Products of Generalised Functions

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Abstract

A new space of generalised functions extending the space D', together with a well defined product, is constructed. The new space of generalized functions is used to prove interesting equalities involving products among elements of D'. A way of multiplying the defined generalised functions with polynomials is also derived.

Key Words: distribution theory, product of distributions.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and in partial differential equation (e.g shock wave solutions in hydrodynamics) see [1]. An important issue, related to product of distributions, is the fact that the product, in the general case, is not well defined in D'. This issue is known as the Schwartz impossibility result (see [1] §1.3). In the Schwartz classical theory, only the product between a smooth function and a distribution is well defined. Historically, products of distributions are addressed by means of algebras of generalised functions developed initially by J. F. Colombeau (see [1] and [2]). In this paper we will propose a new approach to define products of distributions.

In paragraphs from 2 to 6, we construct a new space of generalised functions, extending the space D'. In paragraph 7, we define a products in the above mentioned space of generalised functions. In paragraphs 8, we use the new developed theory to derive interesting equalities involving products among elements of D'. In paragraphs 9, we derive a method to multiply the generalised functions defined in this paper with polynomials.

2 Main generating functions

Definition 1. *We define F to be the set of all the function $f(x)$ having the following characteristics.*

$$\begin{array}{l} 1)\ f(x) \in C^{\infty} \\ 2)\ \lim_{x\to-\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N} \\ 3)\ \lim_{x\to+\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N} \end{array} \tag{1}$$

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Definition 2. *Given any $\xi(x) \in F$ then, if $\xi(x)$ verifies the following conditions:*

$$\int_{-\infty}^{+\infty} \xi(x) x^k dx = \begin{cases} 1 & for\ k = 0 \\ 0 & for\ k \in \mathbb{N} - \{0\} \end{cases} \tag{2}$$

then we call ξ a main generating function of order 0. We also call the derivative $\xi^{(p)}$ with $p \in \mathbb{N}$ a main generating function of order p.

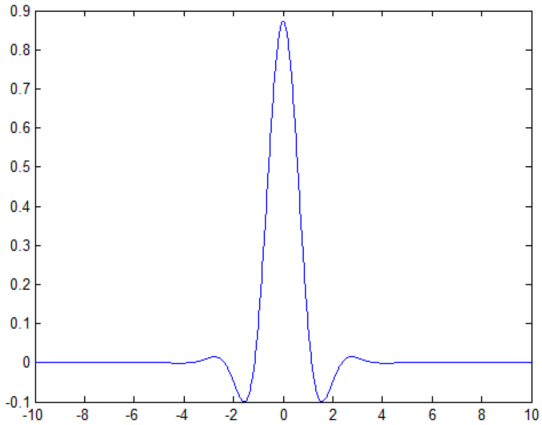

Figure 1: Plot of a ξ function

If we call $H^p \subset F$ the set of all generating functions of order p, then we have:

$$\begin{array}{lll} \xi(x) \in H^0 & \Rightarrow & \alpha\xi(\alpha x) \in H^0 \\ \xi(x) \in H^0 & \Rightarrow & a[\,\alpha_1\xi(\alpha_1 x)\,] + b[\,\alpha_2(\xi(\alpha_2 x)\,] \in H^0 \\ \xi^{(p)}(x) \in H^p & \Rightarrow & \alpha^{p+1}\xi^{(p)}(\alpha x) \in H^p \end{array} \tag{3}$$

with $a + b = 1$, $\alpha_1 > 0$ and $\alpha_2 > 0$.

From the second implication of the (3) it follows that if $\xi_1 \in H^0$, then for any $\rho(\alpha) \in D'$, such that:

$$\int_0^{\infty} \rho(\alpha) d\alpha = 1 \tag{4}$$

we have:

$$\xi_2(x) = \int_0^{\infty} \rho(\alpha)\alpha\xi_1(\alpha x) d\alpha \in H^0 \tag{5}$$

provided that the above integral converges. Note that, given ξ_1 and ξ_2, ρ is not unique. Also, ρ may be continuous, impulsive or mixed. For example, in the second implication of the (3), we have $\rho(\alpha) = a\delta(\alpha - \alpha_1) + b\delta(\alpha - \alpha_2)$.

Finally we note that if $\xi^{(p)} \in H^p$ then:

$$\int_{-\infty}^{+\infty} \xi(x)^{(p)} x^k dx = \begin{cases} 1 & for\ k = p \\ 0 & for\ k \in \mathbb{N} - \{p\} \end{cases} \tag{6}$$

3 New generalised functions

In this paragraph we will define a new class of generalized functions. Generalised functions can be defined by means of the limit of sequences of functions $f_n(x)$. In this paper we will deal only with generalised functions defined by means of the limit of a sequence of the form:

$$\lim_{n\to\infty} n^q f(nx) \tag{7}$$

with $f \in F$. Note that the above sequences are not the most general way to define distributions. For example, there is no sequence of the form (7) converging to $\delta + \delta'$. We will call $f(x)$ the generating function, $n^q f(nx)$ the generating sequence and q the growing index of the generalised function defined by the (7).

