Products of Generalised Functions

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Abstract

An elementary algebra of products of distributions is constructed. The constructed algebra is equivalent, although less general, of the full Colombeau algebra of generalised functions. However, the loss of generality is compensated by the fact that the new algebra of generalised functions is very convenient for practical calculations. An equivalent relation, among elements of the above algebra, is proposed and a linear space of generalised functions is constructed as a partition space of the elementary algebra with respect to the equivalent relation. The new space of generalized functions is used to prove interesting equalities involving products among elements of D'. A way of multiplying the defined generalised functions with polynomials is also given.

Key Words: distribution theory, product of distributions.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and in partial differential equation (e.g shock wave solutions in hydrodynamics) see [1]. An important issue, related to product of distributions, is the fact that the product, in the general case, is not well defined in D'. This issue is known as the Schwartz impossibility result (see [1] §1.3). In the Schwartz classical theory, only the product between a smooth function and a distribution is well defined. Historically, products of distributions are addressed by means of algebras of generalised functions developed initially by J. F. Colombeau (see [1] and [2]). In this paper we will propose a new simplified approach to the above mentioned theory together with a new criterion of weak equivalence between generalised functions.

2 Aim of the paper and notation

Let X be the set, which elements are symbols, defined as follows:

$$X = \{\text{x: the symbol: } a_1 x^{k_1} \delta^{(p_{11})} \cdot \ldots \cdot \delta^{(p_{1m_1})} + a_2 x^{k_2} \delta^{(p_{21})} \cdot \ldots \cdot \delta^{(p_{2m_2})} + \ldots\} \quad (1)$$

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with $a_i \in \mathbb{R}$, $k_i \in \mathbb{N} \cup \{0\}$ and $p_{ij} \in \mathbb{N} \cup \{0\} \cup \{-1\}$ (where $\delta^{(-1)} = u(x)$ is the Heaviside function). Elements of X are, for example, $\delta, \delta^2, u\delta\delta' + 2x\delta\delta''$ and so on. We define the derivative of an element of X to be the symbolic derivative using the Leibniz rule. For example $D(\delta^2 + x\delta) = 2\delta\delta' + \delta + x\delta'$. It is easy to see that if $x \in X$ then $Dx \in X$.

The set X is a derivative algebra of products of distributions. We all agree that the set X is quite useless. However, it already provides a framework where products of distributions makes sense.

The full Colombeau algebra is defined as the quotient algebra (see [1]):

$$\mathcal{G} = \mathcal{M}/\mathcal{I} \tag{2}$$

where $\mathcal{G}$ is an algebra of moderate functions and $\mathcal{I}$ is an ideal of null functions of $\mathcal{G}$. The elements of $\mathcal{G}$ are sequences obtained from the convolution of the functions to be embedded in $\mathcal{G}$ and the sequence $\rho_\epsilon = \frac{1}{\epsilon}\rho(\frac{x}{\epsilon})$ with ρ a suitable function called the mollifier. The meaning of moderate and null functions is made precise in the theory.

Now, it is possible to show that the set X, defined above, can be embedded in $\mathcal{M}$. Unfortunately, it is possible to show that X can also be embedded in $\mathcal{G}$ and separate elements of X are mapped to separate elements of $\mathcal{G}$. The problem is that the criterion to define $\mathcal{I}$ is designed to guarantee that the direct embedding of C^∞ and its embedding thought the convolution lead to the same generalised functions. This criterion has the nice property to guarantee that the final set is an algebra but at the price to be too strict to associate separate elements of X. For example, it is know in literature (compare with [3]) that $\delta^2 \approx -u(x)\delta'$. This point is further discussed in paragraph 4 where a digression on the null functions, with reference to the theory described in this paper, is given.

In order to fix the above problem, some other criteria for associating generalised function have been proposed. For example, if $J(f) = \int_{-\infty}^{+\infty} f(x)\Psi(x)dx$, with $\Psi \in D$, is the Schwartz functional and $h_1, h_2 \in \mathcal{M}$ are generalised functions, a possible criterion is (week equivalence or association, see [1] §3.2 §8.5):

$$J(h_1 - h_2) = 0 \quad \text{as a limit for } \epsilon \to 0 \tag{3}$$

This allows to associate functions like $u(x)$ and $u^2(x)$ but it does not works with the other example given above because $J(\delta^2 - (-u\delta')) = \frac{1}{2}\delta'$ (we wanted it to be zero). Moreover, although this is not really a problem, functions like $x\delta$ are associated to the 0 function (i.e. $J(x\delta) = 0$) .

In this paper we propose to use the following criterion:

$$\frac{J(h_1)}{J(h_2)} = 1 \quad \text{as a limit for } \epsilon \to 0 \tag{4}$$

which solves some of the issues reported above. This may be seen as the dual approach with respect of the one used in the Colombeau algebras (i.e. working on the multiplication rather then the addition). This is shown in paragraphs 3.

