Products of Generalised Functions

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Abstract

A new space of generalised functions extending the space D', together with a well defined product, is constructed. The new space of generalized functions is used to prove interesting equalities involving products among elements of D'. A way of multiplying the defined generalised functions with polynomials is also derived.

Key Words: distribution theory, product of distributions.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and, in partial differential equation, such as shock wave solutions, in hydrodynamics, (see [1]). An important issue, related to product of distributions, is the fact that the product, in the general case, is not well defined in D', issue known as the Schwartz impossibility result (see [1] §1.3) and that only the product between a smooth function and a distribution is well defined. Historically, products of distributions are addressed by means of algebras of generalised functions developed initially by J. F. Colombeau (see [1] and [2]). In this paper we will propose a new approach to define products of distributions.

In paragraphs from 3 to 6, we construct a new space of generalised functions, extending the space D'. In paragraph 7, we use the new space of generalised functions to define a product among distributions. In paragraphs 8, we use the new developed theory to derive interesting equalities involving products among elements of D'. In paragraphs 9, we derive a method to multiply the generalised functions defined in this paper with polynomials.

2 The need for new generalised functions.

In [3] we have proven that:

$$f(g(x))\delta(x) = \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\delta(x) \tag{1}$$

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where the above product has to be intended as:

$$f(g(x))\delta(x) = \frac{1}{b-a} \lim_{n\to\infty} f(g_n(x))g'_n(x) \tag{2}$$

and $g_n(x)$ is a sequence converging to a step discontinuous function jumping, from a to b, in 0. The above equation shows that product of distributions strongly depend from the structure of the various discontinuity which are multiplied (in this case the step discontinuity), where the structure (represented in this case by the function f) has to be intended as the specific sequence of functions used to define the discontinuity.

The above equation, focuses its attention only on the structure of step discontinuities and the way they are modified (by composition with a locally integrable function f). When it comes to Dirac delta functions, it is possible to show that they change their own structure by means of multiplication by step discontinuous functions. Let us consider the function $f(g(x))$ where g is a step discontinuous function, jumping from a to b in 0, and $f \in L^1_{loc}([a,b])$. Since we may define our function as the limit of a sequence of functions $f(g_n(x))$ with $g_n(x) \in C^1$, and since the Leibniz rule may be applied to each term of the sequence, we will suppose that we can apply the Leibniz rule also to its limit. This point will be justified in further paragraphs. We have:

$$Df(g(x)) = (b-a)f'(g(x))\delta(x) \tag{3}$$

from which we see that by multiplying a delta function having structure $g(x)$ (i.e. derivative of a step discontinuous function $g(x)$) by $f'(g(x))$ we get a delta function with structure $f(g(x))$ (i.e. derivative of a step discontinuous function $f(g(x))$).

We have seen that, in a product of distributions, if we change the structure of a term we get a different result. In order to overcame this limitation, we want now to extend the space of distributions D' by adding to it, as separate generalised functions, additional elements representing any possible discontinuity structure needed for describing products of step and delta functions.

We will assume now that all step discontinuous and delta functions, we are dealing with, are all related to the same Heaviside function and their structure can be described by the way they are related to it. From this new point of view, the function f, which before was used to relate distribution structures, became now the structure itself of the distribution. We will say that a step discontinuity has structure f if it is of the form $f(u(x))$. We will say that a delta function has structure f if it is the derivative of a step discontinuous function of structure f. We will consider steps and delta functions, with different structures, as separate generalised functions.
We will use the following notation:

$$\begin{array}{lll} u_{[f(x)]} = f(u(x)) & \textit{step function having structure } f \\ \delta_{[f'(x)]} = f'(u(x))\delta(x) & \textit{delta function having structure } f \end{array} \tag{4}$$

where $u_{[f(x)]}$ and $\delta_{[f(x)]}$ are not normalised (i.e they may have amplitude different from 1) and $u_{[x]} = u(x) \in D'$, $\delta_{[1]} = \delta(x) \in D'$. We will show, with an example at the end of this paragraph, that the above defined generalised functions have components outside D' and therefore there is a need for defining

a larger space of generalised functions including D'. We will do that in the next paragraphs.
Using the (4), we define the multiplication as follows:

$$u_{[f_1]}u_{[f_2]}\cdot\ \dots\ \cdot u_{[f_n]}\delta_{[f_{n+1}]} = \delta_{[f_1 f_2\cdot\ \dots\ \cdot f_n f_{n+1}]} \tag{5}$$

Finally we define a projector operator $P_{D'}$, which project any generalised function of the kind (4), on the space D'. For step discontinuous functions the way $P_{D'}$ works is trivial (e.g. $u^2(x)$ goes to $u(x)$). For delta functions, we apply the (1), we have:

$$P_{D'}\left(\delta_{[f_1 f_2\cdot\ \dots\ \cdot f_n f_{n+1}]}\right) = \left(\int_0^1 f_1 f_2\cdot\ \dots\ \cdot f_n f_{n+1} dx\right)\delta(x) \in D' \tag{6}$$

where the integration is performed between 0 and 1, which is the jump of our reference step discontinuity $u(x)$. Note that the (5) and (6) provide a well defined product of the (4). The product is also commutative and associative since commutative and associative is the product of the f_i functions used in the definition of the (5).

