study the Rindler coordinate theory and understand the Rindler coordinate theory and find the Rindler theory’s mistake, find the new accelerated theory that used the tetrad on the new method. And expand to be the new accelerated theory of the accelerated observer that have the initial velocity.

Finding the Rindler’s coordinate theory, use following the formula about the constant accelerated matter.

\[ x + \frac{c^2}{a_0} = \frac{c^2}{a_0} \cosh\left(\frac{a_0 \tau}{c}\right), t = \frac{c}{a_0} \sinh\left(\frac{a_0 \tau}{c}\right) \]  

\(x\) and \(t\) is the coordinate and the time in the inertial system about the constant accelerated matter. \(a_0\) is the constant acceleration, \(\tau\) is invariable time about the constant accelerated matter, \(c\) is light speed in the inertial system in the free space-time.

In the special relativity, the formula about 2-Dimension inertial coordinate system \(S(t, x)\) and \(S'(t', x')\) is

\[ V = \frac{u + v_0}{1 + \frac{u}{c^2} v_0}, \quad V = \frac{dx}{dt}, \quad u = \frac{dx'}{dt'}, \quad \frac{dx'}{dt'} = \frac{dx}{dt} \sqrt{1 - \frac{v_0^2}{c^2}}, \quad \frac{dt'}{dt} = \frac{dt + \frac{v_0}{c^2} dx'}{\sqrt{1 - \frac{v_0^2}{c^2}}} \]

\[ a = \frac{d}{dt} \left( \frac{V}{\sqrt{1 - \frac{V^2}{c^2}}} \right), \quad a' = \frac{d}{dt'} \left( \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} \right) \]

(1-1)

The velocity \(V\) has the initial velocity \(v_0\) and the velocity \(u\) is the velocity by the pure acceleration \(a'\).

\[ a = \frac{d}{dt} \left( \frac{V}{\sqrt{1 - \frac{V^2}{c^2}}} \right) = \frac{d}{dt} \left( \frac{u + v_0}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = \frac{1}{1 + \frac{v_0}{c^2} u} \frac{d}{dt} \left( \frac{u + v_0}{\sqrt{1 - \frac{u^2}{c^2}}} \right) \]

\[ a(1 + \frac{v_0}{c^2} u) = \frac{d}{dt} \left( \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} \right) + \frac{d}{dt} \left( \frac{v_0}{\sqrt{1 - \frac{u^2}{c^2}}} \right) \]

(1-2)

In this time, if the pure acceleration \(a'\) of the velocity \(u\) is

\[ a' = \frac{d}{dt'} \left( \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} \right), \quad u = \sqrt{1 + \frac{1}{c^2} \left( \int a' dt' \right)^2} \]

(1-3)

Eq(1-2) is
\[
a(1 + \frac{v_0}{c^2} u) = \frac{d}{dt'} \left( \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} \right) + \frac{d}{dt'} \left( \frac{v_0}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = a' + v_0 \frac{d}{dt'} \left( \sqrt{1 + \frac{1}{c^2} \left[ \int a' dt' \right]^2} \right)
\]

\[
= a' + v_0 \frac{\int a' dt'}{\sqrt{1 + \frac{1}{c^2} \left[ \int a' dt' \right]^2}} = a' + \frac{v_0}{c^2} \frac{\int a' dt'}{\sqrt{1 + \frac{1}{c^2} \left[ \int a' dt' \right]^2}} = a' \left( 1 + \frac{v_0}{c^2} \frac{u}{u} \right) \tag{1-4}
\]

Therefore, the acceleration \( a \) about the accelerated matter that has the initial velocity \( v_0 \) in 2-Dimension inertial coordinate system \( S(t, x) \) and the other acceleration \( a' \) about the accelerated matter that has not the initial velocity \( v_0 \) in 2-Dimension inertial coordinate system \( S'(t', x') \) are same.

