

Introduction to Real Clifford Algebras: from $Cl(8)$ to E_8 to Hyperfinite II_1 factors

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Real Clifford Algebras roughly represent the Geometry of Real Vector Spaces of signature (p,q) with the Euclidean Space $(0,q)$ sometimes just being written (q) so that the Clifford algebra $Cl(0,q) = Cl(q)$.

A useful starting place for understanding how they work is to look at the most central example and then extend from it to others.

This paper is only a rough introductory description to develop intuition and is NOT detailed or rigorous - for that see the references.

Real Clifford Algebras have a tensor product periodicity property whereby

$$Cl(q+8) = Cl(q) \times Cl(8)$$

so that if you understand $Cl(8)$ you can understand larger Clifford Algebras such as $Cl(16) = Cl(8) \times Cl(8)$ and so on for as large as you want.

So $Cl(8)$ is taken to be the central example in this paper which has 4 parts:

How $Cl(8)$ works - page 2

What smaller Clifford Algebras inside $Cl(8)$ look like - page 7

How the larger Clifford Algebra $Cl(16)$ gives E_8 - page 9

How larger Clifford Algebras $Cl(8N)$ give in the large N limit a generalized Hyperfinite II_1 von Neumann factor - page 14

References:

Lectures on Clifford (Geometric) Algebras and Applications
Rafal Ablamowicz, Garret Sobczyk (eds) (Birkhauser 2003)
especially lectures by Lounesto and Porteous

Clifford Algebras and Spinors
Pertti Lounesto (Cambridge 2001)

Clifford Algebras and the Classical Groups
Ian R. Porteous (Cambridge 2009)

My Introduction to E_8 Physics at viXra:1108.0027

How Cl(8) works

Cl(8) is a graded algebra with grade k corresponding to dimensionality of vectors from the origin to subspaces of 8-dim space spanned by k basis vectors of 8 orthogonal basis vectors $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ of the 8-dim Euclidean space. In the following construction use only the positive basis vectors (not their mirror image negatives). That is the same as looking at only the all-positive octant of the 8 -dim Euclidean space.

Grade:

0 - vectors from origin to itself - 0-dimensional - 1 point

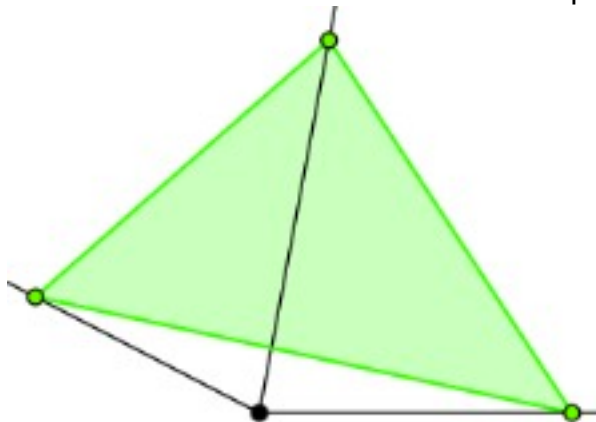
1 - vectors from origin to 1 of the 8 basis vectors - 1-dim - 8 line segments
each of the 8 line segments is a 1-dim simplex
whose outer "face" is a 0-dimensional point.

These 8 basis vectors are the basis vectors of the 8-dim vector space of Cl(8).

2 - vectors from origin to pairs of the 8 basis vectors - 2-dim - 28 triangles defined by pairs of vectors
each of the 28 triangles is a 2-dim simplex (but NOT equilateral)
whose outer "face" is a 1-dimensional line segment.

These 28 bivectors (pairs of vectors) give the 28 planes of rotation of the 28-dim group Spin(8) that includes rotations of the 8-dim vector space of Cl(8).

3 - vectors from origin to triples of the 8 basis vectors - 3-dim - 56 tetrahedra defined by triples of vectors
each of the 56 tetrahedra is a 3-dim simplex (but NOT equilateral)



(image from wikipedia)

whose outer "face" is a 2-dimensional triangle (that IS equilateral).

