

## GAUSSIAN QUADRATURE OF THE INTEGRALS

$$\int_{-\infty}^{\infty} f(x) dx / \cosh x$$

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ABSTRACT. The manuscript delivers nodes and their weights for Gaussian quadratures with a “non-classical” weight in the integrand defined by a reciprocal hyperbolic cosine. The associated monic orthogonal polynomials are constructed; their coefficients turn out to be simple multiples of the coefficients of the Meixner polynomials. A final table shows the abscissae-weight pairs for up to 128 nodes.

### 1. INTRODUCTION

We tabulate the abscissae  $x_i$  and weights  $w_i$  for Gaussian integration with a reciprocal hyperbolic cosine in the integral kernel,

$$(1) \quad \int_{-\infty}^{\infty} f(x) \frac{dx}{\cosh x} = 2 \int_{-\infty}^{\infty} f(x) d[\arctan e^x] \approx \sum_{i=1}^N w_i f(x_i).$$

The function  $1/\cosh x = \operatorname{sech} x$  has the same even parity and inverse quadratic shape for small  $x$  as the Gaussian  $e^{-x^2}$ , but falls off  $\sim e^{-x}$  for large  $x$ . In that sense the Gaussian quadrature shares the symmetry properties with the more familiar Gauss-Hermite quadratures, whereas for large  $x$  the node’s distances, weights and densities resemble those of the Gauss-Laguerre quadratures.

The variable substitutions  $x = \operatorname{arcosh} t$  and  $t = 1/\cos \phi = 1/y$  define a small family of related rules,

$$(2) \quad \int_0^{\infty} f(x) \frac{dx}{\cosh x} = \int_1^{\infty} f(\operatorname{arcosh} t) \frac{dt}{t\sqrt{t^2-1}} = \int_0^{\pi/2} f(\operatorname{arcosh} \frac{1}{\cos \phi}) d\phi \\ = \int_0^1 f(\operatorname{arcosh} \frac{1}{y}) \frac{dy}{\sqrt{1-y^2}}.$$

The last format on the right hand side is usually evaluated with Gauss-Chebyshev quadratures [1, (25.4.38)][8, 6, 5].

### 2. GAUSSIAN INTEGRATION

**2.1. Moments.** The  $w_i$  and  $x_i$  in Eq. 1 are computed with the standard theory from roots of a system of orthogonal polynomials  $p_n$  with norm [2, 4, 9]

$$(3) \quad \langle f, g \rangle \equiv \int_{-\infty}^{\infty} f(x)g(x) \frac{dx}{\cosh x}.$$

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Moments  $\mu$  are defined as

$$(4) \quad \mu_n \equiv \int_{-\infty}^{\infty} \frac{x^n}{\cosh x} dx.$$

The even moments are powers of  $\pi$  multiplied by Euler numbers [3, 3.523.4],

$$(5) \quad \int_0^{\infty} \frac{x^{2n}}{\cosh(ax)} dx = [\pi/(2a)]^{2n+1} |E_{2n}|, \quad a > 0.$$

**Example 1.**

$$(6) \quad \mu_0 = 2 \int_0^{\infty} dx / \cosh x = \pi.$$

$$(7) \quad \mu_2 = 2 \int_0^{\infty} \frac{x^2}{\cosh x} dx \approx 7.751569170074955043869 = \pi^3/4;$$

$$(8) \quad \mu_4 = 2 \int_0^{\infty} \frac{x^4}{\cosh x} dx \approx 95.631151495400454144606659 = 5\pi^5/16.$$

The odd moments are zero because  $\cosh x$  is an even function.

**Remark 1.** *The associated semi-infinite integrals start with [3, 3.521.2]*

$$(9) \quad \int_0^{\infty} \frac{x}{\cosh x} dx = 2C,$$

where  $C \approx 0.915965$  is Catalan's constant [7, A006752]. Expansion of  $1/\cosh(ax)$  in a Taylor series of  $e^{-ax}$  and exchange of summation and integration yields the Mellin transform [3, 3.523.3]

$$(10) \quad \int_0^{\infty} \frac{x^{\tau-1}}{\cosh(ax)} dx = \frac{2}{a^{\tau}} \Gamma(\tau) \sum_{k=0}^{\infty} (-)^k \frac{1}{(2k+1)^{\tau}} \quad \Re \tau > 0, a > 0.$$

So the semi-infinite integrals are essentially Dirichlet beta-functions with some factorial growth.

**Example 2.** [7, A006752, A175572]

$$(11) \quad \beta(2) = \sum_{k \geq 0} \frac{(-)^k}{(2k+1)^2} \approx 0.91596559422;$$

$$(12) \quad \beta(4) = \sum_{k \geq 0} \frac{(-)^k}{(2k+1)^4} \approx 0.988944551741.$$

These  $\beta$ -values turn out to be irrelevant for the assembly of the orthogonal polynomials further down.