Definition 3. *We define the generalised functions $\eta_\xi^{p,q}$ to be the following limit:*

$$\eta_\xi^{p,q}(x) = \lim_{n\to\infty} n^q \xi^{(p)}(nx) \quad \textit{with}\, p \geq 0,\ q \in \mathbb{Z} \tag{8}$$

provided that $\xi^{(p)} \in H^p$.

Note that, for reasons that will be clear further on, it is very important to keep track of the main generating function $\xi^{(p)}$. We do that by using the notation $\eta_\xi^{p,q}$. It is easy to see that:

$$\eta_\xi^{p,p+1}(x) = \delta^{(p)}(x) \tag{9}$$

What kind of generalised function are the $\eta_\xi^{p,q}$? If the sequence of distributions $f_n = n^q \xi^p(nx)$, in the (8), converges to $\eta_\xi^{p,q}$, then $\frac{f_n}{n^{q-p-1}}$ converges to $\delta^{(p)}$. So, with an abuse of notation, we may say that:

$$\eta_\xi^{p,q} = \frac{\delta^{(p)}}{n^{p-q+1}} \tag{10}$$

The $\eta^{p,q}$ are therefore the limit of sequences of functions that are shaped like $\delta^{(p)}$ and that, when we take the limit, grow at a lower or faster rate (according to the sign of p-q+1).

Now, let us see how to determine all the $\eta^{p,q}$ components of a generalised function defined by means of the (7) and having generating function $f(x) \in F$. We will suppose, for the moment, that all $\eta^{p,q}$, have the same main generating function $\xi \in H^0$. We will see, further on, that this turn out to be true. First of all, we note that all the components of the distribution (7) have the same growing index q. We will call this kind of generating functions homogeneous. We have:

$$h = \lim_{n\to\infty} n^q f(nx) = \sum_{p=0}^{\infty} a_p \eta_\xi^{p,q} \tag{11}$$

where the a_p, although not explicitly noted, refer and depend from the function ξ which we suppose known. Now, if $q \geq 1$, h always contains one (and only one) distribution $\eta_\xi^{q-1,q}(x) = \delta^{(q-1)}(x) \in D'$. From the (10) we know that:

$$\lim_{n\to\infty} \frac{n^q f(nx)}{n^{q-p-1}} = a_p \delta^{(p)} \tag{12}$$

So, for the distribution defined by the (11), we can determine the a_p coefficients by applying the Schwartz theory of distribution to our sequence of functions divided by n^{q-p-1}. Let ϕ be a test function and given p, we have:

$$\frac{h}{n^{q-p-1}} = \lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{p+1} f(nx)\phi(x)dx = \lim_{n\to\infty} \sum_{k=0}^{\infty} (-1)^k a_k \frac{\phi^{(k)}(0)}{n^{k-p}} \tag{13}$$

In the right side of the above equation we have two problems which make difficult to evaluate the a_k. The coefficients a_k are mixed up by the summation on k and, given a generic ϕ, this test function may pick up in the same coefficient a_k components related to different $\xi^{(p)}$. To better evaluate all a_p we decide to use, as a test function, x^p. In this way we solve both problems mentioned above since x^p has all derivatives of order i equal to 0 for $i \neq p$ (so the summation will not mix various a_k terms) and x^p will filter out all components $\xi^{(i)}$ with $i \neq p$. Of course a test function should vanish outside a compact interval (compact support) and x^p does not. However, the above requirement is needed to ensure integrability which in our case is ensured by the fact that $f \in F$. So the fact that x^i has not compact support it is not a problem. We have:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{p+1} f(nx) x^p dx = (-1)^p \, a_p \, p! \tag{14}$$

where $p!$ is the value of the p^{th} derivatives of x^p. From the (14) we can easily evaluate the a_p as follows:

$$\begin{aligned} a_p &= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n^{p+1} f(nx) x^p dx \\ &= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n\, f(nx)(nx)^p dx \end{aligned} \tag{15}$$

We note that the right part of the (15), for n that goes to infinity, in the (x, y) plane, shrinks (along x) and grows (along y) like n, which leaves the integral unchanged. For the above reason, the limit of the (15) is simply the value of the integrals for any n. We may as well evaluate it for n=1. We have:

$$a_p = \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} f(x) x^p dx \tag{16}$$

and therefore the a_p coefficients are related to the momenta of f. We are now ready to define our new space of generalised functions.