In this paper, we make also a second point. The Colombeau algebras are a beautiful theory as well as very general since they automatically embed important spaces of functions such as C^∞ and D'. However, working with the Colombeau algebras is, most of the times, very difficult. In this paper we propose an equivalent but more elementary approach for defining a space of generalised functions which is less general of the Colombeau algebras but very convenient for practical calculations since it does not require the definition of moderate and null functions and therefore it gives much less constrains on the mollifier. We believe that, once this approach is properly formalised, most of the problems that can be addressed by means of Colombeau algebras, can also be addressed by means of the space we propose. This is presented in paragraph 3.

In paragraph 4 the concept of structure of a distributions is presented. In paragraph 5 some equalities involving products of elements of distributions are given. In paragraph 6 the theory generalised to the products among generalised function and polynomials.

Preliminary Definitions. Given any function $f \in C^\infty$, with the notation $f^{(p)}$, where $p \in \mathbb{Z}$, we refer to the derivatives of order p of f for $p \geq 0$ and to the function defined by the following recursive formula:

$$f^{(p-1)} = \int_{-\infty}^{x} f^{(p)}(\tau)d\tau \tag{5}$$

for $p < 0$.

Let $f \in C^\infty(\mathbb{R})$ be any function, we say that f is a function of order $p \in \mathbb{Z}$ if:

$$0 < \left| \int_{-\infty}^{+\infty} f^{(-p)}(x)dx \right| < \infty \tag{6}$$

and

$$\left| \int_{-\infty}^{+\infty} f^{(-p-1)}(x)dx \right| = \infty \tag{7}$$

We define $S(\mathbb{R})$ to be the set of all the functions $f(x)$ having the following characteristics.

$$\begin{array}{l} 1)\ f(x) \in C^\infty \\ 2)\ \lim_{x\to-\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N}\cup\{0\} \\ 3)\ \lim_{x\to+\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N}\cup\{0\} \end{array} \tag{8}$$

This functions are known in literature as rapidly decreasing functions on $\mathbb{R}$.

3 An algebra of generalised functions

Definition 1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be any function such that $\phi \in S(\mathbb{R})$, $\phi(x) \geq 0$ for each $x \in \mathbb{R}$, ϕ is a function of order 0 and $\int_{-\infty}^{+\infty} \phi(x)dx = 1$. We define $\boldsymbol{M}(\phi)$ to be the vector space of infinite dimension spanned by the elements:*

$$\eta^{q,p}_{x^k\phi^{(p_1)}\cdot\ldots\cdot\phi^{(p_m)}} \tag{9}$$

where the elements $\eta^{q,p}_{x^k\phi^{(p_1)}\cdot\ldots\cdot\phi^{(p_m)}}$ are the sequences $S_n(x) = n^q g(nx)$ where:

- $q \in \mathbb{Z}$
- $p_1, \cdots, p_m \in \mathbb{Z}$

- $g(x) = x^k \phi^{(p_1)}(x) \cdot \ldots \cdot \phi^{(p_m)}(x)$
- $k \in \mathbb{N} \cup \{0\}$
- *p is the order of $g(x)$*

and where the operation + and the multiplication by a scalar are the usual ones for series.

We will call the above defined elements and their linear combinations η functions. Note that the p in the above notation is redundant since the information on the order of the function g is already given by the function g itself (i.e by the integers m and p_i) and it is independent from ϕ. The p may therefore be omitted in the notation.

Definition 2. *We define the following operations on the elements $\eta_g^{q,p}$ introduced above:*

$$\begin{aligned} &1)\ \eta_{g_1}^{q_1} \cdot \eta_{g_2}^{q_2} = \eta_{g_1 g_2}^{q_1+q_2} && \textit{product term by terms of the sequences} \\ &2)\ \tfrac{d}{dx}\eta_g^q = \eta_{g'}^{q+1} && \textit{derivative term by terms on the sequence} \end{aligned} \tag{10}$$

Note that the operations $(\cdot\,, +)$ defined above are commutative, associative and that the product is distributive with respect of the sum. Note finally that the above definition of derivative of an η function is compliant with the Leibniz rule.
It is easy to see that $\mathbf{M}(\phi)$, with the operations $(\cdot, +)$, is a derivative algebra (with the definition of derivative given above).

The elements of the set $\mathbf{M}(\phi)$ are dependent from the function ϕ of definition 1, however the structure of $\mathbf{M}(\phi)$ is not (although we should formally prove it). So we may see $\mathbf{M}(\phi)$ as the structure of the vector space of definition 1 which is independent from ϕ.

We want to show now that it is possible to map a subset of elements of D' to a subset of elements of $\mathbf{M}(\phi)$. Given $h = \sum_{p=-\infty}^{+\infty} c_p \delta^{(p)} \in D'$, we map h to the element $h = \sum_{p=-\infty}^{+\infty} c_p \eta_{\phi^{(p)}}^{p+1,p} \in \mathbf{M}(\phi)$. We choose this particular mapping because, if we define the following linear transformation:

$$T : \eta \in \mathbf{M}(\phi) \to h \in D' \text{ defined as: } h = \lim_{n\to\infty} \sum_{p=-\infty}^{+\infty} c_p n^{p+1} \phi^{(p)}(nx)(nx) \tag{11}$$

then the above limit exists in D', is independent from the function ϕ of definition 1 and clearly $T(\eta) = h$. With the above mapping we define a relation $\rightleftarrows$ which identify elements of D' with elements of $\mathbf{M}(\phi)$. For example we have:

$$\delta + \delta' \quad \rightleftarrows \quad \eta_\phi^1 + \eta_{\phi'}^2 \tag{12}$$