In this time, if the acceleration \( a' \) is the constant acceleration \( a_0 \), the inertial acceleration in 2-Dimension inertial coordinate system \( S(t, x) \) and in 2-Dimension inertial coordinate system \( S'(t', x') \) is the constant acceleration \( a_0 \).

\[
a_0 = a' = \frac{d}{dt'} \left( \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = a = \frac{d}{dt} \left( \frac{\sqrt{V}}{\sqrt{1 - \frac{V^2}{c^2}}} \right) \tag{1-5}
\]

Hence, Eq(1) is in the 2-Dimension inertial coordinate system \( S'(t', x') \)

\[
x' = \frac{c^2}{a_0} ( \cosh \left( \frac{a_0 \tau}{c} \right) - 1 )
\]

\[
t' = \frac{c}{a_0} \sinh \left( \frac{a_0 \tau}{c} \right) \tag{1-6}
\]

Therefore, in the 2-Dimension inertial coordinate system \( S(t, x) \)

\[
t = \gamma(t' + \frac{v_0}{c^2} x') = \gamma \left( \frac{c}{a_0} \sinh \left( \frac{a_0 \tau}{c} \right) + \frac{v_0}{a_0} \left( \cosh \left( \frac{a_0 \tau}{c} \right) - 1 \right) \right)
\]

\[
x = \gamma(x' + v_0 t') = \gamma \left( \frac{c^2}{a_0} \left( \cosh \left( \frac{a_0 \tau}{c} \right) - 1 \right) + \frac{v_0 c}{a_0} \sinh \left( \frac{a_0 \tau}{c} \right) \right), \quad \beta = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \tag{1-7}
\]

\[
dt = \gamma \left( \cosh \left( \frac{a_0 \tau}{c} \right) + \frac{v_0}{c} \sinh \left( \frac{a_0 \tau}{c} \right) \right) d\tau
\]

\[
dx = \gamma \left( c \sinh \left( \frac{a_0 \tau}{c} \right) + v_0 \cosh \left( \frac{a_0 \tau}{c} \right) \right) d\tau
\]
\[ V = \frac{dx}{dt} = (c \tanh \left( \frac{a_0}{c} \tau \right) + v_0) / (1 + \frac{v_0}{c} \tanh \left( \frac{a_0}{c} \tau \right)) \] 
\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \] (1-8)

II. Additional chapter-I

The tetrad \( e_{a}^{\mu} \) is the unit vector that is each other orthographic and it used the following formula.

\[ e_{a}^{\mu} e_{b}^{\nu} g_{\mu \nu} = \eta_{ab} \quad (2) \]

\( e_{a}^{\mu} \) is

\[ e_{a}^{\mu} = \eta_{ab} g_{\mu \nu} e_{b}^{\nu} \quad (3) \]

and it is \( e_{a}^{\mu} \)'s inverse-metric. And it is

\[ e_{a}^{\mu} e_{b}^{\mu} = \delta_{a}^{b} \quad , \quad e_{a}^{\mu} e_{a}^{\nu} = \delta_{\mu}^{\nu} \]
\[ e_{a}^{\mu} e_{b}^{\nu} \eta_{ab} = g_{\mu \nu} \quad (4) \]

According to the tetrad \( e_{a}^{\mu} \) the flat Minkowski space’s 4-Dimension inertial coordinate system \( S(t, x, y, z) \) transform the 4-Dimension accelerated system \( \xi(\xi^0, \xi^1, \xi^2, \xi^3) \). In this time, the accelerated observer of the 4-Dimension accelerated system \( \xi(\xi^0, \xi^1, \xi^2, \xi^3) \) and the accelerated matter that has the initial velocity \( v_0 \) in 2-Dimension inertial coordinate system \( S(t, x) \) are same. Therefore

\[ d\tau^2 = dt^2 - \frac{1}{c^2}[dx^2 + dy^2 + dz^2] \]
\[ = -\frac{1}{c^2} \eta_{ab} \frac{\partial x^a}{\partial \xi^\mu} \frac{\partial x^b}{\partial \xi^\nu} d\xi^\mu d\xi^\nu \quad (5) \]
\[ = \frac{1}{c^2} \eta_{ab} e_{a}^{\mu} e_{b}^{\nu} d\xi^\mu d\xi^\nu = -\frac{1}{c^2} g_{\mu \nu} d\xi^\mu d\xi^\nu \quad (6) \]

\[ e_{a}^{\mu} = \frac{\partial x^a}{\partial \xi^\mu} \quad (7) \]

\( e_{a}^{\mu}(\tau) \) is the tetrad that if \( \xi^1 = \xi^2 = \xi^3 = 0 \) , \( d\xi^1 = d\xi^2 = d\xi^3 = 0 \). It is not the accelerated system and it is the point’s the accelerate motion. Therefore \( \xi^0 = \tau \) , in this case, it does \( g_{\mu \nu} = \eta_{\mu \nu} \)

