On the Mass-Energy Relationship $E=mc^2$

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Abstract

This paper examines Einstein's derivations of the mass-energy relationship $E=mc^2$ using the Lorentz transformations applied to a slowly accelerating charge of mass m, and also applied to a moving system emitting photons in opposite directions. In the first case, it is shown that Einstein's use of the transformations is inconsistent with the classical definition of electrostatic force, $\vec{F} = q\vec{E}$, and that the expression given by Einstein for the kinetic energy of the charge due to its motion relative to the stationary system must be interpreted as a modification of the classical electrostatic interaction. In the second case, it is shown that Einstein's conclusion does not follow from his example, and that the Lorentz transformations give no information about the kinetic energy of the excitation energy of a system due to its motion relative to a stationary coordinate system.

Einstein in 1905 gave two derivations [1,2] of the mass energy relationship $E=mc^2$. The first was based on the change in kinetic energy of a slowly accelerating charge in an electrostatic field, using the classical definition of force on an electric charge due to an electrostatic field, $\vec{F} = q\vec{E}$. Einstein treats the case of a charge ε, of mass m, under the influence of an electric field in the x-direction, X, constant along the x-axis, so that the classical law of motion, $ma_x = F_x$ becomes

$$m\, d^2x/dt^2 = \varepsilon X \qquad (1)$$

He expresses (1) in a coordinate system moving at a velocity, v, relative to the stationary field, X, and then uses the Lorentz transformations to convert the acceleration on the left side of (1) back to the stationary coordinate system.

The force definition, $\vec{F} = q\vec{E}$, of course, already incorporates the change in velocity of the charge due to the force. That is, the force defined by the right side of the equation is independent of the velocity of the charge relative to the stationary field,

$\vec{E}$. Einstein assumes that, because ε, m, and X are constants in the coordinate system of the field, they are constant also in a coordinate system moving at constant velocity, v, along the x-axis. Therefore (1) is also independent of any transformation to such a coordinate system. That is, the acceleration defined by the right side of (1) is always the same. Using ξ,τ as the coordinates in the moving coordinate system, we then must have

$$d^2\xi/d\tau^2 = d^2x/dt^2 = \varepsilon X/m .$$

Einstein ignores this independence, expresses (1) in a coordinate system moving at a velocity, v, relative to the stationary field, as

$$m\, d^2\xi/d\tau^2 = \varepsilon X' \qquad (2),$$

and then inappropriately transforms it back into the coordinate system of the stationary field with the Lorentz transformations, using them on the only term in the expression that is transformable, which is the acceleration, $d^2\xi/d\tau^2$. His new acceleration in the stationary coordinate system then becomes

$$d^2x'/dt'^2 = (\beta^3)\, d^2x/dt^2, \qquad (\beta = 1/\sqrt{1-v^2/c^2})$$

which he incorrectly substitutes for d^2x/dt^2 in (1), treating both sides of (1) as independent from each other, and gets

$$m\, \beta^3\, d^2x/dt^2 = \varepsilon X \qquad (3)$$

He then presents this as the classical electrostatic law of motion modified by the Lorentz transformations, which it is not. The acceleration on the left side of (1) is not independent of, but rather is defined by the force on the right, so that the acceleration in (1) by definition is εX/m, and cannot be modified without changing the force on the right. That is, if $d^2x/dt^2 = \varepsilon X/m$, then $\beta^3\, d^2x/dt^2 \neq \varepsilon X/m$. By modifying the acceleration on the left side of (1), Einstein necessarily changes the force on the charge from εX to $\varepsilon X\, \beta^3$, so that if we are consistent with the logic of the classical law of motion, the left side of equation (3) must represent a new force, and so (3) must be correctly written as

$$m\, d^2x/dt^2 = \varepsilon X\, \beta^3 \qquad (4)$$

The kinetic energy imparted to the charge by the force in (4) would then be

$$W = \iint_{0,0}^{x,v} \varepsilon X\, \beta^3 dx\, dv.$$

Instead of using this expression, however, Einstein treats the left and right sides of the incorrect expression (3) as independent and equal, then writes incorrectly

$$W = \int \varepsilon\, X\, dx \; = m \int_0^v \beta^3\, v\, dv\ .$$

Evaluating the integral on the right side, he then gets

$$W = mc^2\, (\beta - 1),$$

thereby changing the force on the charge from $F = \varepsilon X$ to $F = m\beta^3\, v$.

So, by an inappropriate application of the Lorentz transformations, and a complete misinterpretation of the classical law of motion, Einstein somehow arrives at an expression for the kinetic energy of the moving charge that evidently represents, at least to some degree, an actual velocity dependence, and possibly a mass dependence, of the electromagnetic interaction not exhibited in the classical model—to me a truly remarkable result. Unfortunately, in my view, the general acceptance of Einstein's representation of this modification as a change in the mass, or effective mass, of the moving charge due merely to its motion relative to an arbitrary coordinate system has effectively prevented any investigation into its actual source, as well as implying an applicability to other forces that may or may not be correct.

In Einstein's second derivation, he examines an excited system of energy E_0 before and after emitting two light pulses of equal energy, ½ L, in opposite directions. He expresses the energy in its own coordinate system, and in a coordinate system moving at a velocity v relative to it. He then attempts to show that the kinetic energy difference before and after emission between the two coordinate systems is

$$K_0 - K_1 = L(\beta - 1) \tag{5}$$

Using the Lorentz transformations, he determines the energy of the emitted light pulses in the moving coordinate system, and then writes the total energy in each coordinate system after emission as

$$E_0 = E_1 + L, \tag{6}$$

$$H_0 = H_1 + \beta L \tag{7}$$

He then defines the kinetic energies

$$H_0 - E_0 = K_0 + C \tag{8}$$

$$H_1 - E_1 = K_1 + C \tag{9}$$

where C is an arbitrary constant. By subtracting the two, he then obtains (5).

However, we can also write E_0 and H_0 more transparently as

$$E_0 = E_1 + E^* \text{ , and}$$

$$H_0 = H_1 + H^* \text{ ,}$$

where E* and H* represent the excitation energy of the system before emission as determined in the two coordinate systems. Then the expressions for K become

$$(H_1 + H^*) - (E_1 + E^*) = K_0 + C \text{ and}$$

$$H_1 - E_1 = K_1 + C \text{ ,}$$

so that

$$K_0 - K_1 = H^* - E^* = L(\beta - 1) .$$

In other words, since $E^* = L$ by assumption, Einstein merely equates the potential energy of excitation as transformed to the moving coordinate system, H^*, to the light pulse energy as transformed to the moving coordinate system, $L\beta$. But we do not know the transformed value of E^*. That is, we have no way of determining whether the potential energy of this excited system behaves as a kinetic energy of mass such that $H^* = L\beta$ without measuring it, or at least by some other argument than Einstein provides here.

[1] A. Einstein, *Annalen der Physik*, 17:891, 1905. 1923 English translation

[2] A. Einstein, *Annalen der Physik*. **18**:639, 1905. 1923 English translation