4 - vectors from origin to 4-tuples of the 8 basis vectors - 4-dim - 70 4-simplexes defined by 4-tuples of vectors

each of the 70 4-simplexes is a 4-dim simplex (but NOT equilateral)

whose outer "face" is a 3-dimensional tetrahedron (that IS equilateral).

5 - vectors from origin to 5-tuples of the 8 basis vectors - 5-dim - 56 5-simplexes defined by 5-tuples of vectors

each of the 56 5-simplexes is a 5-dim simplex (but NOT equilateral)

whose outer "face" is a 4-dim simplex (that IS equilateral).

6 - vectors from origin to 6-tuples of the 8 basis vectors - 6-dim - 28 6-simplexes defined by 6-tuples of vectors

each of the 28 6-simplexes is a 6-dim simplex (but NOT equilateral)

whose outer "face" is a 5-dim simplex (that IS equilateral).

7 - vectors from origin to 7-tuples of the 8 basis vectors - 7-dim - 8 7-simplexes defined by 7-tuples of vectors

each of the 8 7-simplexes is a 7-dim simplex (but NOT equilateral)

whose outer "face" is a 6-dim simplex (that IS equilateral).

8 - vectors from origin to 8-tuples of the 8 basis vectors - 8-dim - 1 8-simplex defined by the unique 8-tuple of vectors

the 8-simplex is an 8-dim simplex (but NOT equilateral)

whose outer "face" is a 7-dim simplex (that IS equilateral).

The total dimension of Cl(8) is

$$1+8+28+56+70+56+28+8+1 = 256 = 2^8 = 16 \times 16$$

The Cl(8) algebra is the algebra of 16x16 matrices of real numbers.

The product of the Cl(8) algebra is the product of 16x16 real matrices but it also has geometric meaning.

If you multiply for example a

grade-2 element = 2-dim simplex with basis vectors { x_3, x_5 }

by a

grade-4 element = 4-dim simplex with basis vectors { x_2, x_6, x_7, x_8 }

then

you get a

grade-6 element = 6-dim simplex with basis vectors { $x_2, x_3, x_5, x_6, x_7, x_8$ }

BECAUSE

the Clifford Algebra product in this case acts like the exterior algebra wedge product (or the cross-product in 3-dim)

so that the product of two independent subspaces of the Cl(8) 8-dim Euclidean space is sort of the span of both subspaces taken together

BUT

If you multiply for example a

grade-2 element = 2-dim simplex with basis vectors { x_2, x_5 }

by

a grade-4 element = 4-dim simplex with basis vectors { x_2, x_6, x_7, x_8 }

then

you get

a grade-4 element = 4-dim simplex with basis vectors { x_5, x_6, x_7, x_8 }

BECAUSE

the Clifford Algebra in this case acts partly like a dot-product

so that in the product of two defined-by-the-same-vector subspaces the two subspaces cancel out to zero (the common basis vector x_2 is cancelled).

In short, Clifford Algebra describes the geometry of vector subspaces and

the geometry is exactly described by matrix algebras like

Cl(8) = 16x16 real matrices R(16).

The 16x16 real matrices $R(16)$ are made up of 16 column vectors each of which is a 16-dim vector that decomposes into two 8-dim vectors.

Since 16 times an $8+8 = 16$ -dim column vector gives all $16 \times 16 = 256$ elements of $R(16) = Cl(8)$ it is useful to regard the 16-dim column vectors as fundamental square-root-type constituents of $Cl(8)$ and to call them $Cl(8)$ spinors.

Since the 16-dim $Cl(8)$ spinors decompose into two 8-dim parts, call them 8-dim +half-spinors and 8-dim -half-spinors and denote them by $\mathbf{8+s}$ and $\mathbf{8-s}$.

In the case of $Cl(8)$ the grade-1 vectors are also 8-dim, denoted by $\mathbf{8v}$, so for $Cl(8)$ we have a **Triality Automorphism $\mathbf{8+s} = \mathbf{8-s} = \mathbf{8v}$** that turns out to be very useful in physics because it gives a relation between +half-spinors, -half-spinors, and vectors.

The equality between +half-spinors and -half-spinors gives a symmetry between fermion particles and antiparticles.

Since gauge bosons are grade-2 bivectors of which $Cl(8)$ has 28, the gauge boson Lagrangian dimension in 8-dim spacetime is 28.