Let us make an example. Consider the product of distributions $sign^2(x)\delta(x)$ (compare with [2] §1.1 ex. iii). By using proposition 1 we find easily that:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) \tag{7}$$

Let us check associativity by using, once again, proposition 1:

$$sign^2(x)\delta(x) = sign(x)[sign(x)\delta(x)] = sign(x)\cdot 0 = 0 \tag{8}$$

we conclude that, in D', our product is not associative. Let us see what happen using the (5):

$$sign(x)[sign(x)\delta(x)] = sign(x)\delta_{[(2x-1)\cdot 1]} = sign(x)[\delta_{[2x]} - \delta_{[1]}] \tag{9}$$

In D', $\delta_{[1]} = \delta$ and $P_{D'}(\delta_{[2x]}) = \delta$. However, as generalised function of the kind (4), they are separate objects and they do not cancel each other. We have eventually:

$$sign^2(x)\delta(x) = P_{D'}\left(\delta_{[(2x-1)^2]}\right) = \frac{1}{3}\delta(x) \tag{10}$$

3 New generalised functions

In this paragraph we will define a new class of generalized functions. Before we proceed, we need a definition. We define F to be the set of all the function $f(x)$ having the following characteristics.

$$\begin{array}{l} 1)\ f(x) \in C^\infty \\ 2)\ \lim_{x\to-\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N} \\ 3)\ \lim_{x\to+\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N} \end{array} \tag{11}$$

Generalised functions can be defined by means of the limit of sequences of functions $f_n(x)$. In this paper we will deal only with generalised functions defined by means of the limit of a sequence of the form:

$$\lim_{n\to\infty} n^q f(nx) \tag{12}$$

with $f \in F$. Note that the above sequences are not the most general way to define distributions. For example, there is no sequence of the form (12) converging to $\delta + \delta'$. We will call $f(x)$ the generating function, $n^q f(nx)$ the generating sequence and q the growing index of the generalised function defined by the (12). Finally, we define the generalised functions $\eta_f^{p,q}$ to be the following limit:

$$\eta_f^{p,q}(x) = \lim_{n\to\infty} n^q f^{(p)}(nx) \quad with\, p \geq 0,\; q \in \mathbb{Z} \tag{13}$$

provided that $f \in F$ and

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \tag{14}$$

We will see in further paragraphs that, in order to have $\eta_f^{p,q}$ to be an interesting mathematical object, we need to define further constrains on f. Note also that, for reasons that will be clear further on, it is very important to keep track of the generating function f. We do that by using the notation $\eta_f^{p,q}$. It is easy to see that:

$$\eta_f^{p,p+1}(x) = \delta^{(p)}(x) \tag{15}$$

What kind of generalised function are the $\eta_f^{p,q}$? If the sequence of distributions f_n converges to $\eta_f^{p,q}$, then $\frac{f_n}{n^{q-p-1}}$ converges to $\delta^{(p)}$. So, with an abuse of notation, we may say that:

$$\eta_f^{p,q} = \frac{\delta^{(p)}}{n^{p-q+1}} \tag{16}$$

The $\eta^{p,q}$ are therefore the limit of sequences of functions that are shaped like $\delta^{(p)}$ and that, when we take the limit, grow at a lower or faster rate (according to the sign of p-q+1).

Now, let us see how to determine all the $\eta^{p,q}$ components of a generalised function defined by means of the (12) and having generating function $f(x) \in F$. We will suppose, for the moment, that all $\eta^{p,q}$, have the same generating function $g \in F$. We will see further on, that this turn out to be true. First of all, we note that all the components of the distribution (12) have the same growing index q. We will call this kind of generating functions homogeneous. We have:

$$h = \lim_{n\to\infty} n^q f(nx) = \sum_{p=0}^{\infty} a_p \eta_g^{p,q} \tag{17}$$

where the a_p, although not explicitly noted, refer and depend from the function g which we suppose known. Now, if $q > 1$, h always contains one (and only one) distribution $\eta_g^{q-1,q}(x) = \delta^{(q-1)}(x) \in D'$. From the (16) we know that:

$$\lim_{n\to\infty} \frac{n^q f(nx)}{n^{q-p-1}} = a_p \delta^{(p)} \tag{18}$$

So, for the distribution defined by the (17), we can determine the a_p coefficients by applying the Schwartz theory of distribution to our sequence of functions divided by n^{q-p-1}. Let ϕ be a test function and given p, we have:

$$\frac{h}{n^{q-p-1}} = \lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{p+1} f(nx)\phi(x)dx = \lim_{n\to\infty} \sum_{k=0}^{\infty} (-1)^k a_k \frac{\phi^{(k)}(0)}{n^{k-p}} \tag{19}$$

In the right side of the above equation, when n goes to infinity, some terms go to infinity (the ones with $k < p$) and some other terms go to 0 (the ones with $k > p$). To better evaluate all a_p we decide to use a test function ϕ that has all derivatives $\phi^{(i)}(0) = 0$ for $i \neq p$. A test function with this characteristic is $\phi(x) = x^p$. Of course a test function should vanish outside a compact interval and x^p does not. However, given the (11), integrability for $|x|$ going to infinity in ensured, and therefore this is not a problem. We have:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{p+1} f(nx) x^p dx = (-1)^p \, a_p \, p! \tag{20}$$

where $p!$ is the value of the p^{th} derivatives of x^p. From the (20) we can easily evaluate the a_p as follows:

$$\begin{aligned} a_p &= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n^{p+1} f(nx) x^p dx \\ &= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n \, f(nx)(nx)^p dx \end{aligned} \tag{21}$$

We note that the right part of the (21), for n that goes to infinity, in the (x, y) plane, shrinks (along x) and grows (along y) like n, which leaves the integral unchanged. For the above reason, the limit of the (21) is simply the value of the integrals for any n. We may as well evaluate it for n=1. We have:

$$a_p = \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} f(x) x^p dx \tag{22}$$

and therefore a_p coefficients are related to the momenta of f. We are now ready to define our new space of generalised functions.

