DYNAMICAL SYSTEMS DETERMINABLE BY DISCRETE SAMPLES

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In this paper we shall define and study an important class of dynamic systems which allow to effectively determine their mathematical model exclusively on experimental basis. The usefulness of these results of mathematical nature, obtained by extending the Whittaker-Shannon sampling theory, will be highlighted through an applied example from the field of optoelectronics.

Keywords: Reconstructible dynamical systems, Whittaker-Shannon sampling theory.

1. INTRODUCTION

The concept of dynamic system allows the mathematical modelling of a very large range of phenomena occurring in nature, from which those considered as belonging to natural sciences are studied within physics, biology and chemistry, where as those belonging to humanities, are studied within social sciences or economics, etc. Thus, it is no need in insisting upon the importance of this concept. However, for those who wish to initiate themselves or who want to bolster knowledge in this direction we recommend the monograph [2] as an initial study. Those who want to enlighten themselves on the potential the notion of dynamic system offers when mathematically modeling phenomena from nature, we recommend, for example, the papers [4, 5, 6, 8, 9], where they can find some interesting applications of the concept of dynamic system, others than the now classical ones. Also, for those interested in news of theoretical nature from the field of dynamic systems we recommend paper [3, 7].

In this paper we shall study the general case of the finite dimensional dynamical systems of input-output type, intuitively defined by the diagram represented in fig.1.

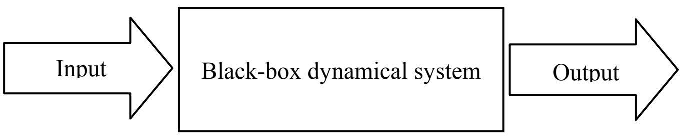

Figure 1. Conceptual model of an abstract input-output dynamical system.

If by $x_1,..,x_n$ we denote the inputs, and by $y_1,..,y_m$ the outputs, then such a system can be mathematically abstracted through a set of m equations

$$\Sigma : \begin{cases} y_1(s_1) = f_1(x_1,..,x_n,s_1) \\ \text{.......................................} \\ y_m(s_m) = f_m(x_1,..,x_n,s_m) \end{cases}, \tag{1}$$

where the set of parameters $\{s_1,..,s_m\}$ expresses the possible states of the system or the operating mode in which it finds itself when receiving the impulses $x_1,..,x_n$. Each parameter s_k, $k = 1,2,..,m$, can be in part a scalar, a vector or a function, depending on the nature of the system that characterizes it.

Remarks :1) It is very important to keep in mind that the problem we would like to approach in this paper has in view the general and very abstract case of dynamical systems used for modeling processes and phenomena occurring in nature, that is especially, those cases in which the functions $f_1 = f_1(x_1,..,x_n,s_1)$, …, $f_m = f_m(x_1,..,x_n,s_m)$ are not explicitly known. For example, the monetary policies redound upon the economical systems through the effects they produce, but this fact does not allow us to know the exact internal mechanisms they trigger (formally expressed by an array of functions $f_1,.., f_m$), although the existence of these mechanisms is assumed.

2. THE COMPATIBILITY CONDITIONS OF THE PROBLEM

If the functions $f_1 = f_1(x_1,..,x_n,s_1)$,.., $f_m = f_m(x_1,..,x_n, s_m)$, have a certain form, for instance they are polynomials, or bandlimited, that is, functions that admit the Fourier transform and whose transforms have a compact support, then the problem of their determination by virtue of samples resulted from practical experiments is possible. Indeed, when we know a priori that the unknown functions $f_1 = f_1(x_1,..,x_n,s_1)$,.., $f_m = f_m(x_1,..,x_n,s_m)$ are polynomials, their determination from the corresponding sample representations can be easily done with the help of the interpolating polynomials. Hereinafter, we shall focus on solving this problem for the case of bandlimited functions because the systems belonging to this class have numerous applications in optoelectronics.