Since $\mathbf{M}(\phi)$ is and algebra, we can use it as our space of generalised functions in which a product makes sense. We extend therefore the relation $\rightleftarrows$ also to products of distributions, whatever they are (to us they are just elements of

$\mathbf{M}(\phi)$), in the obvious way. For example we have:

$$\begin{aligned} \delta + \delta^{'} &\rightleftarrows \eta^1_{\phi} + \eta^2_{\phi'} \\ \delta\delta^{'} + \delta^{'}\delta^{''} &\rightleftarrows \eta^3_{\phi\phi'} + \eta^5_{\phi'\phi''} \\ u(x)\delta(x) &\rightleftarrows \eta^1_{\phi^{(-1)}\phi} \end{aligned} \tag{13}$$

$u(x)$ being the Heaviside function and where the coefficient q of the η functions are evaluated by using the definition of product given above.

We note explicitly that the theory developed in this paper is completely equivalent to the Colombeau algebras and that the set $\mathbf{M}(\phi)$ is basically a subset of the set $\mathcal{M}$, mentioned in the paragraph above, in a disguised form. We show this with an example. We evaluate the generalised function $A(\rho, x) \in \mathcal{M}$ associate to the product δ^2. If $B(\rho, x) \in \mathcal{M}$ is the generalised function associated to δ we have:

$$A(\rho, x) = (B(\rho, x))^2 = \left(\int_{-\infty}^{+\infty} \delta(\tau)\rho_{\epsilon}(x-\tau)d\tau\right)^2 = \left(\frac{1}{\epsilon}\rho\left(\frac{x}{\epsilon}\right)\right)^2 \tag{14}$$

by setting $\epsilon = \frac{1}{n}$ we see that $A(\rho, x) \in \mathcal{M}$ and $\eta^2_{\phi^2} \in \mathbf{M}(\phi)$ are the same sequence. Moreover, the η function are moderate function by construction.

We introduce now a new terminology we will use later in the paper. Given an η function $\eta^{q,p}_g$, we will call $g(x)$ the generating function, q the growth index and p the order of η. Moreover, we will call the elements of $\mathbf{M}(\phi)$, which are sequences, the generating sequence of the relevant η function. Note that, even thought the generating sequence of a specific η function does not converge with n, this is not a problem for us since we are not interested in its limit but rather in the sequence itself.

We define the following equivalence relation between elements of $\mathbf{M}(\phi)$:

Definition 3. *Given two elements η_1 and η_2 of $\boldsymbol{M}(\phi)$ with equal growth indexes q and equal orders p and with generating sequences $S_n(x)$ and $R_n(x)$ we say that the two η functions are two representative of the same element of $\boldsymbol{M}(\phi)$ if, given any test function $\Psi \in D$, we have:*

$$\lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} S_n(x)\Psi(x)dx}{\int_{-\infty}^{+\infty} R_n(x)\Psi(x)dx} = 1 \tag{15}$$

and if the above limit is independent both from Ψ and the function ϕ of definition 1. In this case we use the notation $\eta_1 \approx \eta_2$

Note that, by using the expression representatives of the same element of $\mathbf{M}(\phi)$, we are using a terminology similar to the one used in the Colombeau algebras. Although we are facing a similar situation, the criterion (15), proposed in this paper, is rather different with respect of the Colombeau algebras, and therefore identical terms do not imply identical meanings in the two theories.

Note finally that, in this case, the criterion given by the above definition is used on the elements of $\mathbf{M}(\phi)$ but it is quite a general criterion which can be used to tell if two generalised functions are equivalent.

Example 2.1. Given $\eta_1, \eta_2 \in \mathbf{M}(\phi)$, with $\eta_1 = \delta^2(x)$ and $\eta_2 = u(x)\delta'(x)$, we have:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} n^2\phi^2(nx)\Psi(x)dx}{\int_{-\infty}^{+\infty} n^2\phi^{(-1)}(nx)\phi'(nx)\Psi(x)dx} &= \lim_{n\to\infty} \frac{n\int_{-\infty}^{+\infty} n\phi^2(nx)\Psi(x)dx}{n\int_{-\infty}^{+\infty} n\phi^{(-1)}(nx)\phi'(nx)\Psi(x)dx} \\
&= \frac{\int_{-\infty}^{+\infty} \phi^2(x)dx}{\int_{-\infty}^{+\infty} \phi^{(-1)}(x)\phi'(x)} \frac{\Psi(0)}{\Psi(0)} \qquad (16) \\
&= \frac{\int_{-\infty}^{+\infty} \phi^2(x)dx}{\int_{-\infty}^{+\infty} \phi^{(-1)}(x)\phi'(x)} = -1
\end{aligned}$$

which is independent from ϕ as it is easy to show by integrating by parts $\int_{-\infty}^{+\infty} \phi^2(x)dx$ and using the fact that ϕ vanishes for x that goes to infinity. We conclude that $\delta^2(x) \approx -u(x)\delta'(x)$.