Definition 1. *We define $\mathbb{G}^\eta$ to be space of generalised functions which elements are the limits of series of the type (13), (which we know to be homogeneous generalised functions), or the linear combinations af a finite or infinite numbers of them.*

We also define A to be the set of all sequences of coefficients $a_f = (a_0, a_1, \ldots)$ associated by the (22) to the generating function f.

4 Main generating functions

For a generalised function $h \in \mathbb{G}^\eta$, if q is the growing index, $f \in F$ is the generating functions and $a_f \in A$ are the coefficients of the $\eta_g^{p,q}$, we can fully characterize the structure of a discontinuity (i.e. fully define the relevant generalised function) by providing either (f, q) or (a_f, g, q). Moreover, if $a_f = (a_0, a_1, \ldots)$, then $a_{f'} = (0, a_0, a_1, \ldots)$ and therefore, in $\mathbb{G}^\eta$, the derivative of $h = (f, q)$ is $h' = (f', q + 1)$.

Let $f(x) \in F$ be a generating function for δ in D' and $h \in \mathbb{G}^\eta$ the relevant generalised function defined by the generating sequence $n \, f(nx)$. If we evaluate the coefficients $a_f \in A$ of h, we know that $a_1 = 1$. We also know that all the others coefficient can have any value. We are interested, among all the $f \in F$, to the ones for which a_f is of the form $a_0 = 1$ and $a_p = 0$ for $p > 1$. We give the following definitions:

Definition 2. *Given $\xi(x) \in F$. If $\xi(x)$ verifies the following equations:*

$$\int_{-\infty}^{+\infty} \xi(x) x^p dx = \begin{cases} 1 & for\, p = 0 \\ 0 & for\, p > 0 \end{cases} \tag{23}$$

then we call ξ a main generating function for $\delta(x)$ or main generating function of order 0. We also call the derivative $\xi^{(p)}$ with $p \in \mathbb{N}$ a main generating function of order p.

We have:

$$\lim_{n \to n} n^q \xi^{(p)} = \eta_\xi^{p,q} \;\; with\, p \geq 1,\; q \in \mathbb{Z} \tag{24}$$

and also (compare with the (16) above)

$$\lim_{n \to \infty} n^{p+1} \xi^{(p)} = \delta^{(p)}(x) \;\; in\, \mathbb{G}^\eta \;\; p \geq 0 \tag{25}$$

The (25) states that, if we use main generating functions, we can define delta and delta derivatives that have no components outside D'. In a few words, if we accept generalised function $\eta^{p,q}$ to be real things (i.e. we work in $\mathbb{G}^\eta$), we have also to accept that only sequences $n\xi(nx)$ composed of main generating functions converge to δ.

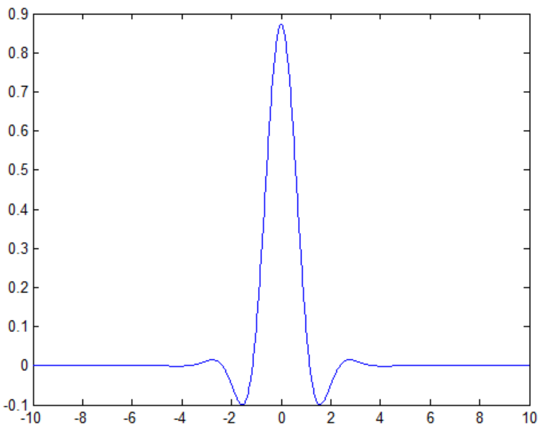

Figure 1: ξ function

The above figure is a plot of a $\xi(x)$.

Given $f \in F$, there is only one element $a_f \in A$. On the contrary, given $a_f \in A$, there exist at least one generating function $g \in F$, with $f \neq g$, such that $a_f = a_g$. In particular, there are several elements of F which are a main generating function of order 0.

If $H^p \subset F$ is the set of all generating functions of order p, then we have:

$$\begin{array}{lll} \xi(x) \in H^0 & \Rightarrow & \alpha\xi(\alpha x) \in H^0 \\ \xi(x) \in H^0 & \Rightarrow & a[\,\alpha_1\xi(\alpha_1 x)\,] + b[\,\alpha_2(\xi(\alpha_2 x)\,] \in H^0 \;\; with\, a+b=1 \\ \xi^{(p)}(x) \in H^p & \Rightarrow & \alpha^{p+1}\xi^{(p)}(\alpha x) \in H^p \\ \xi_1^{(p)}(x) \in H^p & \Rightarrow & \xi_2^{(p-1)}(x) = -\frac{x}{p}\xi_1^{(p)}(x) \in H^{p-1} \;\; with\, p>0 \end{array} \tag{26}$$

with $\alpha_1 > 0$, $\alpha_2 > 0$.