3. THE RESOLUTION OF THE PROBLEM FOR THE CASE OF BANDLIMITED FUNCTIONS

Although the functions $f_1 = f_1(x_1,..,x_n,s_1)$,.., $f_m = f_m(x_1,..,x_n,s_m)$, are not known a priori, we can still determine in an experimental way the next n-rectangular lattice of samples

$$f_k^{\text{sample}}(x_1,..,x_n, s_k) = \operatorname{comb}\left(\frac{x_1}{h_1}\right)\cdots\operatorname{comb}\left(\frac{x_n}{h_n}\right) f_k(x_1,..,x_n, s_k),\ \ k = 1,2,..,m, \tag{2}$$

where $\operatorname{comb}(x) = \sum_{k=-\infty}^{\infty} \delta(x-k)$, δ being the so-called Dirac function. Indeed, let us notice first that the relations (2) can be brought to the equivalent form

$$f_k^{\text{sample}}(x_1,..,x_n, s_k) = \left(\sum_{j_1=-\infty}^{\infty} \delta\left(\frac{x_1}{h_1} - j_1\right)\right)\cdots\left(\sum_{j_n=-\infty}^{\infty} \delta\left(\frac{x_n}{h_n} - j_n\right)\right) f_k(x_1,..,x_n, s_k),\ \ k = 1,2,..,m, \tag{3}$$

and then, by choosing on the $0x_1$ axis the step h_1, on the $0x_2$ axis the step h_2, …, and on the $0x_n$ axis the step h_n, the values of the functions $f_1 = f_1(x_1,..,x_n,s_1)$,.., $f_m = f_m(x_1,..,x_n,s_m)$ in the points $x_1 = h_1 j_1,\ j_1 \in \mathbb{Z}$, $x_2 = h_2 j_2,\ j_2 \in \mathbb{Z}, \dots,\ x_n = h_n j_n,\ j_n \in \mathbb{Z}$ can be experimentally determined even if the analytical expressions of these functions are not known. One amendment must be made on this occasion, that of considering all the functions $f_1 = f_1(x_1,..,x_n,s_1)$,.., $f_m = f_m(x_1,..,x_n,s_m)$ defined on $\mathbb{R}^n$. This requisite is however easy to fulfill under the hypothesis we work, because we can consider that any of these functions is equal to zero in the points in which these functions are not defined.

By carrying out the notations

$$F_k(\omega_1,..,\omega_n, s_k) = \mathcal{F}\left\{f_k(x_1,..,x_n, s_k)\right\},$$

$$F_k^{\text{sample}}(\omega_1,..,\omega_n,s_k) = \mathcal{F}\left\{f_k^{\text{sample}}(x_1,..,x_n, s_k)\right\},\ k = 1,2,..,m,$$

where $\mathcal{F}$ represents the Fourier transform, and by allowing for the relations

$$F_k^{\text{sample}}(\omega_1,..,\omega_n, s_k) = \mathcal{F}\left\{\text{comb}\left(\frac{x_1}{h_1}\right)\cdots\text{comb}\left(\frac{x_n}{h_n}\right)\right\} * F_k(\omega_1,..,\omega_n, s_k),\ k = 1,2,..,m\ ,$$

$$\mathcal{F}\left\{\text{comb}\left(\frac{x_1}{h_1}\right)\cdots\text{comb}\left(\frac{x_n}{h_n}\right)\right\} = h_1\cdots h_n\cdot\text{comb}(h_1\omega_1)\cdots\text{comb}(h_n\omega_n) =$$

$$= \sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty}\delta\left(\omega_1 - \frac{j_1}{h_1},..,\omega_n - \frac{j_n}{h_n}\right),$$

we get

$$F_k^{\text{sample}}(\omega_1,..,\omega_n, s_k) = \sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty}F_k\left(\omega_1 - \frac{j_1}{h_1},..,\omega_n - \frac{j_n}{h_n}\right),\ k = 1,2,..,m. \tag{4}$$

Since the function f_k is bandlimited by hypothesis, its spectrum F_k is non-zero only on a certain finite region R_k (F_k has the support R_k bounded) of the space of frequencies $0\omega_1,..,\omega_n$.