Definition 4. *We define $\boldsymbol{G}(\phi)$ to be the set of partitions of $\boldsymbol{M}(\phi)$ with respect of the equivalence relation given by definition 3.*

The set $\mathbf{G}(\phi)$ is not an algebra but just a vector space. The set $\mathbf{G}(\phi)$ can be effectively seen as a set of generalised function for which an associative and commutative product is well defined and with a definition of derivative (see definition 2) that is compliant with the Leibniz rule and that gives the right answer when applied to elements of $\mathbf{G}(\phi)$ that can be mapped to elements of D'.

We note that the theory developed in this paper is only apparently less general of the Colombeau algebras. For example if $f_1, f_2 \in C^\infty$ and $h_1, h_2 \in \mathbf{G}(\phi)$, the functions $f_1 + h_1$ and $f_2 + h_2$, which have a corresponding generalised function in the Colombeau algebras theory, are not addressed by the theory developed in this paper. However we have:

$$(f_1 + h_1)(f_2 + h_2) = f_1 f_2 + f_1 h_2 + f_2 h_1 + h_1 h_2 \qquad (17)$$

We will show later that we are able to evaluate the product of an η function with polynomials and therefore, since f_1 and f_2 can be Taylor expanded, we are also able to evaluate the above product. Since the above procedure implies the evaluation of a series in $\mathbf{M}(\phi)$, to be able to do that, we should define some notion of convergence in $\mathbf{M}(\phi)$. This is not a difficult task since $\mathbf{M}(\phi)$ is a vector space and therefore each element of it can be represented by a set of coefficients with respect to the base of $\mathbf{M}(\phi)$ and therefore a norm in terms of these coefficients can be easily defined. For example, a notion of equality of elements of $\mathbf{M}(\phi)$ is provided in definition 6 in the next paragraph. Moreover, we will see in paragraph 6 that multiplication of an η function by the monomial x^k decreases the growth index of the η function. So, if we work in $G(\eta)$, since all the components with lower growth index are filtered out by equation (15), only a finite number of terms play a role and we do not even need to worry of the convergence of the whole series.

Before we finish this paragraphs, we give below some examples of η functions. The η functions given below are only examples since, for each element

$n^q g(x)$, there are infinitely many number of elements of $\mathbf{M}(\phi)$ that are the same representative of the element reported in the table below as well as infinitely many elements which are not.

$\eta^{q,p}$	p=-1	p=0	p=1	p=2	p=3
q=5		$\cdots$	$\cdots$	$\cdots$	$\cdots$
q=4		$\cdots$	$\frac{d}{dx}(\delta^3(x))$	$\frac{d^2}{dx^2}(\delta^2(x))$	$\cdots$
q=3		$\delta^3(x)$	$\frac{d}{dx}(\delta^2(x))$	$\delta''(x)$	
q=2	$\cdots$	$\delta^2(x)$	$\delta'(x)$		
q=1	$(\delta^2(x))^{(-1)}$	$\delta(x)$			
q=0	$u(x)$				

Figure 1 : Examples of η functions

Where, for example, the element $\frac{d}{dx}(\delta^2(x))$ has to be read as $2\delta(x)\delta'(x) \in \mathbf{M}(\phi)$ and so on.

4 Structure of a generalised function

We have seen in the previous paragraph that the simplest element $h = \eta^q_{g(x)} \in \mathbf{M}(\phi)$ can be defined by means of its generating function S_n of the type:

$$S_n(x) = n^q g(nx) \tag{18}$$

which most of the times does not converge in D'. We want so determine the structure of such a generalised function. Before we proceed, we need to give a definition which will be used later on:

Definition 5. *Given any function $\xi(x) \in S(\mathbb{R})$, if $\xi(x)$ verifies the following conditions:*

$$\int_{-\infty}^{+\infty} \xi(x) x^k dx = \begin{cases} 1 & for\ k = 0 \\ 0 & for\ 0 < k \leq s\ \ with\ s \in \mathbb{N} \end{cases} \tag{19}$$

then we call ξ a main generating function of order 0. We also call its derivatives $\xi^{(p)}$ with $p \in \mathbb{N}$ and $p < s$ a main generating function of order p. Finally, for each s, we define A_s to be the set of all possible $\xi^{(p)}$ functions relevant to s.

For more details on the A_s sets, see [1] §8.2. In most cases, we will assume that s is large enough for our purpose (i.e. given $\xi^{(p)}$ the main generating function of higher order, we are working with, we have $p < s$) and we will not explicitly mention it in our discussion.

Let us see how to determine all the components, of different order, of a generalised function defined by means of (18) and having generating function $g(x)$. We will suppose, for the moment, that it is possible to find a function $\xi(x) \in A_s$ such that it is possible to express the generating function g as follows:

$$g(x) = \sum_{p=0}^{s} a_p \xi^{(p)} + r(x) \tag{20}$$

where $r(x)$ is a function having all momenta, of order lower then s, equal to 0. We will see, further on, that the above ξ function exists. By multiplying the right hand side of the above equation by x^p, integrating by parts p times and taking into account that $\xi(x)$ vanishes at infinity, we find easily that:

$$a_p = \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} g(x) x^p dx \tag{21}$$

and therefore the a_p coefficients are related to the momenta of g. The equation (21) allow us to evaluate the coefficient a_p of all the components of order lower then s. However, a dependency from s is not explicitly present in the above equations and therefore we can use it to evaluate components of any order just assuming s to be big enough.