From the second implication of the (26) it follows that if $\xi_1 \in H^0$, then for any $\rho(\alpha) \in D'$, such that:

$$\int_0^\infty \rho(\alpha)d\alpha = 1 \tag{27}$$

we have:

$$\xi_2(x) = \int_0^\infty \rho(\alpha)\alpha\xi_1(\alpha x)d\alpha \in H^0 \tag{28}$$

provided that the above integral converges. Note that given ξ_1 and ξ_2, ρ is not unique. Note that ρ may be continuous, impulsive or mixed. For example, in the second implication of the (26), we have $\rho(\alpha) = a\delta(\alpha-\alpha_1) + b\delta(\alpha-\alpha_2)$.

5 Additional remarks on the η functions

Given a function $f \in F$, we say that f is a null function if all the coefficients a_p evaluated by means of the (22) are equal to 0 (i.e. all the momenta of f vanish). We define $N \subset F$ to be the set of all null functions. Of course, null functions are generating functions for $0 \in \mathbb{G}^\eta$.

For example, if ξ_1 and ξ_2 are two separate main generating functions of order 0, then $\xi_1 - \xi_2$ is a null function.

If we choose any $\xi \in H^0$, then given $f \in F$ we can easily evaluate the coefficient a_p by means of the (22). We define the function f_ξ to be:

$$f_\xi = \sum_{p=0}^{\infty} a_p \xi^{(p)} \tag{29}$$

We will call $F_\xi \subset F$ the subset of F of all functions for which $f - f_\xi = 0$. In a few words, F_ξ is the set of all elements of F which can be expressed as a finite or infinite linear combinations of the $\xi^{(p)}$.

We show now an important fact about the coefficient a_p of the (22). Let $\xi \in H^0$ be any main generating function of order 0 and $\eta_\xi^{p,q} \in \mathbb{G}^\eta$ a related generalised function. We have:

$$\lim_{n\to\infty} a_p(\xi)n^q\xi^{(p)}(nx) = a_p(\xi)\eta_\xi^{p,q} \tag{30}$$

If we choose $\xi_\alpha = \xi(\alpha x)$, as a different generating function for of order 0, we have:

$$\lim_{n\to\infty} a_p(\xi_\alpha)n^q\alpha^{p+1}\xi^{(p)}(n\,\alpha x) = a_p(\xi_\alpha)\eta_{\xi_\alpha}^{p,q} \tag{31}$$

If the (30) and (31) are the same generalised function then:

$$\lim_{n\to\infty} a_p(\xi)n^q\xi^{(p)}(nx) = \lim_{n\to\infty} \frac{a_p(\xi_\alpha)}{\alpha^{q-p-1}}(n\alpha)^q\xi^{(p)}(n\,\alpha x) \tag{32}$$

since

$$\lim_{n\to\infty} n^q \xi^{(p)}(nx) = \eta_\xi^{p,q} = \lim_{n\to\infty} (n\alpha)^q \xi^{(p)}(n\,\alpha x) \tag{33}$$

because the left and the right side limit of the (33) are the same function growing and shrinking at the same rate with n, and therefore converge to the same generalised function $\eta_\xi^{p,q}$, we conclude that:

$$a_p(\xi_\alpha) = \alpha^{q-p-1} a_p(\xi) \tag{34}$$

and therefore:

$$\eta_{\xi_\alpha}^{p,q} = \frac{1}{\alpha^{q-p-1}} \eta_\xi^{p,q} \tag{35}$$

From the (35), it is clear that if we want to use the $\eta^{p,q}$ notation we have always to specify the reference main generating function ξ since, for any specific element of $\mathbb{G}^\eta$, this has an impact on the amplitudes of the $\eta^{p,q}$ (i.e. the amplitude of the coefficients). This is why we use the notation $\eta_\xi^{p,q} \in \mathbb{G}^\eta$.

Note that if $p = q - 1$ then, as expected, the η notation is independent from ξ since $\eta^{p,p+1} = \delta^{(p)}$.

We conclude this paragraph by finding a relation similar to the (34) but valid in the most general case. Given $\eta_{\xi_1}^{p,q}$, if we choose any other $\xi_2 \in H^0$, evaluated using the (28), as the reference generating function, then by using both the (34) and the (28) we find that the relationship between the coefficients of $\eta_{\xi_2}^{p,q}$ and $\eta_{\xi_1}^{p,q}$ is the follows:

$$a_p(\xi_2) = \sigma_{12}^{p,q}\, a_p(\xi_1) \tag{36}$$

where:

$$\sigma_{12}^{p,q} = \int_0^\infty \rho(\alpha) \alpha^{q-p-1} d\alpha \tag{37}$$

We are now ready to see an example. Given a Gaussian distribution $f_{\xi_1}(x) \in F_{\xi_1}$ defined as follows:

$$f_{\xi_1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{38}$$

we want to represent the generalised function $h \in \mathbb{G}^\eta$, having generating function f_{ξ_1} and grooving index $q = 1$, by means of the $\eta^{p,q}$ notation. Using the (22) we have:

$$h = \lim_{n\to\infty} n\, f_{\xi_1}(nx) = \delta(x) + \frac{1}{2}\eta_{\xi_1}^{2,1} + \frac{1}{8}\eta_{\xi_1}^{4,1} + R\left(\eta^{6,1}\right) \tag{39}$$

where $R(\eta^{6,1})$ means that, to have the above equality exact, we need to add components of growing index 1 and order ≥ 6. If $\xi_2 \in H^0$ is a different main generating function, we can represent the same generalised function h by means of the $\eta_{\xi_2}^{p,q}$. To do that, we need to find the function ρ which allows as to evaluate ξ_2 from ξ_1 as defined by the (28) and then, using the (37), we have:

$$h = \delta(x) + \frac{1}{2}\,\sigma_{12}^2\,\eta_{\xi_2}^{2,1} + \frac{1}{8}\,\sigma_{12}^4\,\eta_{\xi_2}^{4,1} + R\left(\eta^{6,1}\right) \tag{40}$$