According to Eq (1-7), Eq (6), Eq (7)
\[ e_0^\alpha (\tau) = \frac{\partial x^\alpha}{c \partial \xi^0} = \frac{1}{c} \frac{dx^\alpha}{d\tau} = \gamma \cosh \left( \frac{a_0}{c} \tau \right) + \frac{v_0}{c} \gamma \sinh \left( \frac{a_0}{c} \tau \right), 0, 0 \]  \[ \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \]  

(8)

About \( Y \)-axis’s and \( Z \)-axis’s orientation

\[ e_2^\alpha (\tau) = (0, 0, 1, 0) \]  \[ e_3^\alpha (\tau) = (0, 0, 0, 1) \]  

(9)  (10)

And the other unit vector \( e_1^\alpha (\tau) \) has to satisfy the tetrad condition, Eq (4)

\[ e_1^\alpha (\tau) = \gamma \sinh \left( \frac{a_0}{c} \tau \right) + \frac{v_0}{c} \gamma \cosh \left( \frac{a_0}{c} \tau \right), 0, 0 \]  

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \]  

(11)

III. Additional chapter-II

According to the new accelerated system’s need, \( e_\mu^\alpha (\xi^0) \) is used by Eq(8), Eq(9), Eq(10), Eq(11) that used \( \xi^0 \) instead of \( \tau \) and \( e_0^\alpha (\xi^0) \)’s term multiply \( \frac{a_0}{c} \).

The vector \( e_0^\alpha (\xi^0) \)

\[ e_0^\alpha (\xi^0) = \frac{\partial x^\alpha}{c \partial \xi^0} \]

\[ = \left( e^{0 \xi^0} \gamma \cosh \left( \frac{a_0}{c} \xi^0 \right) + e^{0 \xi^0} \frac{v_0}{c} \gamma \sinh \left( \frac{a_0}{c} \xi^0 \right) \right) \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \]  

(11-1)

\[ \frac{\partial e_0^\alpha (\xi^0)}{\partial \xi^1} = \frac{\partial^2 x^\alpha}{\partial \xi^1 \partial \xi^0} = \frac{\partial e_1^\alpha (\xi^0)}{c \partial \xi^0} \]  

(11-2)

Therefore, the vector \( e_1^\alpha (\xi^0) \) is

\[ e_1^\alpha (\xi^0) = \frac{\partial x^\alpha}{\partial \xi^1} = \left( e^{0 \xi^1} \gamma \sinh \left( \frac{a_0}{c} \xi^0 \right) + e^{0 \xi^1} \frac{v_0}{c} \gamma \cosh \left( \frac{a_0}{c} \xi^0 \right) \right) \]

\[ e^{0 \xi^1} \gamma \cosh \left( \frac{a_0}{c} \xi^0 \right) + e^{0 \xi^1} \frac{v_0}{c} \gamma \sinh \left( \frac{a_0}{c} \xi^0 \right), 0, 0 \]  

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \]  

(12)

About \( Y \)-axis’s and \( Z \)-axis’s orientation, the unit vector \( e_2^\alpha (\xi^0) \) and \( e_3^\alpha (\xi^0) \) is
\( e^\alpha_2 (\xi^0) = \frac{\partial x^\alpha}{\partial \xi^2} = (0,0,1,0) \) (13), \( e^\alpha_3 (\xi^0) = \frac{\partial x^\alpha}{\partial \xi^3} = (0,0,0,1) \) (14)

The differential coordinate transformation is

\[
\begin{align*}
\text{dx}^\alpha &= \frac{\partial x^\alpha}{\partial \xi^\mu} \text{d}\xi^\mu = \frac{\partial x^\alpha}{\partial \xi^0} \text{d}\xi^0 + \frac{\partial x^\alpha}{\partial \xi^1} \text{d}\xi^1 + \frac{\partial x^\alpha}{\partial \xi^2} \text{d}\xi^2 + \frac{\partial x^\alpha}{\partial \xi^3} \text{d}\xi^3 \\
&= e^\alpha_0 (\xi^0) \text{c} \text{d}\xi^0 + e^\alpha_1 (\xi^0) \text{d}\xi^1 + e^\alpha_2 (\xi^0) \text{d}\xi^2 + e^\alpha_3 (\xi^0) \text{d}\xi^3
\end{align*}
\]

\[
\text{cdt} = \gamma [e^{\alpha_0} (\xi^0) \{ \cosh \left( \frac{a_0 \xi^0}{c} \right) + \frac{v_0}{c} \sinh \left( \frac{a_0 \xi^0}{c} \right) \} \text{d}\xi^0 \\
+ e^{\alpha_1} (\xi^0) \{ \sinh \left( \frac{a_0 \xi^0}{c} \right) + \frac{v_0}{c} \cosh \left( \frac{a_0 \xi^0}{c} \right) \} \text{d}\xi^1 ]
\]