From (4) we deduce that the support of the sampled function F_k^{sample} is

$$\bigcup_{j_1=-\infty}^{\infty}\cdots\bigcup_{j_n=-\infty}^{\infty}T_{\left(\frac{j_1}{h_1},\frac{j_2}{h_2},..,\frac{j_n}{h_n}\right)}(R_k),$$

where $T_{\left(\frac{j_1}{h_1},\frac{j_2}{h_2},..,\frac{j_n}{h_n}\right)}$ represents the vector translation $\left(\frac{j_1}{h_1},\frac{j_2}{h_2},..,\frac{j_n}{h_n}\right)$ in the space of the frequencies. The smaller the parameters $h_1,..,h_n$ (the steps of the divisions) will be (that is when the samples are sufficiently close together) the larger the distance between the different spectral sub-regions $T_{\left(\frac{j_1}{h_1},\frac{j_2}{h_2},..,\frac{j_n}{h_n}\right)}(R_k)$ will be, so that, we can suppose that, by conveniently choosing these parameters, the regions $T_{\left(\frac{j_1}{h_1},\frac{j_2}{h_2},..,\frac{j_n}{h_n}\right)}(R_k),\ j_1,..,j_n\in\mathbb{Z}$, will be disjoint two by two. Due to this, the exact recovery of the original spectrum F_k from F_k^{sample} can be accomplished by passing this sampled function through a linear filter that transmits the term corresponding to $j_1=\cdots=j_n=0$ from (4) without distortion, while perfectly excluding all other terms. Thus, at the output of this filter we find an exact replica of the original spectrum data F_k and consequently of the original data f_k.

In order to determine the necessary spacing between the samples, let $2b_{1,k},..,2b_{n,k}$ be the dimensions, in the direction of the axes $0\omega_1,..,0\omega_n$, of the smallest n-parallelepiped, which includes the region R_k. As the different sub-regions $T_{\left(\frac{j_1}{h_1},\frac{j_2}{h_2},..,\frac{j_n}{h_n}\right)}(R_k)$ are situated at the distances $\frac{1}{h_1}$ in the direction of the $0\omega_1$ axis, $\frac{1}{h_2}$ in the direction of the $0\omega_2$ axis,… and $\frac{1}{h_n}$ in the direction of the $0\omega_n$ axis, respectively, one from another, the separation between these spectral regions is ensured if

$$h_1\le\frac{1}{2b_{1,k}},..,\ h_n\le\frac{1}{2b_{n,k}}.$$

At this point the only thing left is to indicate the transfer function (the filter) through which the data of the sample used (F_k^{sample}) must pass through. The filter searched has the expression

$$\Phi_k(\omega_1,..,\omega_n)=\text{rect}\left(\frac{\omega_1}{2b_{1,k}}\right)\cdots\text{rect}\left(\frac{\omega_n}{2b_{n,k}}\right).$$

Consequently, in order to rebuild F_k from F_k^{sample} we only have to apply the filter H_k to F_k^{sample}, that is, we have the relation

$$F_k(\omega_1,..,\omega_n,s_k)=F_k^{\text{sample}}(\omega_1,..,\omega_n,s_k)\,\text{rect}\left(\frac{\omega_1}{2b_{1,k}}\right)\cdots\text{rect}\left(\frac{\omega_n}{2b_{n,k}}\right).$$

By applying to this relation the inverse Fourier transform $\mathcal{F}^{-1}$ we find that

$$f_k(x_1,..,x_n,s_k)=\left[\text{comb}\left(\frac{x_1}{h_1}\right)\cdots\text{comb}\left(\frac{x_n}{h_n}\right)f_k(x_1,..,x_n,s_k)\right]*\phi_k(x_1,..,x_n),\ k=1,2,..,m, \tag{5}$$

where

$$\phi_k(x_1,..,x_n)=\mathcal{F}^{-1}\left\{\Phi(\omega_1,..,\omega_n)\right\}=2^n b_{1,k}\cdots b_{n,k}\text{sinc}(2b_{1,k}x_1)\cdots\text{sinc}(2b_{n,k}x_n)$$