Note that the (40) has generating function:

$$f_{\xi_2} \in F_{\xi_2} = \sum_{p=0}^{\infty} \sigma_{12}^p\, a_p\, \xi_2^{(p)} \tag{41}$$

also note that $f_{\xi_1} \neq f_{\xi_2}$ and both f_{ξ_1} and f_{ξ_2} are generating functions for the same generalised function $h \in \mathbb{G}^\eta$ which is:

$$h = \lim_{n\to\infty} n\, f_{\xi_1}(nx) = \lim_{n\to\infty} n\, f_{\xi_2}(nx) \tag{42}$$

6 Transformations in F

Before we proceed, we note briefly that it is certainly possible to find elements of F which do not belong to any F_ξ, a good example of that are the elements of N (null momenta functions).

Now, given the $\xi_1, \xi_2 \in H^0$ and the σ from the (37), we define

$$\tau_{12}^{q} = (\sigma_{12}^{0,q},\, \sigma_{12}^{1,q},\, \cdots) \tag{43}$$

to be a transformation in F such that:

$$\tau_{12}^{q} : f_1 \in F_{\xi_1} \to f_2 \in F_{\xi_2} \tag{44}$$

which transforms any element of F_{ξ_1} in the relevant element of F_{ξ_2} such that the two elements are generating functions for the same element of $\mathbb{G}^\eta$. We also define T to be the set of all separate τ functions. Note that, for example, $\tau^{q}_{\xi_1\xi_1}$ and $\tau^{q}_{\xi_2\xi_2}$, having the same σ components, are the same element in T. It is easy to show that T has the structure of an Abelian group where the operation is composition of transformations and:

$$\begin{array}{lll} 1) & \tau^{q}_{\xi\xi} = (1,\, 1,\, \cdots) & \textit{is the } 0 \textit{ element} \\ 2) & -\tau^{q}_{\xi_1\xi_2} = \tau^{q}_{\xi_2\xi_1} & \textit{with } \sigma_{21}^{p,q} = (\sigma_{12}^{p,q})^{-1} \end{array} \tag{45}$$

Now, let $f_{\xi_1}, g_{\xi_1} \in F_{\xi_1}$ be two generating functions for $h_1, h_2 \in \mathbb{G}^\eta$ of growing indexes q_1 and q_2. Let also $f_{\xi_2}, g_{\xi_2} \in F_{\xi_2}$, be the relevant generating functions (taking into account the growing indexes) for the same generalised functions, h_1 and h_2. If $f_{\xi_1} g_{\xi_1} \in F_{\xi_v}$ and $f_{\xi_2} g_{\xi_2} \in F_{\xi_w}$ then we state that:

$$\tau_{12}^{q_1}(f_{\xi_1}) \cdot \tau_{12}^{q_2}(g_{\xi_1}) = \tau_{vw}^{q_1+q_2}(f_{\xi_1} \cdot g_{\xi_1}) \tag{46}$$

The (46) tells us that we can transform f and g from F_{ξ_1} to F_{ξ_2} and then multiply them or multiply them and then transform the product from F_{ξ_v} to F_{ξ_w}. In both cases we get the same function.

Unfortunately we do not have a formal prove for the (46). However, numerical evidences (see appendix) suggest that the (46) is true.

An important question is whether ξ_v and ξ_w depend only from ξ_1 and ξ_2 and are independent from the function f and g. We believe this is likely to be the case although we do not have a formal proof of it. However, for the (46) to be true, this assumption is not required nor it is ever used throughout the paper and therefore, for the time being, we will not spend more time on it.

7 Product of generalised functions in $\mathbb{G}^\eta$.

Let us see now, how to use the theory developed in the previous paragraphs to define the product of generalised functions in $\mathbb{G}^\eta$.

Definition 3. *Given k homogeneous generalised functions $h_i \in \mathbb{G}^\eta$ with generating functions $f_i \in F_{\xi_i}$ and growing indexes q_i, we define the product h of the h_i, to be the limit of the product of the generating sequences $n^{q_i} f_i(nx)$:*

$$h = \lim_{n\to\infty} n^{q_1 \cdot \ldots \cdot q_k} f_1(nx) \cdot \ldots \cdot f_k(nx) \tag{47}$$

Note that, the product $f_1(x) \cdot \ldots \cdot f_k(x) \in F_{\xi_v}$ where, in general, $\xi_i \neq \xi_v$ for each i.

Since any generalised function h_i is well defined when the relevant generating function and growing index is given, then commutativity, associativity and applicability of the Leibniz rule in $\mathbb{G}^\eta$, for the product defined above, is ensured by the commutativity, associativity and applicability of the Leibniz rule for the relevant generating sequences.