Definition 4. *We define $\mathbb{G}^\eta$ to be the space of generalised functions which elements are the limits of sequences of the type (8), (which we know to be homogeneous generalised functions), or the linear combinations af a finite or infinite numbers of them.*

We note that, since by definition we have:

$$\lim_{n\to\infty} n^{p+1} \xi^{(p)} = \delta^{(p)}(x) \;\; in\; \mathbb{G}^\eta \;\; p \geq 0 \tag{17}$$

then the (17) states that, if we use main generating functions, we can define delta and delta derivatives that have no components outside D'. In a few words, if we accept generalised function $\eta^{p,q}$ to be real things (i.e. we work in $\mathbb{G}^\eta$), we have also to accept that only sequences $n\xi(nx)$ composed of main generating functions converge to δ.

4 Definition of the F_ξ sets

In the previous paragraph we have assumed that the coefficient a_p are all related to the same main generating function ξ. We will show that the above assumption is true by finding an algorithm to determine the base function ξ of a generating function f.
We give the following algorithm:

- Given a function $f \in F$, we determine the coefficient a_p by means of the (16).

- ($a_0 \neq 0$) We suppose $a_0 = 1$ (if not, we can always divide f by $1/a_0$), clearly $f_1 = f(x) - a_1 f'(x)$ is a generating function for an element of $\mathbb{G}^\eta$ which has no $\eta^{1,q}$ component. Iterating the process on f_1 (i.e. we evaluate the new a_p coefficients and we remove the term of order 2) we get a function f_2 which is a generating function for an element of $\mathbb{G}^\eta$ which has no $\eta^{1,q}$ and $\eta^{2,q}$ components. By keeping iterating on f_n, we can remove all the $\eta^{p,q}$ components with $p > 0$ and we get eventually the generating function for $\eta^{0,q} \in \mathbb{G}^\eta$ which is, of course, a ξ function. We have:

$$\xi(x) = f - a_1 f' - (a_2 - a_1^2) f'' - (a_3 - a_1^2 - a_2 a_1 + a_1^3) f''' - \dots \tag{18}$$

- ($a_0 = 0$) If a_p is the first coefficient different from 0 then we can use the same algorithm of point 2 but, this time, to determine $\xi^{(p)}$. We can then integrate $\xi^{(p)}$ p times to get ξ.

Now, by starting from ξ and the a_p, we can reverse the algorithm and we see that we eventually construct a function f which is the generating function for an element $h \in \mathbb{G}^\eta$ which has all components η_ξ. This proves that our assumption, according to which the (16) picks up η components all related to the same ξ function, is true.

Definition 5. *We define $F_\xi \subset F$ to be the set of the function $f \in F$ for which by applying the above described algorithm, we get the same main generating function $\xi \in H^0$.*

It is easy to see that, given any $\xi_1, \xi_2 \in H^0$ with $\xi_1 \neq \xi_2$, then $F_{\xi_1} \cap F_{\xi_2} = \emptyset$.

Now, given a function $f \in F$, we say that f is a null function if all the coefficients a_p evaluated by means of the (16) are equal to 0 (i.e. all the momenta of f vanish).

Definition 6. *We define $N \subset F$ to be the set of all null functions.*

For example, if ξ_1 and ξ_2 are two separate main generating functions of order 0, then $\xi_1 - \xi_2$ is a null function.

Finally we note that, it is possible to find functions $f \in F$ which do not belong to any F_ξ. Good examples of that are the elements of N.

5 Additional remarks on the η functions

We show now an important fact about the coefficient a_p of the (16). Let $\xi \in H^0$ be any main generating function of order 0 and $\eta_\xi^{p,q} \in \mathbb{G}^\eta$ a related generalised function. We have:

$$\lim_{n\to\infty} a_p(\xi) n^q \xi^{(p)}(nx) = a_p(\xi)\eta_\xi^{p,q} \tag{19}$$

If we choose $\xi_\alpha = \xi(\alpha x)$, as a different generating function for of order 0, we have:

$$\lim_{n\to\infty} a_p(\xi_\alpha) n^q \alpha^{p+1} \xi^{(p)}(n\,\alpha x) = a_p(\xi_\alpha)\eta_{\xi_\alpha}^{p,q} \tag{20}$$