Since in 8-dim spacetime fermions have Lagrangian dimension $7/2$ the full fermion term of the Lagrangian also has dimension 28.

Therefore, in the high-energy 8-dim spacetime Lagrangian the boson and fermion terms cancel due to the Triality Supersymmetry of boson Lagrangian dimension = 28 = fermion Lagrangian dimension.

Once you understand the $Cl(8)$ example you can extend the model to $Cl(N)$ for any N and also extend it to spaces with any signature (p,q) for $p+q = N$ where p is the number of dimensions of negative signature and q is the number of dimensions of positive signature in the vector space over which the Clifford Algebra $Cl(p,q)$ is defined.

Clifford Algebras for Euclidean spaces $Cl(0,q)$ are also denoted $Cl(q)$, such as **$Cl(8) = Cl(0,8) = R(16)$ and $Cl(16) = Cl(0,16) = R(256)$**

For some (p,q) the Clifford Algebras are matrix algebras over complex C or quaternionic H as for example

$$Cl(4) = Cl(1,3) = H(2)$$

$$Cl(2) = H$$

$$Cl(1) = C$$

What smaller Clifford Algebras inside $Cl(8)$ look like

Here is a table of all Clifford Algebras $Cl(p,q)$ smaller than $Cl(8) = Cl(0,8) = R(16)$

	$q \rightarrow$							
p	R	C	H	²H	H(2)	C(4)	R(8)	²R(8)
\downarrow	²R	R(2)	C(2)	H(2)	²H(2)	H(4)	C(8)	R(16)
	R(2)	²R(2)	R(4)	C(4)	H(4)	²H(4)	H(8)	C(16)
	C(2)	R(4)	²R(4)	R(8)	C(8)	H(8)	²H(8)	H(16)
	H(2)	C(4)	R(8)	²R(8)	R(16)	C(16)	H(16)	²H(16)
	²H(2)	H(4)	C(8)	R(16)	²R(16)	R(32)	C(32)	H(32)
	H(4)	²H(4)	H(8)	C(16)	R(32)	²R(32)	R(64)	C(64)
	C(8)	H(8)	²H(8)	H(16)	C(32)	R(64)	²R(64)	R(128)

from Ian Porteous's book "Clifford Algebras and the Classical Groups" (Cambridge 1995, 2009).

Some of the smaller Clifford Algebras are particularly useful in physics.

Here is how some of them fit inside $Cl(8) = Cl(0,8)$:

$$Cl(8) = Cl(1,7) = R(16) = H(2) \times H(2) = Cl(1,3) \times Cl(1,3)$$

Spin(8) Triality Group and F4

$$Cl(2,6) = Cl(3,5) = H(8)$$

$$Cl(6) = R(8) = H \times H(2)$$

$$Cl(2,4) = Cl(1,5) = H(4)$$

Conformal SU(2,2) Group

$$Cl(4) = Cl(1,3) = H(2)$$

Minkowski Lorentz Group

$$Cl(3) = R(4) = H + H$$

$$Cl(2) = H$$

$$Cl(1) = C$$

$$Cl(0) = R$$

How the larger Clifford Algebra Cl(16) gives E8

By Periodicity the tensor product $Cl(8) \times Cl(8) = Cl(16)$ with graded structure

				1
				16
				120
				560
				1820
				4368
				8008
				11440
				12870
1	1			12870
8	8			11440
28	28			8008
56	56			4368
70	70	x	=	1820
56	56			560
28	28			120
8	8			16
1	1			1
Cl(8)	x	Cl(8)	=	Cl(16)

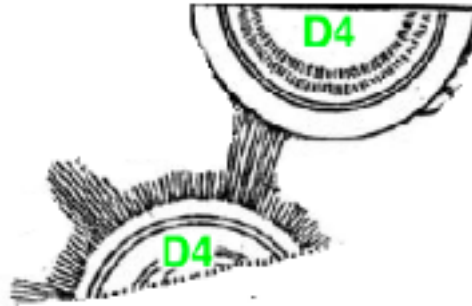
The 16-dim vector space of $Cl(16)$ comes from $Cl(8) \times Cl(8)$
 as grade $(0+1) = (1+0) = 1$
 and dimension $1 \times 8 + 8 \times 1 = 16$

The 120-dim bivector space of $Cl(16)$ comes from $Cl(8) \times Cl(8)$
 as grade $(0+2) = (2+0) = (1+1) = 2$
 and dimension $1 \times 28 + 28 \times 1 + 8 \times 8 = 28 + 28 + 64$

The 28 bivectors in each of the $Cl(8)$ generate the D_4 Lie Algebra $Spin(8)$.
 The 120 bivectors in $Cl(16)$ generate the D_8 Lie Algebra $Spin(16)$.

