An Elementary Proof of Legendre's Conjecture

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February 3, 2013.

ABSTRACT

We prove the Legendre's conjecture: given an integer, $n > 0$, there is always one prime, p, such that $n^2 < p < (n+1)^2$, using the prime-counting function and the Bertrand's Postulate.

1. INTRODUCTION

The Legendre's conjecture, named after Adrien-Marie Legendre (1752-1833), asserts that: There is always one prime number between a square number and the next. Algebraically speaking, given an integer, $n > 0$, there is always one prime, p, such that $n^2 < p < (n+1)^2$. Put yet another way, $\pi((n+1)^2) - \pi(n^2) > 0$, where $\pi(x)$ is the prime-counting function.

This conjecture was considered unproved when it was listed in Landau's problems, in 1912.

Chen Jingrun (1933-1996) proved a slightly weaker version of the conjecture: there is either a prime $n^2 < p < (n+1)^2$ or a semiprime $n^2 < pq < (n+1)^2$, where q is one prime unequal to p.

2. LEMMAS AND THEOREMS

LEMMA 1. (Bertrand's Postulate, actually a Theorem) *For any integer* $n > 3$, *there always exists, at least, one prime number,* p, *with* $n < p < 2n - 2$.

A weaker, but more elegant formulation is:

LEMMA 2. (Weak Bertrand's Postulate) *For every* $n > 1$ *there is always, at least, one prime number,* p, *such that* $n < p < 2n$.

THEOREM 1. *For* $n \geq 5$ *and* $n \in \mathbb{Z}_+$, *then*

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}, \tag{1}$$

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left[\frac{2\pi\Gamma(k)}{k}\right] + \cos\left[\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right]}{2 - 2\cos\left(\frac{2\pi}{k}\right)}, \tag{2}$$

$$\pi(n) = 2 - \sum_{k=5}^{n} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\}, \tag{3}$$

(4) $$\pi(n) = 2 + \sum_{k=5}^{n} \frac{\left(1 - e^{-\frac{2\pi i \Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi i}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}.$$

Proof. Part 1. In [1, pp. 427], H. Laurent noted that

(5) $$f(z) = \frac{e^{\frac{2\pi i \Gamma(z)}{z}} - 1}{e^{-\frac{2\pi i}{z}} - 1} = \begin{cases} 0, if\ z\ is\ composite \\ 1, if\ z\ is\ prime \end{cases},$$

for $z \geq 5$ and $z \in \mathbb{Z}_+$.

Observe that

$$f(z) = e^{\frac{2\pi i}{z}} \frac{\cos\left[\frac{2\pi\Gamma(z)}{z}\right] + i\sin\left[\frac{2\pi\Gamma(z)}{z}\right] - 1}{1 - e^{\frac{2\pi i}{z}}}$$

$$= e^{\frac{2\pi i}{z}} \frac{\cos\left[\frac{2\pi\Gamma(z)}{z}\right] + i\sin\left[\frac{2\pi\Gamma(z)}{z}\right] - 1}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$

$$= -\left[\cos\left(\frac{2\pi}{z}\right) + i\sin\left(\frac{2\pi}{z}\right)\right] \frac{1 - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] - i\sin\left[\frac{2\pi\Gamma(z)}{z}\right]}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right) - i\sin\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$
$$- \left\{\frac{i\sin\left(\frac{2\pi}{z}\right) - i\cos\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}\right\}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)} - \frac{i\left\{\sin\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right) - \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right)\right\}}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right] + i\left\{\sin\left(\frac{2\pi}{z}\right) - \sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]\right\}}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}.$$

Using the identity

$$\frac{a + bi}{c - di} = \frac{ac - bd}{c^2 + d^2} + i\frac{bc + ad}{c^2 + d^2},$$

we find the following real part

$$\Re[f(z)] = \frac{1 - \cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] + \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)} \tag{6}$$

$$= -\csc\left(\frac{\pi}{z}\right)\sin\left[\frac{\pi\Gamma(z)}{z}\right]\cos\left\{\frac{\pi[\Gamma(z)+1]}{z}\right\} \tag{7}$$

$$= \begin{cases} 0, if\ z\ is\ composite \\ 1, if\ z\ is\ prime \end{cases},$$

for $z \geq 5$ and $z \in \mathbb{Z}_+$.

The imaginary part is the following

$$\Im[f(z)] = -\frac{\sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right] - \sin\left[\frac{2\pi\Gamma(z)}{z}\right] - \sin\left(\frac{2\pi}{z}\right)}{2 - 2\cos\left(\frac{2\pi}{z}\right)} \tag{8}$$

$$= \frac{\sin\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right] - \sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)}.$$

From (6) and (8), it follows that

$$f(z) = \frac{1 - \cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] + \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)} + i\frac{\sin\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right] - \sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)} \tag{9}$$

$$= \frac{1 - \cos\left(\frac{2\pi}{z}\right) + i\sin\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] + i\sin\left[\frac{2\pi\Gamma(z)}{z}\right] + \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right] - i\sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)}$$

$$= \frac{1 - e^{-\frac{2\pi i}{z}} - e^{-\frac{2\pi i\Gamma(z)}{z}} + e^{-\left[\frac{2\pi i\Gamma(z)}{z} + \frac{2\pi i}{z}\right]}}{2 - 2\cos\left(\frac{2\pi}{z}\right)}$$

$$= \frac{-1\left(e^{-\frac{2\pi i}{z}} - 1\right) + e^{-\frac{2\pi i\Gamma(z)}{z}}\left(e^{-\frac{2\pi i}{z}} - 1\right)}{2 - 2\cos\left(\frac{2\pi}{z}\right)}$$

$$= \frac{\left(1 - e^{-\frac{2\pi i\Gamma(z)}{z}}\right)\left(1 - e^{-\frac{2\pi i}{z}}\right)}{2 - 2\cos\left(\frac{2\pi}{z}\right)} = \begin{cases} 0, if\ z\ is\ composite \\ 1, if\ z\ is\ prime \end{cases},$$

for $z \geq 5$ and $z \in \mathbb{Z}_+$.