If the (19) and (20) are the same generalised function then:

$$\lim_{n\to\infty} a_p(\xi) n^q \xi^{(p)}(nx) = \lim_{n\to\infty} \frac{a_p(\xi_\alpha)}{\alpha^{q-p-1}} (n\alpha)^q \xi^{(p)}(n\,\alpha x) \tag{21}$$

since

$$\lim_{n\to\infty} n^q \xi^{(p)}(nx) = \eta_\xi^{p,q} = \lim_{n\to\infty} (n\alpha)^q \xi^{(p)}(n\,\alpha x) \tag{22}$$

because the left and the right side limit of the (22) are the same function growing and shrinking at the same rate with n, and therefore converge to the same generalised function $\eta_\xi^{p,q}$, we conclude that:

$$a_p(\xi_\alpha) = \alpha^{q-p-1} a_p(\xi) \tag{23}$$

and therefore:

$$\eta_{\xi_\alpha}^{p,q} = \frac{1}{\alpha^{q-p-1}} \eta_\xi^{p,q} \tag{24}$$

From the (24), it is clear that if we want to use the $\eta^{p,q}$ notation we have always to specify the reference main generating function ξ since, for any specific element of $\mathbb{G}^\eta$, this has an impact on the amplitudes of the $\eta^{p,q}$ (i.e. the amplitude of the coefficients). This is why we use the notation $\eta_\xi^{p,q} \in \mathbb{G}^\eta$.

Note that if $p = q - 1$ then, as expected, the η notation is independent from ξ since $\eta^{p,p+1} = \delta^{(p)}$.

We conclude this paragraph by finding a relation similar to the (23) but valid in the most general case. Given $\eta_{\xi_1}^{p,q}$, if we choose any other $\xi_2 \in H^0$, evaluated using the (5), as the reference generating function, then by using both the (23) and the (5) we find that the relationship between the coefficients of $\eta_{\xi_2}^{p,q}$ and $\eta_{\xi_1}^{p,q}$ is the follows:

$$a_p(\xi_2) = \sigma_{12}^{p,q}\, a_p(\xi_1) \tag{25}$$

where:

$$\sigma_{12}^{p,q} = \int_0^\infty \rho(\alpha)\alpha^{q-p-1} d\alpha \tag{26}$$

We are now ready to see an example. Given a Gaussian distribution $f_{\xi_1}(x) \in F_{\xi_1}$ defined as follows:

$$f_{\xi_1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{27}$$

we want to represent the generalised function $h \in \mathbb{G}^\eta$, having generating function f_{ξ_1} and grooving index $q = 1$, by means of the $\eta^{p,q}$ notation. Using the (16) we have:

$$h = \lim_{n\to\infty} n\, f_{\xi_1}(nx) = \delta(x) + \frac{1}{2}\eta_{\xi_1}^{2,1} + \frac{1}{8}\eta_{\xi_1}^{4,1} + R\left(\eta^{6,1}\right) \tag{28}$$

where $R(\eta^{6,1})$ means that, to have the above equality exact, we need to add an infinite number components of growing index 1 and order ≥ 6. If $\xi_2 \in H^0$ is a different main generating function, we can represent the same generalised function h by means of the $\eta_{\xi_2}^{p,q}$. To do that, we need to find the function ρ which allows as to evaluate ξ_2 from ξ_1 as defined by the (5) and then, using the (26), we have:

$$h = \delta(x) + \frac{1}{2}\,\sigma_{12}^2\,\eta_{\xi_2}^{2,1} + \frac{1}{8}\,\sigma_{12}^4\,\eta_{\xi_2}^{4,1} + R\left(\eta^{6,1}\right) \tag{29}$$

Note that the (29) has generating function $f_{\xi_2} \in F_{\xi_2}$ with $f_{\xi_1} \neq f_{\xi_2}$ and both f_{ξ_1} and f_{ξ_2} generating functions for the same generalised function $h \in \mathbb{G}^\eta$ which is:

$$h = \lim_{n\to\infty} n\,f_{\xi_1}(nx) = \lim_{n\to\infty} n\,f_{\xi_2}(nx) \tag{30}$$

6 Transformations in F

Definition 7. *Given $\xi_1, \xi_2 \in H^0$ and the σ from the (26), we define*

$$\tau_{12}^q = (\sigma_{12}^{0,q},\, \sigma_{12}^{1,q},\, \cdots) \tag{31}$$

to be a transformation in F such that:

$$\tau_{12}^q : f_1 \in F_{\xi_1} \to f_2 \in F_{\xi_2} \tag{32}$$

which transforms any element of F_{ξ_1} in the relevant element of F_{ξ_2} such that the two elements are generating functions for the same element of $\mathbb{G}^\eta$. We also define T to be the set of all separate τ functions.