Therefore 120-dim D8 contains:

two copies of 28-dim D4

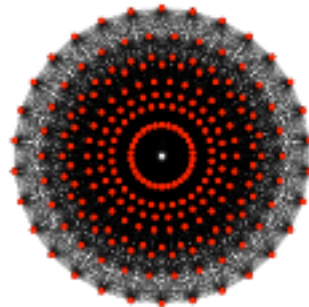


plus

a 64-dim structure that is the product of two 8-dim Cl(8) vector spaces each of which is half of the 16-dim D8 vector space so that effectively the 64-dim structure is the square of the rank 8 of D8 which is also



the rank of 248-dim E8 whose 240 Root Vectors can be seen as 8 concentric circles of 30 Root Vectors (Wikipedia image)



such that there exists the symmetric space

$$D8 / D4 \times D4 = 8 \times 8 = 64\text{-dim rank } 8 \text{ Grassmannian}$$

and

$$248\text{-dim E8} = 120\text{-dim D8} + 128\text{-dim D8} + \text{half-spinor} =$$

$$= D4 \times D4 + 8 \times 8 + 128\text{-dim D8} + \text{half-spinor D8+s}$$

The spinors of $Cl(16) = Cl(8) \times Cl(8)$ come from the spinors of $Cl(8)$:

$$\begin{aligned}
 & (\mathbf{8+s} + \mathbf{8-s}) \times (\mathbf{8+s} + \mathbf{8-s}) = \\
 & = (\mathbf{8+s} \times \mathbf{8+s} + \mathbf{8+s} \times \mathbf{8-s} + \mathbf{8-s} \times \mathbf{8+s} + \mathbf{8-s} \times \mathbf{8-s}) = \\
 & = (\mathbf{8+s} \times \mathbf{8+s} + \mathbf{8-s} \times \mathbf{8-s}) + (+ \mathbf{8+s} \times \mathbf{8-s} + \mathbf{8-s} \times \mathbf{8+s}) = \\
 & = (\mathbf{64++} + \mathbf{64--}) + (\mathbf{64+-} + \mathbf{64-+}) = 128 + 128 = 256\text{-dim } Cl(16) \text{ spinors}
 \end{aligned}$$

If you try to combine all $128+128 = 256$ of the $Cl(16)$ D8 spinors with the 120-dim $Cl(16)$ D8 Lie Algebra you will see that they will fail to make a nice Lie Algebra

but

if you take only the $(\mathbf{64++} + \mathbf{64--}) = 128\text{-dim } Cl(16)$ D8 +half-spinors $D8+s$



and combine them with the 120-dim $Cl(16)$ D8 Lie Algebra they DO form the $128+120 = 248\text{-dim } E8$ Lie Algebra.

Although $E8$, like all Lie Algebras, can be written in terms of commutators, the 128-dim D8 +half-spinor part of $E8$ can be written as anticommutators, a property that $E8$ in $Cl(16)$ inherits from $F4$ in $Cl(8)$. Therefore, the 120-dim D8 part of $E8$ physically represents boson and vector spacetime with commutators

and

the 128-dim D8 +half-spinor part of $E8$ physically represents fermions with anticommutators. Further, since it is made up of $(\mathbf{64++} + \mathbf{64--})$ it represents 8 spacetime components of 8 fermion particles plus 8 spacetime components of 8 fermion antiparticles.