We will show now, with a specific example, how to use the (47) to define a product of generalised functions which is independent from the chosen ξ_i. Suppose we want to evaluate the product h of the two generalised functions $h_1, h_2 \in \mathbb{G}^\eta$. We choose any $\xi_1 \in H^0$ and we find the relevant generating function $f_{\xi_1}, g_{\xi_1} \in F_{\xi_1}$ and the growing indexes q_1, q_2. We know also that $f_{\xi_1} \cdot g_{x_1} \in F_{\xi_v}$. We have:

$$h_{\xi_1} = \lim_{n\to\infty} n^{q_1+q_2} f_{\xi_1}(nx)\, g_{\xi_1}(nx) \tag{48}$$

Suppose now that we want to choose a different generating function of order 0 $\xi_2 \in H^0$ for which we find the generating functions $f_{\xi_2} \in F_{\xi_2}$ and $g_{\xi_2} \in F_{\xi_2}$ relevant to h_1 and h_2. We know also that $f_{\xi_2} \cdot g_{\xi_2} \in F_{\xi_w}$. We have:

$$h_{\xi_2} = \lim_{n\to\infty} n^{q_1+q_2} f_{\xi_2}(nx)\, g_{\xi_2}(nx) \tag{49}$$

given the (46) then we have:

$$\begin{aligned} h_{\xi_2} &= \lim_{n\to\infty} n^{q_1}\, \tau_{12}^{q_1}\left(f_{\xi_1}(nx)\right) n^{q_2}\, \tau_{12}^{q_2}\left(g_{\xi_1}(nx)\right) \\ &= n^{q_1+q_2}\, \tau_{vw}^{q_1+q_2}\left(f_{\xi_1}(nx)\, g_{\xi_1}(nx)\right) \end{aligned} \tag{50}$$

from which we see that h_{ξ_1} and h_{ξ_2} are the same generalised function in $\mathbb{G}^\eta$ and therefore the above product is well defined.

8 Equalities in D'

By using the above defined product, we can prove interesting equalities involving products among elements of D'. We will see an example in this paragraph.

Note that from now on, we will choose a specific main generating function $\xi \in H^0$, once and forever. We will perform all our calculations with generating functions in F_ξ and we will give all the final results in terms of $\eta_\xi^{p,f}$. Since the underling ξ is always the same, we will drop the ξ notation from the $\eta^{p,q}$ functions and a_p coefficients. When we write $\eta^{p,q}$, we really mean $\eta_\xi^{p,q}$ and when we write a_p, we really mean $a^p(\xi)$.

Before we proceed we need to see how to represent step discontinuous functions by using elements of $\mathbb{G}^{\eta}$. Let $\xi \in F$ be a main generating function for δ. We define the following function:

$$\chi(x) = \int_{-\infty}^{x} \xi(t)dt \tag{51}$$

to be a main generating function for $u(x)$, the Heaviside function, where we use a growing rate $q = 0$. Also if $f \in C^{\infty}$ is a function and χ is a main generating function for u, then we define $f(\chi(x)))$ to be a generating function for $f(u(x))$.

Of course $\chi(x)$ and $f(\chi(x)))$ are not in F. However we are interested in multiplication of a step discontinuous functions with elements of $\mathbb{G}^{\eta}$ and therefore in multiplying $\chi(x)$ and $f(\chi(x)))$ with elements of F so that we eventually get a generating function, for our product, which is in F.

Now, given a generalised function $f(u(x)))$, there are always $\beta, \gamma \in \mathbb{R}$ such that:

$$[f(\chi(x)) - \beta - \gamma\chi(x)] \in F \tag{52}$$

By applying the (22) to the (52), we can evaluate $f(u(x))$ in terms of elements of $\mathbb{G}^{\eta}$ as follows:

$$f(g(x)) = \beta + \gamma u(x) + \sum_{p=0}^{\infty} a_p \eta^{p,0} \tag{53}$$

For example:

$$u^2(x) = u(x) + \sum_{p=0}^{\infty} a_p \eta^{p,0} \tag{54}$$

$$sign^2(x) = (2u(x) - 1)^2 = 1 + \sum_{p=0}^{\infty} a_p \eta^{p,0} \tag{55}$$

By comparing the (55) with the (9), we can finally see why the functions $\delta_{[2x]}$ and $\delta_{[1]}$, present in that equation, are different and do not cancel each other.

Note that, in the following example we will use the notation introduced in (16) ($\eta^{p,q}$ expressed in the $\delta^{(p)}/n^k$ notation) and, since we do not have ξ in a closed form, the coefficients of the $\eta^{p,q}$ will be evaluated numerically.
We want to evaluate $u(x)\delta'(x)$:

$$u(x)\delta'(x) \to n^2\ \chi(nx)\xi'(nx) \tag{56}$$

From which we have:

$$u(x)\delta'(x) = a_0 n\delta(x) + \frac{1}{2}\,\delta'(x) + a_2\,\frac{\delta^{(2)}}{n} + R\left(\frac{\delta^{(4)}}{n^3}\right) \tag{57}$$

We want to remove the $n\delta$ term. To do that, we evaluate the product $\delta^2(x)$:

$$\delta^2(x) \to n^2\ \xi^2(nx) \tag{58}$$

From which we have:

$$\delta^2(x) = b_0 n\delta(x) + b_2\,\frac{\delta^{(2)}}{n} + b_4\,\frac{\delta^{(4)}}{n^3} + R\left(\frac{\delta^{(6)}}{n^5}\right) \tag{59}$$