Note that, for example, $\tau_{\xi_1\xi_1}^q$ and $\tau_{\xi_2\xi_2}^q$, having the same σ components, are the same element in T. It is easy to show that T has the structure of an Abelian group where the operation is composition of transformations and:

$$\begin{array}{lll} 1) & \tau_{\xi\xi}^q = (1,\, 1,\, \cdots) & \textit{is the 0 element} \\ 2) & -\,\tau_{\xi_1\xi_2}^q = \tau_{\xi_2\xi_1}^q & \textit{with } \sigma_{21}^{p,q} = (\sigma_{12}^{p,q})^{-1} \end{array} \tag{33}$$

Now, let $f_{\xi_1}, g_{\xi_1} \in F_{\xi_1}$ be two generating functions for $h_1, h_2 \in \mathbb{G}^\eta$ of growing indexes q_1 and q_2. Let also $f_{\xi_2}, g_{\xi_2} \in F_{\xi_2}$, be the relevant generating functions (taking into account the growing indexes) for the same generalised functions, h_1 and h_2. If $f_{\xi_1}g_{\xi_1} \in F_{\xi_v}$ and $f_{\xi_2}g_{\xi_2} \in F_{\xi_w}$ then we state that:

$$\tau_{12}^{q_1}\left(f_{\xi_1}\right) \cdot \tau_{12}^{q_2}\left(g_{\xi_1}\right) = \tau_{vw}^{q_1+q_2}\left(f_{\xi_1} \cdot g_{\xi_1}\right) \tag{34}$$

The (34) tells us that we can transform f and g from F_{ξ_1} to F_{ξ_2} and then multiply them or multiply them and then transform the product from F_{ξ_v} to F_{ξ_w}. In both cases we get the same function.

Unfortunately we do not have a formal prove for the (34). However, numerical evidences (see appendix) suggest that the (34) is true.

An important question is whether ξ_v and ξ_w depend only from ξ_1 and ξ_2 and are independent from the function f and g. We believe this is likely to be the case although we do not have a formal proof of it. However, for the (34) to be true, this assumption is not required nor it is ever used throughout the paper and therefore, we will not spend more time on it.

7 Product of generalised functions in $\mathbb{G}^\eta$.

Let us see now, how to use the theory developed in the previous paragraphs to define the product of generalised functions in $\mathbb{G}^\eta$.

Definition 8. *Given k homogeneous generalised functions $h_i \in \mathbb{G}^\eta$ with generating functions $f_i \in F_{\xi_i}$ and growing indexes q_i, we define the product h of the h_i, to be the limit of the product of the generating sequences $n^{q_i} f_i(nx)$:*

$$h = \lim_{n\to\infty} n^{q_1\cdot\ldots\cdot q_k} f_1(nx)\cdot\ldots\cdot f_k(nx) \tag{35}$$

Note that, the product $f_1(x)\cdot\ldots\cdot f_k(x) \in F_{\xi_v}$ where, in general, $\xi_i \neq \xi_v$ for each i.

Since any generalised function h_i is well defined when the relevant generating function and growing index is given, then commutativity, associativity and applicability of the Leibniz rule in $\mathbb{G}^\eta$, for the product defined above, is ensured by the commutativity, associativity and applicability of the Leibniz rule for the relevant generating sequences.

We will show now, with a specific example, how to use the (35) to define a product of generalised functions which is independent from the chosen ξ_i. Suppose we want to evaluate the product h of the two generalised functions $h_1, h_2 \in \mathbb{G}^\eta$. We choose any $\xi_1 \in H^0$ and we find the relevant main generating function $f_{\xi_1}, g_{\xi_1} \in F_{\xi_1}$ and the growing indexes q_1, q_2. We know also that $f_{\xi_1}\cdot g_{x_1} \in F_{\xi_v}$. We have:

$$h_{\xi_1} = \lim_{n\to\infty} n^{q_1+q_2} f_{\xi_1}(nx)\, g_{\xi_1}(nx) \tag{36}$$

Suppose now that we want to choose a different main generating function of order 0 $\xi_2 \in H^0$ for which we find the generating functions $f_{\xi_2} \in F_{\xi_2}$ and $g_{\xi_2} \in F_{\xi_2}$ relevant to h_1 and h_2. We know also that $f_{\xi_2}\cdot g_{\xi_2} \in F_{\xi_w}$. We have:

$$h_{\xi_2} = \lim_{n\to\infty} n^{q_1+q_2} f_{\xi_2}(nx)\, g_{\xi_2}(nx) \tag{37}$$

given the (34) then we have:

$$\begin{aligned} h_{\xi_2} &= \lim_{n\to\infty} n^{q_1}\, \tau_{12}^{q_1}\left(f_{\xi_1}(nx)\right) n^{q_2}\, \tau_{12}^{q_2}\left(g_{\xi_1}(nx)\right) \\ &= n^{q_1+q_2}\, \tau_{vw}^{q_1+q_2}\left(f_{\xi_1}(nx)\, g_{\xi_1}(nx)\right) \end{aligned} \tag{38}$$

from which we see that h_{ξ_1} and h_{ξ_2} are the same generalised function in $\mathbb{G}^\eta$ and therefore the above product is well defined.

8 Equalities in D'

By using the above defined product, we can prove interesting equalities involving products among elements of D'. We will see an example in this paragraph.