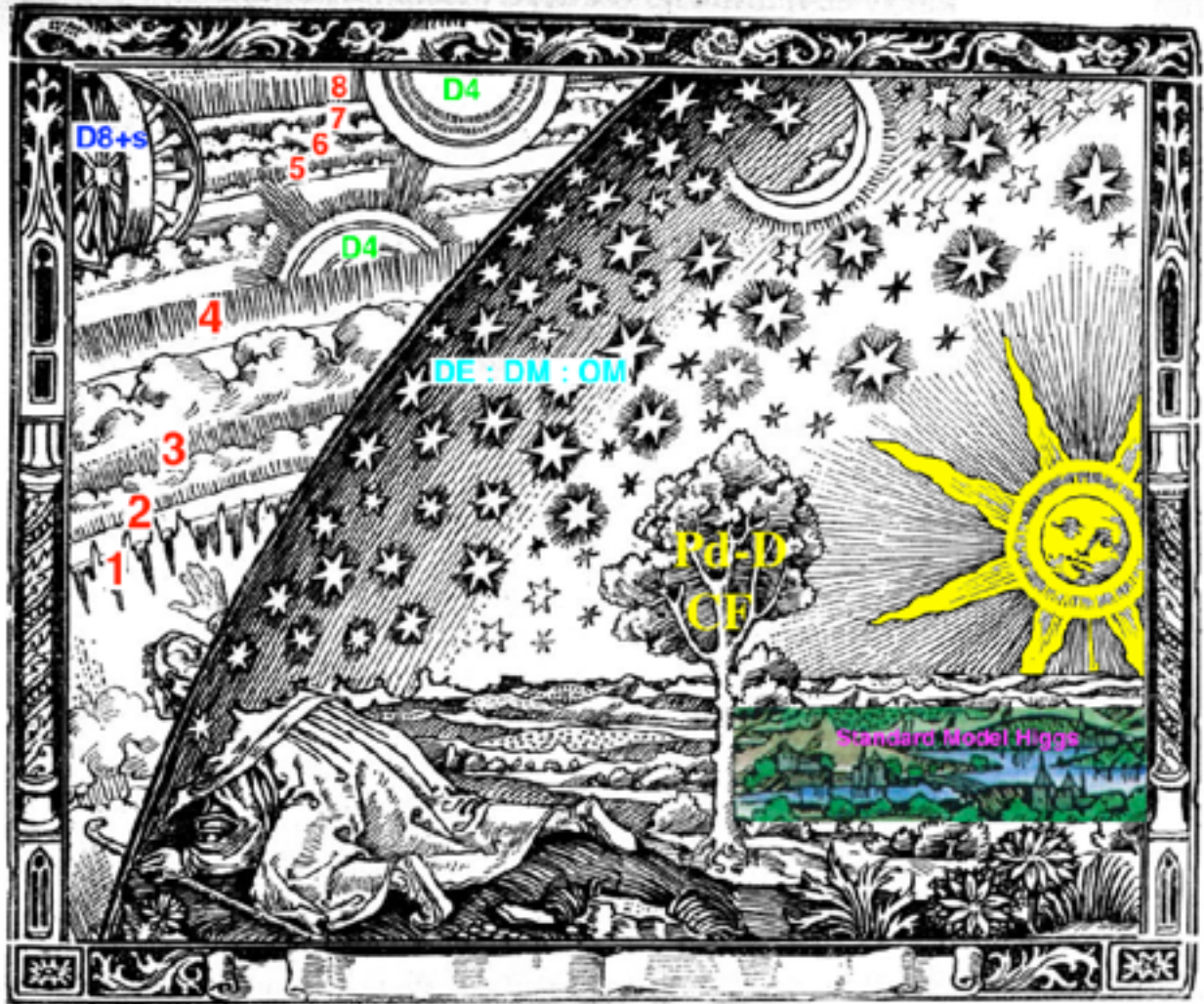
The 8 first-generation fermion types are

electron , red up quark , green up quark , blue up quark ;

blue down quark , green down quark , red down quark, neutrino

Second and Third fermion generations, Higgs, and 3 mass states of Higgs and Tquark emerge as consequences of the Octonion / Quaternion transition from 8-dim Spacetime of the Inflationary Era of our Universe to 4-dim Physical Spacetime + 4-dim CP2 Internal Symmetry Space.