As

$$\operatorname{comb}\left(\frac{x_1}{h_1}\right)\cdots\operatorname{comb}\left(\frac{x_n}{h_n}\right)f_k\left(x_1,..,x_n,\,s_k\right)=$$

$$=h_1\cdots h_n\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty}f_k\left(j_1h_1,..,j_nh_n,s_k\right)\delta\left(x_1-j_1h_1,..,\,x_n-j_nh_n\right),$$

the relation (5) becomes

$$f_k\left(x_1,..,x_n,\,s_k\right)=$$

$$=2^n b_{1,k}\cdots b_{n,k}\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty}f_k\left(j_1h_1,..,j_nh_n,s_k\right)\operatorname{sinc}\left[2b_{1,k}\left(x_1-j_1h_1\right)\right]\cdots\operatorname{sinc}\left[2b_{n,k}\left(x_n-j_nh_n\right)\right].$$

Considering $h_1=\frac{1}{2b_{1,k}},..,\,h_n=\frac{1}{2b_{n,k}}$, the last relation can be written as

$$f_k\left(x_1,..,x_n,\,s_k\right)=$$

$$=\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty}f_k\left(\frac{j_1}{2b_{1,k}},..,\frac{j_n}{2b_{n,k}},\,s_k\right)\operatorname{sinc}\left[2b_{1,k}\left(x_1-\frac{j_1}{2b_1}\right)\right]\cdots\operatorname{sinc}\left[2b_{n,k}\left(x_n-\frac{j_n}{2b_{n,k}}\right)\right],$$

or

$$f_k\left(x_1,..,x_n,s_k\right)=\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty}r_k^{j_1\cdots j_n}\operatorname{sinc}\left[2b_{1,k}\left(x_1-\frac{j_1}{2b_{1,k}}\right)\right]\cdots\operatorname{sinc}\left[2b_{n,k}\left(x_n-\frac{j_n}{2b_{n,k}}\right)\right],$$

where

$$r_k^{j_1\cdots j_n}=f_k\left(\frac{j_1}{2b_{1,k}},..,\frac{j_n}{2b_{n,k}},\,s_k\right),\;k=1,\,2,..,m\,.$$

4. AN IMPORTANT SUBCLASS OF BANDLIMITED FUNCTIONS

Let $s=s\left(x_1,..,x_n\right)$ and $t=t\left(x_1,..,x_n\right)$ be two absolutely integrable functions on $\mathbb{R}^n$, from which at least one of them is bandlimited. Then the function $f=s*t$, where

$$\left(s*t\right)\left(x_1,..,x_n\right)=\int\limits_{\mathbb{R}^n}s\left(\xi_1,..,\xi_n\right)t\left(x_1-\xi_1,..,x_n-\xi_n\right)d\xi_1\cdots d\xi_n\,,$$

represents the convolution product of functions s and t, is also a bandlimited function because

$$\mathsf{F}\left(f\right)=\mathsf{F}\left(s\right)\mathsf{F}\left(t\right),$$

where, just as earlier, $\mathcal{F}$ symbolizes the *n*-dimensional Fourier transform, and from the transforms $\mathcal{F}(s)$, or $\mathcal{F}(t)$ at least one has a compact support. This property allows us to put forward the subclass of dynamical systems of the form (1) for which the functions $f_1,..,f_m$ have the particular form

$$f_k(x_1,..,x_n,s_k) = (s_k * t_k)(x_1,..,x_n),\ k = 1,..,m,$$

where each pair of functions s_k and t_k, $k = 1,..,m$, verify properties similar to the functions *s* and *t* whereof we have discussed earlier.

Remark: The dynamical systems of this subclass are important due to their applications in different scientific areas.

The proof of this statement will be discussed hereinafter.

5.ONE ELOQUENT EXAMPLE OF THE USE OF THE MATHEMATICAL APPARATUS DEVELOPPED IN THIS PAPER

In order to illustrate the usefulness of the mathematical apparatus developed in this paper we shall study hereinafter a problem from the field of optoelectronics, underlining the inherent fact that the area of applicability of the presented theorem does not limit itself only to the field mentioned in the chosen example.