Note that from now on, we will choose a specific main generating function $\xi \in H^0$, once and forever. We will perform all our calculations with generating functions in F_ξ and we will give all the final results in terms of $\eta_\xi^{p,f}$. Since the underling ξ is always the same, we will drop the ξ notation from the $\eta^{p,q}$

functions and a_p coefficients. When we write $\eta^{p,q}$, we really mean $\eta_\xi^{p,q}$ and when we write a_p, we really mean $a^p(\xi)$.

Before we proceed we need to see how to represent step discontinuous functions by using elements of $\mathbb{G}^\eta$. Let $\xi \in F$ be a main generating function for δ. We define the following function:

$$\chi(x) = \int_{-\infty}^{x} \xi(t)dt \tag{39}$$

to be a main generating function for $u(x)$, the Heaviside function, where we use a growing rate $q = 0$. Also if $f \in C^\infty$ is a function and χ is a main generating function for u, then we define $f(\chi(x)))$ to be a generating function for $f(u(x))$.

Of course $\chi(x)$ and $f(\chi(x)))$ are not in F. However we are interested in multiplication of a step discontinuous functions with elements of $\mathbb{G}^\eta$ and therefore in multiplying $\chi(x)$ and $f(\chi(x)))$ with elements of F so that we eventually get a generating function, for our product, which is in F.

Now, given a generalised function $f(u(x)))$, there are always $\beta, \gamma \in \mathbb{R}$ such that:

$$[f(\chi(x)) - \beta - \gamma\chi(x)] \in F \tag{40}$$

By applying the (16) to the (40), we can evaluate $f(u(x))$ in terms of elements of $\mathbb{G}^\eta$ as follows:

$$f(g(x)) = \beta + \gamma u(x) + \sum_{p=0}^{\infty} a_p \eta^{p,0} \tag{41}$$

For example:

$$u^2(x) = u(x) + \sum_{p=0}^{\infty} a_p \eta^{p,0} \tag{42}$$

$$sign^2(x) = (2u(x) - 1)^2 = 1 + \sum_{p=0}^{\infty} a_p \eta^{p,0} \tag{43}$$

Note that, in the following example we will use the notation introduced in (10) ($\eta^{p,q}$ expressed in the $\delta^{(p)}/n^k$ notation) and, since we do not have ξ in a closed form, the coefficients of the $\eta^{p,q}$ will be evaluated numerically.
We want to evaluate $u(x)\delta'(x)$:

$$u(x)\delta'(x) \to n^2\, \chi(nx)\xi'(nx) \tag{44}$$

From which we have:

$$u(x)\delta'(x) = a_0 n\delta(x) + \frac{1}{2}\,\delta'(x) + a_2\,\frac{\delta^{(2)}}{n} + R\left(\frac{\delta^{(4)}}{n^3}\right) \tag{45}$$

We want to remove the $n\delta$ term. To do that, we evaluate the product $\delta^2(x)$:

$$\delta^2(x) \to n^2\, \xi^2(nx) \tag{46}$$

From which we have:

$$\delta^2(x) = b_0 n\delta(x) + b_2\,\frac{\delta^{(2)}}{n} + b_4\,\frac{\delta^{(4)}}{n^3} + R\left(\frac{\delta^{(6)}}{n^5}\right) \tag{47}$$

Where b_3 and b_5 vanish (evaluated numerically, are smaller, in module, then 10^{-15}). For any ξ, $a_0 = -b_0$ (evaluated numerically, have opposite sign and are equal in module with an error smaller, then 10^{-14}). By substituting the value $n\delta$ from the (47) in the (45), we have eventually:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) + R\left(\frac{\delta^{(2)}}{n}\right) \tag{48}$$

or,(compare with [3]), as an equality among products of elements of D' (i.e. ignoring the higher order terms):

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) \tag{49}$$

We can get to the same results by using the Leibniz rule (which we know to work with our definition of product). We evaluate the product of $u(x)\delta(x)$. We have:

$$u(x)\delta(x) \to n\;\chi(nx)\xi(nx) \tag{50}$$

From which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) + R\left(\frac{\delta'}{n}\right) \tag{51}$$

by taking the derivatives of both sides we have:

$$\delta^2(x) + u(x)\delta'(x) = \frac{1}{2}\delta'(x) + R\left(\frac{\delta^{(2)}}{n}\right) \tag{52}$$

as expected. More examples can be found in the appendix.

9 Products with polynomials

From the (15) is possible to show that given any $\xi_1^{(p)} \in H^p$ we have:

$$\xi_2^{(p-1)}(x) = -\frac{x}{p}\xi_1^{(p)}(x) \in H^{p-1}\;\; with\, p > 0 \tag{53}$$

The above equality gives us an hint on how to extend to the concept of main generating functions and define main generating functions of negative orders.