\[
\text{dx} = \gamma [e^{\alpha_0} (\xi^0) \{ \sinh \left( \frac{a_0 \xi^0}{c} \right) + \frac{v_0}{c} \cosh \left( \frac{a_0 \xi^0}{c} \right) \} \text{d}\xi^0 \\
+ e^{\alpha_1} (\xi^0) \{ \cosh \left( \frac{a_0 \xi^0}{c} \right) + \frac{v_0}{c} \sinh \left( \frac{a_0 \xi^0}{c} \right) \} \text{d}\xi^1 ] (16), \quad \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}.
\]

\[
dy = d\xi^2, \quad dz = d\xi^3 (17)
\]

Therefore if it integrates Eq(15), Eq(16) and Eq (17) ,finally the new accelerated system’s coordinate transformation of the accelerated observer with the initial velocity is found.

\[
\begin{align*}
\text{ct} &= \gamma \frac{c^2}{a_0} e^{\alpha_0} (\xi^0) \{ \sinh \left( \frac{a_0 \xi^0}{c} \right) + \frac{v_0}{c} \cosh \left( \frac{a_0 \xi^0}{c} \right) \} - \gamma \frac{V_r}{a_0} (18) \\
x &= \gamma \frac{c^2}{a_0} e^{\alpha_0} (\xi^0) \{ \cosh \left( \frac{a_0 \xi^0}{c} \right) + \frac{v_0}{c} \sinh \left( \frac{a_0 \xi^0}{c} \right) \} - \gamma \frac{c^2}{a_0} (19),
\end{align*}
\]

\[
y = z = \xi^3 (20), \quad \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}
\]

In this time, if t=0, then x=the accelerated system coordinate X(ksai 1), but Eq(19) is not. The theory(has exp(aX/c^2)) has this problem.

Therefore, the new inverse-coordinate transformation of the new accelerated system of the accelerated observer with the initial velocity is

\[
\frac{(ct + \gamma \frac{V_r}{a_0})}{a_0} = \frac{\tanh \left( \frac{a_0 \xi^0}{c} \right) + \frac{v_0}{c} \tanh \left( \frac{a_0 \xi^0}{c} \right)}{1 + \frac{v_0}{c} \tanh \left( \frac{a_0 \xi^0}{c} \right)}
\]
\[
\xi^0 = \frac{c}{a_0} \tanh^{-1}\left[\frac{(ct + \gamma \frac{v_0c}{a_0})}{c}\right] \quad (20-1), \quad \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}
\]

\[
\xi^1 = \frac{c^2}{2a_0} \ln \left|\frac{a_0}{c^2} \sqrt{\left(\frac{x + \gamma \frac{c^2}{a_0}\right)^2 - \left(\frac{ct + \gamma \frac{v_0c}{a_0}\right)^2}\right)}\right| \quad (20-2), \quad \xi^2 = \gamma, \xi^3 = z \quad (20-3), \quad \gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}
\]

### IV. Additional chapter-III

Therefore, the invariable time \( d\tau \) of the new accelerated theory of the accelerated observer with the initial velocity is by Eq(15), Eq(16), Eq(17)

\[
d\tau^2 = dt^2 - \frac{1}{c^2} \left[dx^2 + dy^2 + dz^2\right] = e^{\frac{2a_0}{c^2} \xi^0} \left[\left(d\xi^0\right)^2 - \frac{1}{c^2} \left(e^{\frac{2a_0}{c^2} \xi^0} (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2\right)\right] (21)
\]

Hence, the invariable time \( d\tau \) of the new accelerated theory of the accelerated observer that has the initial velocity \( v_0 \) is not related to the initial velocity \( v_0 \).