Could the $D4 \times D4 + 8 \times 8 + 128\text{-dim } D8 + \text{half-spinor } D8+s$ structure of E8 have been depicted by Flammarion on page 163 of his 1888 book "L'Atmosphere Meteorologie Populaire" ?

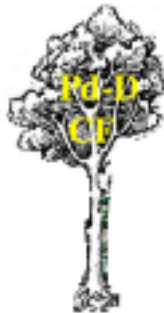


Flammarion's Naive Missionary Explorer sees the intersection of Terrestrial Physics and AstroPhysics as a window to the Realm of Terrestrial-AstroPhysics Unification through E8 Physics.

As to Terrestrial Physics: a Standard Model Higgs has been observed by the LHC near Lake Geneva which looks like this part of the Flammarion Engraving (colorization from image on goodnewsfromdrjoe blog)



and progress has been made toward understanding Palladium-Deuterium Cold Fusion, planting a seed that can grow into a productive Tree of Energy



As to AstroPhysics: Hot Fusion was known to be the Energy Source of the Sun



and our Universe was shown to have a ratio **DE : DM : OM** of Dark Energy to Dark Matter to Ordinary Matter roughly 3/4 : 1/5 : 1/20

How larger Clifford Algebras $Cl(8N)$ give in the large N limit a generalized Hyperfinite II_1 von Neumann factor

As to Clifford Algebras larger than $Cl(0,8)$

there is periodicity theorem that

$Cl(0,n+8) = M(16, Cl(0,n))$ of all 16×16 matrices

whose entries are from $Cl(0,n)$

and $Cl(p,q+4) = Cl(p,q) \times Cl(0,4) = Cl(p,q) \times H(2)$

and $Cl(p,q) = Cl(p+4,q-4)$

and $Cl(p+8,q) = Cl(p,q) \times Cl(0,8) = Cl(p,q) \times H(2) \times H(2) = Cl(p,q) \times R(16)$

and $Cl(p,q+8) = Cl(p,q) \times Cl(0,8) = Cl(p,q) \times H(2) \times H(2) = Cl(p,q) \times R(16)$

whereby

the tensor product of n copies of $Cl(8)$

$$Cl(8) \times \dots (n \text{ times tensor product}) \dots \times Cl(8) = Cl(8n)$$

and

any really large Clifford Algebra can

be embedded in the tensor product of a lot of $Cl(8)$ Clifford Algebras.

Since the E_8 Physics classical Lagrangian is Local,

it is necessary to patch together Local Lagrangian Regions

to form a Global Structure describing

a Global E_8 Algebraic Quantum Field Theory (AQFT).

Mathematically,

this is done by embedding E_8 into the $Cl(16)$ Clifford Algebra and

using a copy of $Cl(16)$ to represent each Local Lagrangian Region.

A Global Structure is then formed

by taking the tensor products of the copies of $Cl(16)$.

Due to Real Clifford Algebra 8-periodicity, $Cl(16) = Cl(8) \times Cl(8)$

and any Real Clifford Algebra, no matter how large, can be embedded in a tensor product of factors of $Cl(8)$, and therefore of $Cl(8) \times Cl(8) = Cl(16)$.

Just as the completion of the union of all tensor products of 2×2 complex Clifford algebra matrices produces the usual Hyperfinite II₁ von Neumann factor that describes creation and annihilation operators on the fermionic Fock space over $C^{(2n)}$ (see John Baez's Week 175),

we can take

the completion of the union of all tensor products of $Cl(16) = Cl(8) \times Cl(8)$ to produce a generalized Hyperfinite II₁ von Neumann factor that gives a natural Algebraic Quantum Field Theory structure for E₈ Physics.

In each tensor product $Cl(16) \times \dots \times Cl(16)$ each of the $Cl(16)$ factors represents a distinct Local Lagrangian Region.

Since each Region is distinguishable from any other, each factor of the tensor product is distinguishable so that the AQFT has Maxwell-Boltzmann Statistics.

Within each Local Lagrangian Region $Cl(16)$ there lives a copy of E₈. Each 248-dim E₈ has indistinguishable boson and fermion particles. The 120-dim bosonic part has commutators and Bose Statistics and the 128-dim fermionic part has anticommutators and Fermi Statistics.