As mentioned above, we need to justify our assumption that it is possible to find a $\xi \in A_s$ such that we can write g as in (20). This can be done by giving a constructive algorithm to evaluate the required ξ. Given g and the a_p coefficients, if we evaluate $g_1 = g/a_0$, we have a function with the same base function ξ of g and with first momentum equal to 1. Now, if we we evaluate the a_p^1 coefficient of g_1 and $g_2 = g_1 - a_1^1 g_1'$, we get a function with the same base function ξ of g, with first momentum equal to 1 and second momentum equal to 0. Iterating the process s times we get eventually our required ξ function in A_s.

Now, given (20), it is easy to see the distribution h associated to the sequence (18), it is actually the sum of infinite components in $\mathbf{M}(\phi)$ of same growth index q and increasing order p as follows:

$$h \rightleftarrows n^q g(nx) = \sum_{p=0}^{s} a_p n^q \xi^p(nx) + n^q r(nx) \tag{22}$$

We will call this kind of generating functions homogeneous. We note that the a_p coefficients, although not explicitly noted, refer and depend from the function ξ.

Since a_p do not depend on s, we may say that the above generalised function h is composed of infinite components which are η functions of same growth index and increasing order and, whereas we are interested in the component of order p, we can evaluate it by choosing a ξ function in A_s with $s > p$. In a few words, h can be expressed as:

$$h = \sum_{p=0}^{s} \eta_{\xi^{(p)}}^{q,p} + R(\eta^{q,s}) \tag{23}$$

Where, with the notation $R\left(\eta^{q,s}\right)$ we mean that, to have the above equality exact, we need to add an infinity number of η components of growth index q

and order $p \geq s$. The equation (23) is what we call the structure of a generalised function (compare with [4]).

It is possible to show that there are no null functions, according to the definition of null functions in Colombeau algebras theory, inside the set $\mathbf{M}(\phi)$. For example, for the simplest $\eta^q_{g(x)}$ functions of definition 1, the condition of being a null function is that $\phi \in A_s \implies g \in A_s$ which is not possible. The subset of $\mathbf{M}(\phi)$ composed of null functions is an empty set and $\mathbf{M}(\phi)$ cannot be factorised. As a consequence of that, $\mathbf{M}(\phi)$ is isomorphic to the set X of paragraph 2.

We are now ready for the following definition:

Definition 6. *Given two homogeneous generalised functions h_1, h_2, with same growth index and defined by means of generating functions having the same base function ξ, then if it is possible to find an integer k such that:*

$$\begin{cases} a_p(h_1) = a_p(h_2) & for\, p = k \\ a_p(h_1) = a_p(h_2) = 0 & for\, p < k \end{cases} \tag{24}$$

we say that h_1 and h_2 are representatives of the same generalised function and we use the notation $h_1 \approx h_2$. Moreover, if:

$$a_p(h_1) = a_p(h_2) \quad for\, each\, p \tag{25}$$

then we say that the two generating function are equal and we use the notation $h_1 = h_2$.

Note that the criterion given by the above definition is equivalent to the criterion given by (15).

We have seen above that the a_p coefficients depend on the underlying ϕ function of definition 1. We will call the a_p relative coefficients. We may want to express the structure of a distributions by means of b_p coefficients which are independent from the function f. In this case we would have b_p which are absolute coefficients. The lower component of a generalised function can be expressed in an absolute way by using equation 15. Other components can be expressed in an absolute way iff are of the kind $\eta^{q,p}$ with $q = p + 1$. We will see this with some examples in paragraph 5.

If we express the generalised function in terms of the b_p coefficients then we have a representation which is independent from ϕ. In this case we can express the structure of our generalised function as follows:

$$h = \sum_{p=0}^{l} b_p \hat{\eta}^{q,p} + R(\eta^{q,l+1})$$

where the hat on the η means that we are using η functions that have, a part from the component of order p, components of order lower than $l+1$ that vanish.

It is worth here to point out explicitly that the above theory is also applicable to generalised functions of order $p < 0$. In this case, however, we need to extend the definition of the a_p to generating function of negative order. Given any function $g \in C^0$ and an integer $p < 0$ we define the relevant a_p coefficient of g as follows:

$$a_p = \int_{-\infty}^{+\infty} g^{(|p|)} dx \tag{26}$$

For example, for the Heaviside function $u(x)$, if we use the generating function $\phi^{(-1)}$ with growth index $q = 0$, the above defined product keeps working. In particular, we have proven in [4] that:

$$f(u(x))\delta(x) = \left(\int_0^1 f(x)dx \right) \delta(x) \tag{27}$$

and, for $f = x^k$ and $k \in \mathbb{N}$, the above statement is a particular case of the (15) given in this paper.