In [1] the mathematical modeling of the diffraction phenomenon at boundaries is done by relating a physical system composed of a monochromatic light source *S*, an opaque screen (*OpS*) endued with an aperture (*A*) through which the light can pass and an observation screen (*ObS)* on which the light signal emitted by the source *S* will be analyzed to a three orthogonal axes frame. If $0xyz$ is the frame chosen for the representation of this system, then in order to make a choice, we shall place the screen *OpS* in the plane $z = 0$ so that the center of the aperture *A* coincides with the center of the axes of coordinates. The light source is considered to be placed behind this screen (somewhere on the negative semi-axis of the 0z axis) and the observation screen *ObS* is placed in the plane $z = z_0$, $z_0 > 0$. See Figure 2.

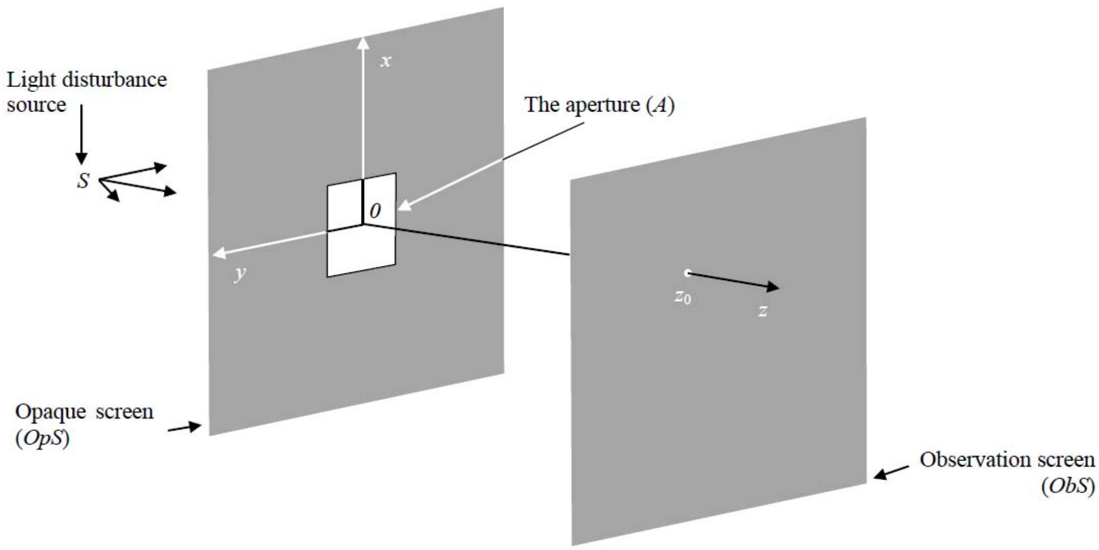

Figure 2. The coordinate system used for modeling the diffraction at boundaries.

Knowing that the complex field across the aperture *A* is represented by $\mathsf{U}(x,y,0)$, we want to determine the value of this field in any point $(x,\ y,\ z_0)$ of the observation screen *OpS*.

In the theory of diffraction at boundaries developed in [1], the value of the field U in one point (x,y,z_0) of the plane $z=z_0$ is described as the response of a spatial dynamical system of the form

$$\mathsf{U}(x,y,z_0)=\iint_{\mathbb{R}^2}\mathsf{U}(\xi,\eta,0)h(x-\xi,y-\eta,z_0)\,d\xi d\eta\,. \tag{6}$$

Within this expression the function $\mathsf{U}(\cdot,\cdot,0)$ describes the state of the system subjected to the experiment (the boundary conditions imposed to the system), the coordinates x,y,z_0, $z_0>0$, of the point in which we want to evaluate the value of the field U, represent the inputs of the system, $h(\cdot,\cdot,\cdot)$ represents the transfer function of the system, and the value $\mathsf{U}(x,y,z_0)$ represents the response (the output) of the system to the imposed boundary conditions (i.e. the inputs).

Remark: In order to facilitate the connection between the notions presented in this paragraph and the theory enunciated in [1], that we now refer to, in this paragraph we have changed the notation adopted earlier in favor of the one from [1].