To do that, we define a function $\xi^{[-p]} \in H^{[-p]} \subset F$ tu be a main generating function of order $-p$ if it is possible to find $f \in F$ such that $\xi^{[-p)]}x^p = f$ for each $x \in C - \{0\}$ (i.e. f goes to 0 in 0^+ and 0^- at least like x^p) and:

$$\int_{-\infty}^{+\infty} \xi^{[-p)]}(x)x^k dx = \begin{cases} 1 & for\; k = -p \\ 0 & for\; k > -p \end{cases} \tag{54}$$

In analogy with the definition of F_ξ, we define the set $F_{[\xi]}$ in the obvious way. Note that the notation $\xi^{[-p]}$ may be misleading since, although the derivative of $\xi^{[-p]}$ is a $\xi^{[-p+1]}$, the $\xi^{[-p]}$, both for p positive and negative, are always null functions. So, for example, the derivative of $\xi^{[-1]}$ is $\xi^{[0]}$ which is different from $\xi^{(0)}$ which, in turn, is the derivative of $\chi(x) \notin F$.

From the (54) we see that:

$$\xi_2^{[-1]}(x) = -x\xi_1^{(0)}(x) \in H^{[-1]} \tag{55}$$

and

$$\xi_2^{[p-1]}(x) = \frac{x}{p-1}\xi_1^{[p]}(x) \in H^{[p-1]} \;\; with\, p < 0 \tag{56}$$

Definition 9. *We define $\eta^{[p],q}$ to be the following generalised functions:*

$$\eta^{[p],q} = \lim_{n\to\infty} n^q \xi^{[p]} \; with\, p, q \in \mathbb{Z} \tag{57}$$

Note that, due to the way $\mathbb{G}^\eta$ has been defined, it already contains the $\eta^{[p],q}$. Now, by using the definitions of $\eta^{p,q}$ and $\eta^{[p],q}$, it is easy to prove the following equalities:

$$\begin{array}{lll} \eta^{p-1,q-1} = -\frac{x}{p}\eta^{p,q} & & p > 0 \\ \eta^{[-1],q-1} = -x\,\eta^{0,q} & & p = 0 \\ \eta^{[p-1],q-1} = -\frac{x}{p}\eta^{[p],q} & & p > 0 \\ \eta^{[p-1],q-1} = \frac{x}{p-1}\eta^{[p],q} & & p < 0 \end{array} \tag{58}$$

We will briefly prove only the first equality of the (58). Multiplying a generalised function by x is equivalent to multiply its generating sequence by x before taking the limit. Given the generating sequence $n^q\, x\, \xi^{(p)}$ and by using the (15), we can evaluate the coefficients as:

$$\begin{aligned} a_k(n^q\, x\, \xi^{(p)}) &= \lim_{n\to\infty} \frac{(-1)^k}{k!} \int_{-\infty}^{+\infty} n\, x\, \xi^{(p)}(nx)(nx)^k dx \\ &= \lim_{n\to\infty} \frac{(-1)^k}{k!}\frac{1}{n} \int_{-\infty}^{+\infty} n\, \xi^{(p)}(nx)(nx)^{k+1} dx \\ &= a_{k-1}(n^q\, \xi^{(p-1)}) \\ &= -k\, a_k(n^{q-1}\, \xi^{(p-1)}) \end{aligned} \tag{59}$$

from which the first of the (58) follows. In particular:

$$\delta^{(p-1)} = -\frac{x}{p}\delta^{(p)} \; for\, p > 0 \tag{60}$$

which is a well known result in literature (compare with [3]).

Note that, $x\xi_1^{[0]} = -\xi_2^{[[-1]]}$ and the derivative of $\xi^{[[-1]]}$ is equal to $\xi^{[[0]]} \neq \xi^{[0]}$, with obvious meaning of the notation. By iterating the process we can define the $\xi^{[[\cdots p \cdots]]}$, all in F, and the $\eta^{[[\cdots p \cdots]],q}$, all in $\mathbb{G}^\eta$, with as many square brackets as we want. These are all new generalised functions present in $\mathbb{G}^\eta$ and which arise naturally from the theory we have developed.

As a final remark we note that the function $f(x) = \xi(x) + \xi^{[-1]}$ verifies the (2) although it contains a $\xi^{[-1]}$ component. Since a good ξ function should not have any $\xi^{[[\cdots p \cdots]]}$ components in it, then we need to add this condition to the (2) in order to have a complete set of conditions for the definition of our ξ functions.

We will use now the theory developed in this paragraph to discuss a well known example in the theory of product of distributions (compare with [2] §1.1 ex. i). If $vp\frac{1}{x}$ is the Cauchy principal value of $\frac{1}{x}$ then we have:

$$0 = (\delta(x) \cdot x) \cdot vp\frac{1}{x} = \delta(x) \cdot \left(x \cdot vp\frac{1}{x} \right) = \delta(x) \tag{61}$$

which is absurd.