Since for the Heaviside function $u(x)$ we have generating functions $\phi^{(-1)}$ and growth index $q = 0$, we may say that:

$$u(x) = \eta^{0,-1}_{\phi^{(-1)}} = \delta^{(-1)} \tag{28}$$

5 Equalities and examples of products in D'

By using the above defined product, we can prove interesting equalities involving products among elements of D'. We will see some examples in this paragraph.

For each example we choose the function ϕ of definition 1 and we find some identities among product of elements of D' by evaluating the components of the structure of a given product (i.e. the coefficients b_p defined in the previous paragraph) with respect of some reference functions. In the following examples we will use, as reference functions, the functions given in table 1. Note that we do not need to stick to the reference functions given in table 1 since we have many possibility in choosing them and therefore the identities evaluated in this paragraphs are just a possible way, among many, to evaluate the given products. At the end of each example we should show that the coefficients b_p do not depend on ϕ. This is trivial and, most of the time, is left to the reader. Note that when proving that the b_p coefficients are independent from ϕ, we never need the hypothesis that $\phi \in C^\infty$. For the above reason, for practical calculations, we can use a much less smooth function. In all examples below we will use just C^1 and C^2 class functions.

Example 6.1: Evaluate the following product:

$$u(x)\delta'(x) \tag{29}$$

We need to choose the function ϕ of definition 1. For this this example it is enough to choose any C^1 class function. We choose the most simple one which is a triangular window centred in the origin with base 2 and hight 1:

$$\phi(x) = (x+1)u(x+1) - 2xu(x) + (x-1)u(x-1) \tag{30}$$

we have $q = q_1 + q_2 = 2$ and $g(x) = \phi^{(-1)}(x)\phi^{(1)}(x)$ and therefore:

$$u(x)\delta'(x) \rightleftarrows n^2 g(nx) = n^2 \; \phi^{(-1)}(nx)\phi^{(1)}(nx) \tag{31}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{aligned} b_0 &= \frac{\int_{-\infty}^{+\infty} g(x)dx}{\int_{-\infty}^{+\infty} \phi^2(x)dx} = \frac{-\frac{2}{3}}{\frac{2}{3}} = -1 \quad \text{coeff. of } \eta^{2,0} = \delta^2 \\ b_1 &= a_1 = \int_{-\infty}^{+\infty} xg(x)dx = \tfrac{1}{2} \quad \text{coeff. of } \eta^{2,1} = \delta' \end{aligned} \tag{32}$$

where $b_1 = a_1$ because for $p = 1$, $p + 1 = q$ and therefore the coefficient a_1 is independent from ϕ. We have:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) + R\left(\eta^{2,2}\right) \tag{33}$$

We have also that b_0 is independent from ϕ because, integrating bay parts, we have:

$$\int_{-\infty}^{+\infty} \phi^{(-1)}(x)\phi^{(1)}(x)dx = -\int_{-\infty}^{+\infty} \phi^2(x)dx \tag{34}$$

Note that, as mentioned above, the above prove requires just $\phi \in C^1$.

We may also express $u(x)\delta'(x)$ as an equality among products of elements of D' (compare with [3]), by ignoring the higher order terms:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) \tag{35}$$

There is a second way to get to the same result. First we evaluate the product of $u(x)\delta(x)$ by sing the sual method shown above. We have:

$$u(x)\delta(x) \rightleftarrows n\ \phi^{(-1)}(nx)\phi(nx) \to q = 1 \tag{36}$$

from which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) + R\left(\eta^{1,1}\right) \tag{37}$$

We use the Leibniz rule, which we know to work with our definition of product. By taking the derivatives of both sides of the above equality we have:

$$\delta^2(x) + u(x)\delta'(x) = \frac{1}{2}\delta'(x) + R\left(\eta^{2,2}\right) \tag{38}$$

as expected.

Finally, there is a third way to get to the same result. First we use (21) for $u(x)\delta'(x)$. We apply it to $g = \phi(x)^{(-1)}\phi(x)^{(1)}$ with $q = 2$:

$$a_p = \frac{(-1)^p}{p!}\int_{-\infty}^{+\infty} g(x)x^p dx = \begin{cases} \frac{1}{p!}\left(\frac{1}{p+3} - \frac{2}{p+2}\right) & for\, p\, even \\ \frac{1}{p!}\,\frac{1}{p+1} & for\, p\, odd \end{cases} \tag{39}$$

and therefore, taking into account that $\eta^{2,1} = \delta'$:

$$u(x)\delta'(x) = -\frac{2}{3}\eta^{2,0}_{\xi_1} + \frac{1}{2}\delta'(x) - \frac{3}{20}\eta^{2,2}_{\xi_1''} + R\left(\eta^{2,3}\right) \tag{40}$$

The a_p coefficients above refer to a ξ_1 base function of g which is unknown.

Then we use (21) for $\delta^2(x)$. We apply it to $g = \phi^2(x)$ with $q = 2$:

$$a_p = \frac{(-1)^p}{p!}\int_{-\infty}^{+\infty} \phi^2(x)x^p dx = \begin{cases} \frac{2}{p!}\left(\frac{1}{p+3} - \frac{2}{p+2} + \frac{1}{p+1}\right) & for\, p\, even \\ \\ 0 & for\, p\, odd \end{cases} \tag{41}$$

and therefore:

$$\delta^2(x) = \frac{2}{3}\eta^{2,0}_{\xi_2} + \frac{17}{60}\eta^{2,2}_{\xi_2''} + R\left(\eta^{2,4}\right) \tag{42}$$

The a_p coefficients above refer to a ξ_2 base function of $\phi^2(x)$ which is unknown.