Where b_3 and b_5 vanish (evaluated numerically, are smaller, in module, then 10^{-15}). For any ξ, $a_0 = -b_0$ (evaluated numerically, have opposite sign and are equal in module with an error smaller, then 10^{-14}). By substituting the value $n\delta$ from the (59) in the (57), we have eventually:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) + R\left(\frac{\delta^{(2)}}{n}\right) \tag{60}$$

or,(compare with [4]), as an equality among products of elements of D' (i.e. ignoring the higher order terms):

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) \tag{61}$$

We can get to the same results by using the Leibniz rule. We evaluate the product of $u(x)\delta(x)$. We have:

$$u(x)\delta(x) \to n\ \chi(nx)\xi(nx) \tag{62}$$

From which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) + R\left(\frac{\delta'}{n}\right) \tag{63}$$

by taking the derivatives of both sides we have:

$$\delta^2(x) + u(x)\delta'(x) = \frac{1}{2}\delta'(x) + R\left(\frac{\delta^{(2)}}{n}\right) \tag{64}$$

as expected. More examples can be found in the appendix.

9 Products with polynomials

From the forth implication of the (26) we know that given any $\xi_1 \in H^0$ we have:

$$\xi_2^{(p-1)}(x) = -\frac{x}{p}\xi_1^{(p)}(x) \in H^{p-1}\ \ with\, p > 0 \tag{65}$$

The above equality gives us an hint on how to extend to the concept of main generating functions and define main generating functions of negative orders.

To do that, we define a function $\xi^{[-p]} \in H^{[-p]} \subset F$ tu be a main generating function of order $-p$ if it is possible to find $f \in F$ such that $\xi^{[-p)]}x^p = f$ for each $x \in C - \{0\}$ (i.e. f goes to 0 in 0^+ and 0^- at least like x^p) and:

$$\int_{-\infty}^{+\infty} \xi^{[-p)]}(x)x^k dx = \left\{ \begin{array}{ll} 1 & for\, k = -p \\ 0 & for\, k > -p \end{array} \right. \tag{66}$$

In analogy with the definition of F_ξ (linear combinations of derivative of ξ), we define the set $F_{[\xi]}$ in the obvious way (linear combinations of derivatives of $\xi^{[-p]}$). Note that the notation $\xi^{[-p]}$ may be misleading since, although the derivative of $\xi^{[-p]}$ is a $\xi^{[-p+1]}$, the $\xi^{[-p]}$, both for p positive and negative, are always null functions. So, for example, the derivative of $\xi^{[-1]}$ is $\xi^{[0]}$ which is different from $\xi^{(0)}$ which, in turn, is the derivative of $\chi(x) \notin F$.

From the (66) we see that:

$$\xi_2^{[-1]}(x) = -x\xi_1^{(0)}(x) \in H^{[-1]} \tag{67}$$

and

$$\xi_2^{[p-1]}(x) = \frac{x}{p-1}\xi_1^{[p]}(x) \in H^{[p-1]} \ \ with\, p < 0 \tag{68}$$

We define the following generalised functions:

$$\eta^{[p],q} = \lim_{n\to\infty} n^q \xi^{[p]} \ with\, p, q \in \mathbb{Z} \tag{69}$$

Note that, due to the way $\mathbb{G}^\eta$ has been defined, it already contains the $\eta^{[p],q}$. Now, by using the definitions of $\eta^{p,q}$ and $\eta^{[p],q}$, it is easy to prove the following equalities:

$$\begin{array}{lll} \eta^{p-1,q-1} = -\frac{x}{p}\eta^{p,q} & p > 0 \\ \eta^{[-1],q-1} = -x\,\eta^{0,q} & p = 0 \\ \eta^{[p-1],q-1} = -\frac{x}{p}\eta^{[p],q} & p > 0 \\ \eta^{[p-1],q-1} = \frac{x}{p-1}\eta^{[p],q} & p < 0 \end{array} \tag{70}$$

We will briefly prove only the first equality of the (70). Multiplying a generalised function by x is equivalent to multiply its generating sequence by x before taking the limit. Given the generating sequence $n^q\, x\, \xi^{(p)}$ and by using the (21), we can evaluate the coefficients as:

$$\begin{aligned} a_k(n^q\, x\, \xi^{(p)}) &= \lim_{n\to\infty} \frac{(-1)^k}{k!} \int_{-\infty}^{+\infty} n\, x\, \xi^{(p)}(nx)(nx)^k dx \\ &= \lim_{n\to\infty} \frac{(-1)^k}{k!}\frac{1}{n} \int_{-\infty}^{+\infty} n\, \xi^{(p)}(nx)(nx)^{k+1} dx \\ &= a_{k-1}(n^q\, \xi^{(p-1)}) \\ &= -k\, a_k(n^{q-1}\, \xi^{(p-1)}) \end{aligned} \tag{71}$$

from which the first of the (70) follows. In particular:

$$\delta^{(p-1)} = -\frac{x}{p}\delta^{(p)} \ for\, p > 0 \tag{72}$$

which is a well known result in literature (compare with [4]).