Part 2. The prime counting function is the function counting the number of prime numbers less than or equal to some real number x. It is denoted by $\pi(x)$. From above definition, we have

$$\pi(x) = \sum_{p \le x} 1.$$

With the restriction for the positive integers and greater than or equal to five, it follows that

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}$$

by (5),

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left[\frac{2\pi\Gamma(k)}{k}\right] + \cos\left[\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right]}{2 - 2\cos\left(\frac{2\pi}{k}\right)}$$

by (6),

$$\pi(n) = 2 - \sum_{k=5}^{n} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\}$$

by (7),

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{\left(1 - e^{-\frac{2\pi i \Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi i}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}$$

by (9). □

COROLLARY 1. *For* $x \in \mathbb{R}_{\ge 5}$, *then*

$$\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}, \tag{10}$$

$$\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left[\frac{2\pi\Gamma(k)}{k}\right] + \cos\left[\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right]}{2 - 2\cos\left(\frac{2\pi}{k}\right)}, \tag{11}$$

$$\pi(x) = 2 - \sum_{k=5}^{\lfloor x \rfloor} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\}, \tag{12}$$

$$\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{\left(1 - e^{-\frac{2\pi i \Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi i}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}. \tag{13}$$

Proof. Is obvious by the definition of floor function: $\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \le x\}$ and previous Theorem. □

THEOREM 2. (Legendre's Theorem) *There is a prime number, p, between* n^2 *and* $(n+1)^2$ *for every positive integer* n*.*

Proof. Part 1. Observe that, by use of (1), we encounter

$$(14)\quad \pi((n+1)^2) = 2 + \sum_{k=5}^{(n+1)^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} = \left[2 + \sum_{k=5}^{2n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right] + \sum_{k=2n+1}^{n^2+2n+1} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$= \pi(2n) + \sum_{k=2n+1}^{n^2+2n+1} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}.$$

and

$$(15)\quad \pi(n^2) = 2 + \sum_{k=5}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} = \left[2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right] + \sum_{k=n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$= \pi(n) + \sum_{k=n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}.$$

Subtracting (15) to (14), it follows that

$$(16)\quad \pi((n+1)^2) - \pi(n^2) = \pi(2n) + \sum_{k=2n+1}^{n^2+2n+1} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \pi(n) - \sum_{k=n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=2n+1}^{n^2+2n+1} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \sum_{k=n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=2n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} + \sum_{k=n^2+1}^{n^2+2n+1} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \sum_{k=n+1}^{2n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \sum_{k=2n+1}^{n^2} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=n^2+1}^{n^2+2n+1} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} - \sum_{k=n+1}^{2n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}.$$

By (5) we have the inequality

$$(17)\qquad 0 = \min_{k\in\mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right) \leq \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} \leq \max_{k\in\mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right) = 1.$$

From (16) and (17), it follows that

$$\pi(2n) - \pi(n) + \sum_{k=n^2+1}^{n^2+2n+1} \min_{k\in\mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right) - \sum_{k=n+1}^{2n} \min_{k\in\mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right) \leq \pi((n+1)^2) - \pi(n^2)$$

$$\leq \pi(2n) - \pi(n) + \sum_{k=n^2+1}^{n^2+2n+1} \max_{k\in\mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right) - \sum_{k=n+1}^{2n} \max_{k\in\mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right),$$

$$\pi(2n) - \pi(n) + \sum_{k=n^2+1}^{n^2+2n+1} 0 - \sum_{k=n+1}^{2n} 0 \leq \pi((n+1)^2) - \pi(n^2) \leq \pi(2n) - \pi(n) + \sum_{k=n^2+1}^{n^2+2n+1} 1 - \sum_{k=n+1}^{2n} 1,$$

$$\pi(2n) - \pi(n) \leq \pi((n+1)^2) - \pi(n^2) \leq \pi(2n) - \pi(n) + 2n + 1 - n,$$

$$\pi(2n) - \pi(n) \leq \pi((n+1)^2) - \pi(n^2) \leq \pi(2n) - \pi(n) + n + 1,$$

for $n \in \mathbb{Z}_{\geq 5}$.

Part 2. For $n = 1$, $\pi(2^2) - \pi(1^2) = \pi(4) - \pi(1) = 2 - 0 = 2 > 0$; for $n = 2$, $\pi(3^2) - \pi(2^2) = \pi(9) - \pi(4) = 4 - 2 = 2 > 0$; for $= 3$, $\pi(4^2) - \pi(3^2) = \pi(16) - \pi(9) = 6 - 4 = 2 > 0$, for $n = 4$, $\pi(5^2) - \pi(4^2) = \pi(25) - \pi(16) = 9 - 6 = 3 > 0$. This completes the proof. □

References

[1] Dickson, Leonard Eugene, *History of the Theory of Numbers, Volume I: Divisibility and Primality*, Dover, 2005.