Since $D(\phi^2(x)) = 2\phi(x)\phi'(x)$, we have that $\xi_1 = \xi_2$ and therefore the a_p notations of the two generalised functions above are comparable each other. For the above reason we can add and subtract them in the a_p notation. By adding them we have:

$$u(x)\delta'(x) + \delta^2(x) = \frac{1}{2}\,\delta'(x) + R\left(\eta^{2,2}\right) \tag{43}$$

as expected.

Example 6.2: Evaluate the following product:

$$u(x)\delta^{''}(x) \tag{44}$$

Before we start we need to choose the function ϕ of definition 1. In this example we need a C^2 class functions, we choose the following function:

$$\phi(x) = \frac{3}{2}\left((x+1)^2u(x+1) - 4xu(x) - (x-1)^2u(x-1)\right) \tag{45}$$

We have $q = q_1 + q_2 = 3$ and $g(x) = \phi^{(-1)}(x)\phi^{(2)}(x)$. and therefore:

$$u(x)\delta^{''}(x) \rightleftarrows n^3g(nx) = n^3\ \phi^{(-1)}(nx)\phi^{(2)}(nx) \tag{46}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{array}{ll} a_0 = \int_{-\infty}^{+\infty} g(x)dx = 0 & \text{coeff. of } \eta^{3,0} = \delta^3 \\ \\ b_1 = \frac{\int_{-\infty}^{+\infty} xg(x)dx}{\int_{-\infty}^{+\infty} x\frac{d}{dx}\phi^2(x)dx} = -\frac{3}{2} & \text{coeff. of } \eta^{3,1} = (\delta')^2 \\ \\ b_2 = a_2 = \int_{-\infty}^{+\infty} x^2g(x)dx = \frac{1}{2} & \text{coeff. of } \eta^{3,2} = \delta^{''} \end{array} \tag{47}$$

where $b_2 = a_2$ because for $p = 2$, $p+1 = q$ and therefore the coefficient a_2 is independent from ϕ. We have:

$$u(x)\delta^{''}(x) = -\frac{3}{2}\eta^{3,1}_{(\phi^2)'} + \frac{1}{2}\delta^{''} + R\left(\eta^{3,3}\right) \tag{48}$$

We see that $u(x)\delta^{''}(x) \notin D'$ since its component $\delta^{''}$ is negligible with respect of $\eta^{3,1}_{(\phi^2)'}$ and therefore $u(x)\delta^{''}(x) \approx -\frac{3}{2}\eta^{3,1}_{(\phi^2)'}$.

Example 6.3: Evaluate the following product:

$$\delta(x)\delta'(x) \tag{49}$$

Before we start we need to choose the function ϕ of definition 1. In this example we need just C^1 class functions, we choose once again (30) of the previous example.
We have $q = q_1 + q_2 = 3$ and $g(x) = \phi(x)\phi^{(1)}(x)$. and therefore:

$$\delta(x)\delta'(x) \rightleftarrows n^3 g(nx) = n^3\ \phi(nx)\phi^{(1)}(nx) \tag{50}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{array}{ll} a_0 = \int_{-\infty}^{+\infty} g(x)dx = 0 & \text{coeff. of } \eta^{3,0} = \delta^3 \\ b_1 = \frac{\int_{-\infty}^{+\infty} xg(x)dx}{\int_{-\infty}^{+\infty} \frac{d}{dx}\phi^2(x)xdx} = \frac{1}{2} & \text{coeff. of } \eta^{3,1} = (\delta^2)' \\ a_2 = \int_{-\infty}^{+\infty} x^2 g(x)dx = 0 & \text{coeff. of } \eta^{3,2} = \delta'' \end{array} \tag{51}$$

we have:

$$\delta(x)\delta'(x) = \frac{1}{2}\eta^{3,1}_{(\phi^2)'} + R\left(\eta^{3,3}\right) \tag{52}$$

Once again, there is a second way to get the same result. By taking twice the derivative of both sides of (37), and rearranging the terms we get:

$$\delta(x)\delta'(x) = -\frac{1}{3}u(x)\delta''(x) + \frac{1}{6}\delta''(x) + R\left(\eta^{3,3}\right) \tag{53}$$

We see easily that, taking into account (48), (52) and (53) are in perfect agreement.

Example 6.4: Evaluate the following product:

$$sign^2(x)\delta(x) \tag{54}$$

We have:

$$sign^2(x)\delta(x) \rightleftarrows n\,(2\phi^{(-1)}(nx) - 1)^2\phi(nx) \to q = 1 \tag{55}$$

which is actually the sum of three products one of which is trivial. We have:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) + R\left(\eta^{1,1}\right) \tag{56}$$

compare with [2] §1.1 ex. iii and with [4].

6 Products with polynomials

Now we want to extend our product of distributions to products involving polynomial.