Note that, $x\xi_1^{[0]} = -\xi_2^{[[-1]]}$ and the derivative of $\xi^{[[-1]]}$ is equal to $\xi^{[[0]]} \neq \xi^{[0]}$, with obvious meaning of the notation. By iterating the process we can define the $\xi^{[[\cdots p \cdots]]}$, all in F, and the $\eta^{[[\cdots p \cdots]],q}$, all in $\mathbb{G}^\eta$, with as many square brackets as we want. These are all new generalised functions present in $\mathbb{G}^\eta$ and which arise naturally from the theory we have developed.

We will use now the theory developed in this paragraph to discuss a well known example in the theory of product of distributions (compare with [2] §1.1 ex. i). If $vp\frac{1}{x}$ is the Cauchy principal value of $\frac{1}{x}$ then we have:

$$0 = (\delta(x)\cdot x)\cdot vp\frac{1}{x} = \delta(x)\cdot\left(x\cdot vp\frac{1}{x}\right) = \delta(x) \tag{73}$$

which is absurd.

By using our theory we know that $x\delta(x) = -\eta^{[-1],0} \neq 0$. We have:

$$0 = (x\cdot\delta(x) + \eta^{[-1],0})\cdot\frac{1}{x} = \delta(x) + \frac{1}{x}\eta^{[-1],0} = \delta(x) - \delta(x) \tag{74}$$

a results that makes us to feel much more comfortable.

Appendix

A.1 Examples of product of distributions

We will use the notation introduced in (16).
Example 1:

$$\delta(x)\delta'(x) \tag{75}$$

By taking twice the derivative of both sides of the (63), and rearranging the terms we get:

$$\delta(x)\delta'(x) = \frac{1}{6}\delta^{(2)}(x) - \frac{1}{3}u(x)\delta^{(2)}(x) + R\left(\frac{\delta^{(3)}}{n}\right) \tag{76}$$

Example 2: evaluated numerically (compare with paragraph 2 above)

$$sign^2(x)\delta(x) \to n\,(2\chi(nx) - 1)^2\xi(nx) \tag{77}$$

from which we have:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) + R\left(\frac{\delta^{(2)}}{n^2}\right) \tag{78}$$

A.2 Numerical evidences in support to the (46)

First of all, to perform our numerical analysis, we need to choose suitable ξ functions. Let $f(x)$ be the following Gaussian distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{79}$$

then we define $\xi_1(x)$ to be:

$$\xi_1(x) = f(x) - \frac{1}{2}f^{(2)}(x) + \frac{1}{8}f^{(4)}(x) - \frac{1}{48}f^{(6)}(x) \tag{80}$$

which is a very good approximation of a ξ function and it is derived from the Gaussian distribution by removing the first 3 higher order $\eta^{p,1}(x)$ components (compare with the (39) above). Also we define $\xi_2(x)$ to be:

$$\xi_1(x) = 0.3\,\xi_1(x) + 0.7\,\xi_2(2x) \tag{81}$$

By means of the (22) we evaluate numerically the coefficients a_p of the two products $\xi \cdot \xi$ and $\xi \cdot \xi'$, generating functions for δ^2 and $\delta \cdot \delta'$. We have:

	a_0	a_1	a_2
q-p-1	1	0	-1
$\xi_1 \cdot \xi_1$	0.747850786175440	0	0.064630940223461
$\xi_2 \cdot \xi_2$	1.164372758468304	0	0.025251755987242

Table 1 (δ^2)

	a_0	a_1	a_2	a_3
q-p-1	2	1	0	-1
$\xi_1 \cdot \xi_1'$	0	0.373925393087720	0	0.032315470111731
$\xi_2 \cdot \xi_2'$	0	0.582186379234155	0	0.012625877993621

Table 2 ($\delta \cdot \delta'$)

The coefficients in the table 1 and 2 above, have been evaluated by integrating the functions numerically in the interval [-10,10] on 5000 points.

Our argument in support of the (46) is that, the ratios between the a_p coefficients are the $\sigma_{vw}^{p,q}$ and, if the (46) is true, these ratios have to be independent from the specific $\xi^{(p)}$ functions that have been multiplied.

We know that the σ depends only from $q - p - 1$ and therefore, if the (46) is true then, for example, $\sigma_{vw}^{0,2}$ evaluated from table 1 will be equal to $\sigma_{vw}^{1,3}$ evaluated from table 2 although they refer to the product of different functions. Note that, for our analysis to be correct, we make the assumptions that ξ_v and ξ_w are the same sets in both multiplications of table 1 and 2. Although this has not been proven in the general case, in this case it is certainly true since, for example, $D\xi_1^2{=}2\xi_1 \cdot \xi_1'$ and therefore ξ_1^2 and $\xi_1 \cdot \xi_1'$ belong to the same space F_{ξ_v}.

We show in the following tables the results we have found in our analysis:

	$\sigma_{vw}^{0,2} = \sigma_{vw}^{1,3}$	$\sigma_{vw}^{2,2} = \sigma_{vw}^{3,3}$
table 1	$\frac{a_0(\xi_w)}{a_0(\xi_v)} = 1.556958660728280$	$\frac{a_2(\xi_w)}{a_2(\xi_v)} = 0.390706926124457$
table 2	$\frac{a_1(\xi_w)}{a_1(\xi_v)} = 1.556958660728288$	$\frac{a_3(\xi_w)}{a_3(\xi_v)} = 0.390706926124451$
difference	$7.9 \cdot 10^{-15}$	$6.0 \cdot 10^{-15}$

Table 3

We conclude that the numerical evidences suggest the (46) is true.

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