About \( x \)-axis’s light speed,

\[
dy = d\xi^2 = dz = d\xi^3 = 0.
\]

\[
cdt = dx, \quad ct = x, \quad c d\xi^0 = d\xi^1, \quad c \xi^0 = \xi^1 \quad (22)
\]

In this time, use the accelerated system’s coordinate transformation, Eq(18), Eq(19)

\[
ct = \gamma \frac{c^2}{a_0} e^{\frac{a_0}{c^2} \xi^0} \left\{ \sinh\left(\frac{a_0}{c} \xi^0\right) + \frac{v_0}{c} \cosh\left(\frac{a_0}{c} \xi^0\right) \right\} - \gamma \frac{v_0c}{a_0}
\]

\[
= \gamma \frac{c^2}{a_0} e^{\frac{a_0}{c} \xi^0} \left\{ e^{\frac{a_0}{c} \xi^0} - \frac{1}{2} \right\} + \frac{v_0}{c} \left( e^{\frac{a_0}{c} \xi^0} + \frac{1}{2} \right) - \gamma \frac{v_0c}{a_0}
\]

\[
= \gamma \frac{c^2}{a_0} \left\{ e^{\frac{2a_0}{c} \xi^0} - 1 + \frac{v_0}{c} \left( e^{\frac{a_0}{c} \xi^0} + 1\right) \right\}
\]
\[ \frac{c^2}{a_0} \left\{ \frac{2a_0 e^{a_0 t}}{c} - 1 \right\} \frac{v_0}{c} \left\{ \frac{2a_0 e^{a_0 t}}{c} - 1 \right\} \]

\[ = \gamma \frac{c^2}{a_0} \left\{ \cosh\left(\frac{a_0 e^{a_0 t}}{c}\right) + \frac{v_0}{c} \sinh\left(\frac{a_0 e^{a_0 t}}{c}\right) \right\} - \gamma \frac{c^2}{a_0} \]

\[ = \gamma \frac{c^2}{a_0} \left\{ \frac{2a_0 e^{a_0 t}}{c} + \frac{v_0}{c} e^{a_0 t} - \frac{2a_0 e^{a_0 t}}{c} - \frac{v_0}{c} e^{a_0 t} \right\} - \gamma \frac{c^2}{a_0} \]

\[ = \gamma \frac{c^2}{a_0} \left\{ \left(\frac{e^{a_0 t}}{c} + 1\right) + \frac{v_0}{c} \frac{e^{a_0 t}}{c} - \left(\frac{e^{a_0 t}}{c} + 1\right) \right\} \]

\[ = \gamma \frac{c^2}{a_0} \left\{ \frac{e^{a_0 t}}{c} - 1 + \frac{v_0}{c} \frac{e^{a_0 t}}{c} - \left(\frac{e^{a_0 t}}{c} + 1\right) \right\} \tag{23} \]

**V. Conclusion**

About the curvature tensor \( R^\rho_{\mu \nu \lambda} \).

The inertial system \( S(t, x, y, z) \)'s the curvature tensor \( R^{\delta}_{\alpha \beta \Gamma} = 0 \)

The new accelerated system \( \tilde{\xi}(\tilde{\xi}^0, \tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3) \)'s the curvature tensor \( R^\rho_{\mu \nu \lambda} \) has to zero.

If compute the new accelerated system's the curvature tensor \( R^\rho_{\mu \nu \lambda} \)

\[ d\tau^2 = e^{\alpha_0 t} \left( d\tilde{\xi}^0 \right)^2 + \frac{1}{c^2} \left[ e^{\alpha_0 t} \left( d\tilde{\xi}^1 \right)^2 + (d\tilde{\xi}^2)^2 + (d\tilde{\xi}^3)^2 \right] \]

\[ g_{00} = -e^{\alpha_0 t}, \quad g_{11} = e^{\alpha_0 t}, \quad g_{22} = g_{33} = 1 \]

\[ g^{00} = -e^{-\alpha_0 t}, \quad g^{11} = e^{-\alpha_0 t}, \quad g^{22} = g^{33} = 1 \]

\[ \Gamma^0_{00} = \frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial \tilde{\xi}^0} = \frac{a_0}{c^2}, \quad \Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} g^{00} \frac{\partial g_{01}}{\partial \tilde{\xi}^1} = \frac{a_0}{c^2}, \quad \Gamma^1_{11} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial \tilde{\xi}^1} = \frac{a_0^2}{c^4} \]

\[ R^1_{001} = \Gamma^1_{001} - \Gamma^0_{011} - \Gamma^0_{001} = \frac{a_0^2}{c^4} \]

\[ -R^0_{001} = R^1_{010} = -\Gamma^0_{001} - \Gamma^0_{100} = \frac{a_0^2}{c^4} \]

otherwise \( R^\rho_{\mu \nu \lambda} = 0 \)

Therefore, new accelerated system is in the flat Minkowski space.

Hence, it found the new accelerated theory that used the tetrad on the new method. And it expanded to be the new accelerated theory of the accelerated observer that have the initial velocity.
Reference