We note that x^k with $k \in \mathbb{N}$ can be expressed as the limit of the following sequence of functions:

$$x^k = \lim_{n\to\infty} n^{-k}(nx)^k \tag{57}$$

and therefore, it is the limit of a sequence of functions of the kind (18) with generating function $g = x^k$ and growth index $q = -k$.

From the above limit, we see immediately that the product of a generalised function with a monomial of degree k, lowers the growth index of the generalised function by k. Given the generalised function h defined by (18), then we have that $x^k h$ can be associated to the following sequence:

$$x^k h \rightleftarrows n^{q-k}(nx)^k g(nx) \tag{58}$$

Let us see what happens, to the order and the amplitude of a generalised function, when we multiply it by x. The generalization to multiplication by x^k is trivial.

For $p > 0$, from (21) is possible to show that given any $\xi_1^{(p)} \in A_s$ with $p < s$ we have:

$$a_{p-1}\left(\xi^{(p-1)}(x)\right) = a_p\left(-\frac{x}{p}\xi^{(p)}(x)\right) \;\; for\, p > 0 \tag{59}$$

From which we see clearly that the product of a generalised function of order p, with x, lower the order of the generalised function by 1. To sum up, we have:

$$\eta^{q-1,p-1} = -\frac{x}{p}\eta^{q,p} \;\; for\, p > 0 \tag{60}$$

and in particular:

$$\delta^{(p-1)} = -\frac{x}{p}\delta^{(p)} \;\; for\, p > 0 \tag{61}$$

which is a well known result in literature (compare with [3]).

For $p = 0$, the situation is a bit more complex. It is possible to show that:

$$a_1\left(\xi^{(1)}(x)\right) = a_0\left(-x\,\xi^{(0)}(x)\right) \tag{62}$$

From with we have:

$$\eta^{q-1,1} = -x\,\eta^{q,0} \tag{63}$$

and in particular:

$$\eta^{0,1} = -x\delta(x) \tag{64}$$

we see that the order $p = 0$ cannot be further lowered by multiplying by x. If we keep multiplying a generalised function of order 0 with x, the growth index keep decreasing but the order toggles between 0 and 1.

The case $p < 0$ will not be addressed in this paper.

We can now define some more examples of reference functions to be used for performing our multiplications:

$\eta^{q,p}$	p=-1	p=0	p=1	p=2	p=3
q=2		$\delta^2(x)$	$\delta'(x)$	$-(x\delta^2(x))'$	$-(x\delta(x))''$
q=1		$\delta(x)$	$-x\delta^2(x)$	$-(x\delta(x))'$	$\cdots$
q=0	$u(x)$	$-(x\delta^2(x))^{(-1)}$	$-x\delta(x)$	$-(x^3\delta^2(x))'$	$\cdots$
q=-1	$-(x\delta^2(x))^{(-2)}$	$-(x\delta(x))^{(-1)}$	$-x^3\delta^2(x)$	$-(x^3\delta(x))'$	
q=-2	$-(x\delta(x))^{(-2)}$	$-(x^3\delta^2(x))^{(-1)}$	$-x^3\delta(x)$	$\cdots$	
q=-3	$\cdots$	$\cdots$	$\cdots$		

Figure 2 : Examples of η functions for $q - p \leq 1$

Example 7.1: Evaluate the following product:

$$x^2\delta^2(x) \tag{65}$$

Once again we need to choose the function ϕ and once again we choose (30) of the previous examples.
If $q_1 = 2$ is the growth index of $\delta^2(x)$, we have $q = q_1 - 2 = 0$ and $g(x) = x^2\phi^2(x)$. and therefore:

$$x^2\delta^2(x) = \lim_{n\to\infty} n^0 g(nx) = (nx)^2\phi^2(nx) \tag{66}$$

We can now evaluate the first coefficient of the structure of our generalised function:

$$b_0 = \frac{\int_{-\infty}^{+\infty} g(x)dx}{\int_{-\infty}^{+\infty} -(x\phi^2(x))^{(-1)}dx} = 1 \quad \text{coeff. of } \eta^{0,0} = -(x\delta^2(x))^{(-1)} \tag{67}$$

which is independent from ϕ. We have:

$$x^2\delta(x) = -(x\delta^2(x))^{(-1)} + R\left(\eta^{1,0}\right) \tag{68}$$

We will use now the theory developed above to discuss a well known example in the theory of product of distributions (compare with [2] §1.1 ex. i).

Example 7.2: If $vp\frac{1}{x}$ is the Cauchy principal value of $\frac{1}{x}$ then we have:

$$0 = (\delta(x)\cdot x)\cdot vp\frac{1}{x} = \delta(x)\cdot\left(x\cdot vp\frac{1}{x}\right) = \delta(x) \tag{69}$$

which is absurd.
By using our theory we know that $x\delta(x) = -\eta^{0,1}_{x\phi} \neq 0$. We have:

$$0 = (x\cdot\delta(x) + \eta^{0,1}_{x\phi})\cdot\frac{1}{x} = \delta(x) + \frac{1}{x}\eta^{0,1}_{x\phi} = \delta(x) - \delta(x) \tag{70}$$

a results that now makes sense.

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