By using our theory we know that $x\delta(x) = -\eta^{[-1],0} \neq 0$. We have:

$$0 = (x \cdot \delta(x) + \eta^{[-1],0}) \cdot \frac{1}{x} = \delta(x) + \frac{1}{x}\eta^{[-1],0} = \delta(x) - \delta(x) \tag{62}$$

a results that makes us to feel much more comfortable.

Appendix

A.1 Examples of product of distributions

We will use the notation introduced in (10).
Example 1:

$$\delta(x)\delta'(x) \tag{63}$$

By taking twice the derivative of both sides of the (51), and rearranging the terms we get:

$$\delta(x)\delta'(x) = \frac{1}{6}\delta^{(2)}(x) - \frac{1}{3}u(x)\delta^{(2)}(x) + R\left(\frac{\delta^{(3)}}{n}\right) \tag{64}$$

Example 2: evaluated numerically.

$$sign^2(x)\delta(x) \to n\,(2\chi(nx) - 1)^2\xi(nx) \tag{65}$$

from which we have:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) + R\left(\frac{\delta^{(2)}}{n^2}\right) \tag{66}$$

A.2 Numerical evidences in support to the (34)

First of all, to perform our numerical analysis, we need to choose suitable ξ functions. Let $f(x)$ be the following Gaussian distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{67}$$

then we define $\xi_1(x)$ to be:

$$\xi_1(x) = f(x) - \frac{1}{2}f^{(2)}(x) + \frac{1}{8}f^{(4)}(x) - \frac{1}{48}f^{(6)}(x) \tag{68}$$

which is a very good approximation of a ξ function and it is derived from the Gaussian distribution by removing the first 3 higher order $\eta^{p,1}(x)$ components (compare with the (28) and the algorithm (18) above). Also we define $\xi_2(x)$ to be:

$$\xi_1(x) = 0.3\,\xi_1(x) + 0.7\,\xi_2(2x) \tag{69}$$

By means of the (16) we evaluate numerically the coefficients a_p of the two products $\xi \cdot \xi$ and $\xi \cdot \xi'$, generating functions for δ^2 and $\delta \cdot \delta'$. We have:

	a_0	a_1	a_2
q-p-1	1	0	-1
$\xi_1 \cdot \xi_1$	0.747850786175440	0	0.064630940223461
$\xi_2 \cdot \xi_2$	1.164372758468304	0	0.025251755987242

Table 1 (δ^2)

	a_0	a_1	a_2	a_3
q-p-1	2	1	0	-1
$\xi_1 \cdot \xi_1'$	0	0.373925393087720	0	0.032315470111731
$\xi_2 \cdot \xi_2'$	0	0.582186379234155	0	0.012625877993621

Table 2 ($\delta \cdot \delta'$)

The coefficients in the table 1 and 2 above, have been evaluated by integrating the functions numerically in the interval [-10,10] on 5000 points.

Our argument in support of the (34) is that, the ratios between the a_p coefficients are the $\sigma_{vw}^{p,q}$ and, if the (34) is true, these ratios have to be independent from the specific $\xi^{(p)}$ functions that have been multiplied.

We know that the σ depends only from $q-p-1$ and therefore, if the (34) is true then, for example, $\sigma_{vw}^{0,2}$ evaluated from table 1 will be equal to $\sigma_{vw}^{1,3}$ evaluated from table 2 although they refer to the product of different functions. Note that, for our analysis to be correct, we make the assumptions that ξ_v and ξ_w are the same sets in both multiplications of table 1 and 2. Although this has not been proven in the general case, in this case it is certainly true since, for example, $D\xi_1^2$=$2\xi_1 \cdot \xi_1'$ and therefore ξ_1^2 and $\xi_1 \cdot \xi_1'$ belong to the same space F_{ξ_v}.

We show in the following tables the results we have found in our analysis:

	$\sigma_{vw}^{0,2} = \sigma_{vw}^{1,3}$	$\sigma_{vw}^{2,2} = \sigma_{vw}^{3,3}$
table 1	$\frac{a_0(\xi_w)}{a_0(\xi_v)} = 1.556958660728280$	$\frac{a_2(\xi_w)}{a_2(\xi_v)} = 0.390706926124457$
table 2	$\frac{a_1(\xi_w)}{a_1(\xi_v)} = 1.556958660728288$	$\frac{a_3(\xi_w)}{a_3(\xi_v)} = 0.390706926124451$
difference	$7.9 \cdot 10^{-15}$	$6.0 \cdot 10^{-15}$

Table 3

We conclude that the numerical evidences suggest the (34) is true.

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[4] V. Nardozza. *Product of Distributions Applied to Discrete Differential Geometry.* www.vixra.org/abs/1211.0099 (2012) version v8 or most recent.