By taking the results obtained in [1], under the hypothesis that the distance z_0 between the planes having the equations $z=0$ and $z=z_0$ is at least several wavelengths longer (than the wavelength of signal emitted by the source *S*), so that the evanescent waves, which occur in the immediate neighborhood of the slit *A*, may be neglected, the determination of the transfer function *h* is done by the relation below

$$h(x,y,z_0)=\int_{\mathbb{R}^2}H\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z_0\right)e^{2\pi i\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)}d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda},$$

where the function *H* is described by the relation

$$H\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda},z\right)=\begin{cases}e^{2\pi i\frac{z}{\lambda}\sqrt{1-\alpha^2-\beta^2}} & ,\ \alpha^2+\beta^2<\dfrac{1}{\lambda^2}\\ 0, & \text{otherwise}\end{cases}, \tag{7}$$

from which the parameters $\alpha,\ \beta,\ \gamma=\sqrt{1-\alpha^2-\beta^2}$, $\alpha^2+\beta^2\le 1$, symbolize the directions cosines of an hypothetical direction of propagation of the light signal, and λ is the wavelength of this signal.

Due to the relation (7) the transfer function *h* is bandlimited. Hence, the optical system (6) integrates itself in the class of systems defined in the previous paragraph. From this remark, we conclude that phenomena of diffraction can be entirely reconstructed by virtue of some discrete samples which can be determined experimentally. This result has a tremendous importance in practice, because it also allows modeling the diffraction phenomena in the case of diffraction gratings, for which the theoretical apparatus is unknown or whose determination is very laborious, provided that the grating are bounded (i.e. there exists a strictly positive number *R* so that the entire network can be included in the disc of radius *R*). This latter condition is imposed by the requirement that the transfer functions of the corresponding dynamic systems to be bandlimited.

REFERENCES

1. GOODMAN, J.,W., *Introduction to Fourier Optics*, McGRAW-HILL, Inc, 1968.
2. KALMAN, R. E., FALB, P. L., ARBIB, M. A., *Topics in mathematical system theory*, McGraw-Hill, New York, 1969.
3. ŞANDRU, O. I., VLADAREANU, L., ŞANDRU, A. , *A new method of aproaching the problems of optimal control*, Proceedings of 10 th WSEAS Int. Conf. on MATHEMATICAL and COMPUTATIONAL METHODS in SCIENCE and ENGINEERING (MACMESE'08), Bucharest, Romania, November 7-9, 2008, Recent Advances in Mathematical and Computational Methods in Science and Engineering, Part II, 390-393, ISSN: 1790-2769.
4. ŞANDRU, O. I., ŞANDRU, I. M. D., "*Regarding marketing problems as dynamic system theory problems*", Proceedings of the International Conference on MANAGEMENT, MARKETING and FINANCES (MMF '09), WSEAS International Conferences, University of Houston, USA, April 30-May 2, 2009, 183 – 187, ISSN: 1790-2769, ISBN: 978-960-474-073-4.
5. ŞANDRU, O. I., ŞANDRU, I. M. D., "*Managing competition between universities by increasing the quality of the educational process*", Annals of DAAAM for 2009 & Proceedings of 20th DAAAM International Symposium, 1555 – 1556, ISBN 978-3-901509-70-4, ISSN 1726-9679.
6. ŞANDRU, O. I., ŞANDRU, I. M. D., "*Regarding education quality management problems as dynamic system theory problems*", Annals of DAAAM for 2009 & Proceedings of 20th DAAAM International Symposium, 1557 - 1558, ISBN 978-3-901509-70-4, ISSN 1726-9679.
7. ŞANDRU, O. I., VLADAREANU, L., ŞANDRU, A. , "Invertible dynamic systems", Proceedings of the European Computing Conference, ECC'11, WSEAS International Conferences Paris, France, April 28-30, 2011, ISBN: 978-960-474-297-4, 317-320.
8. VLADAREANU, L., ION, I., ŞANDRU, O. I., VELEA, L. M., and MUNTEANU, M. S., *The Actuators Control by Probabilistic Mathematical Modelling*, WSEAS TRANSACTIONS on SYSTEMS and CONTROL, Issue 6, Volume 3, June 2008, 547 - 557, ISSN: 1991-8763.
9. VLADAREANU, L., ŞANDRU, O. I., L. VELEA, L. M., YU, H., *Actuator control in continuous flux using Winer filters*, PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A, Volume10, Number 1/2009, ISSN : 1454-9069.