**2.2. Recurrence.** The set of orthogonal (monic) polynomials  $p_n(x)$  is bootstrapped from

$$(13) \quad p_0(x) = 1; \quad p_1(x) = x; \quad p_{n+1}(x) = xp_n(x) - b_n p_{n-1}(x);$$

$$(14) \quad b_n = \frac{\langle xp_n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} = (n\pi/2)^2.$$

The vanishing of the odd moments induces

- that the general term on the right hand side  $x - a$  reduces to  $x$ ,

TABLE 1. Reduced coefficients of the orthogonal polynomials. [7, A060338]

$n$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	-1	0	1						
3	0	-5	0	1					
4	9	0	-14	0	1				
5	0	89	0	-30	0	1			
6	-225	0	439	0	-55	0	1		
7	0	-3429	0	1519	0	-91	0	1	
8	11025	0	-24940	0	4214	0	-140	0	1
9	0	230481	0	-122156	0	10038	0	-204	0
10	-893025	0	2250621	0	-463490	0	21378	0	-285
11	0	-23941125	0	14466221	0	-1467290	0	41778	0
12	108056025	0	-296266266	0	70548511	0	-4054028	0	76263

- a parity  $p_n(x) = (-1)^n p_n(-x)$ .

The first of the orthogonal monic polynomials are

$$(15) \quad p_2 = x^2 - \frac{\pi^2}{4};$$

$$(16) \quad p_3 = x^3 - \frac{5\pi^2}{4}x;$$

$$(17) \quad p_4 = x^4 - \frac{7\pi^2}{2}x^2 + \frac{9\pi^4}{16};$$

$$(18) \quad p_5 = x^5 - \frac{15\pi^2}{2}x^3 + \frac{89\pi^4}{16}x;$$

$$(19) \quad p_6 = x^6 - \frac{55\pi^2}{4}x^4 + \frac{439\pi^4}{16}x^2 - \frac{225\pi^6}{64};$$

$$(20) \quad p_7 = x^7 - \frac{91\pi^2}{4}x^5 + \frac{1519\pi^4}{16}x^3 - \frac{3429\pi^6}{64}x;$$

$$(21) \quad p_8 = x^8 - 35\pi^2x^6 + \frac{2107\pi^4}{8}x^4 - \frac{6235\pi^6}{16}x^2 + \frac{11025}{256}\pi^8.$$

One may reduce this information to the integer triangle in Table 1, which displays in row  $n$  and column  $i$  the value of  $[x^i](2/\pi)^{n-i}p_n(x)$ , the coefficient in front of  $x^i$  of  $p_n(x)$  multiplied by  $(2/\pi)^{n-i}$ . The diagonal of the triangle contains ones, the first sub-diagonal zeros, the second sub-diagonal the negated square pyramidal numbers [7, A000330], and the third sub-diagonal zeros. The first column are alternatingly signed squares of double factorials, [7, A001818].

**2.3. Abscissae and Weights.** The standard further steps are

- normalization of the polynomials such that their norm is unity,

$$(22) \quad p_n^*(x) \equiv \frac{p_n(x)}{\sqrt{\langle p_n, p_n \rangle}},$$

- computation of all zeros  $x_i$  of  $p_N(x)$  at some degree  $N$ .

- computation of the weights  $w_i$  by

$$(23) \quad w_i = -\frac{[x^{N+1}]p_{N+1}^*}{[x^N]p_N^*} \frac{1}{p_{N+1}^*(x_i)p_N^{*\prime}(x_i)},$$

where  $[x^{N+1}]p_{N+1}^*$  and  $[x^N]p_N^*$  are the leading coefficients of the two polynomials after normalization, and where the prime at  $p'$  denotes the derivative with respect to  $x$ .

The final table shows in each line a pair  $(x_i, w_i)$  for  $N$  between 3 and 128. Because the  $p_n$  are even or odd in our case, the  $x_i$  emerge in pairs of the same weight; only the non-negative values need to be shown.

N=3

0.000000000000000000000000000000e+00 2.513274122871834590770114706624e+00  
3.512407365520363196578187321602e+00 3.141592653589793238462643383280e-01

N=4

1.290965896441180073070246826572e+00 1.530492283332155902198177146094e+00  
5.733848834599546320086127816633e+00 4.030404346274071703314454554585e-02

N=5

0.000000000000000000000000000000e+00 2.259122807075806373725945803707e+00  
2.869916769199472541729524465267e+00 4.370369936830449412430525948935e-01  
8.110832925540868955822877150824e+00 4.197929573948491125296194892976e-03

N=6

1.164992698884809070388844640060e+00 1.488812172738848586373384889168e+00  
4.712388980384689857693965074919e+00 8.159980918415047372630242553973e-02  
1.058977065965418878577677478990e+01 3.843448718975591316343769324640e-04

N=7

0.000000000000000000000000000000e+00 2.110886402412039551302983480629e+00  
2.568054183115606545325647437951e+00 5.031080598664201488408040644868e-01  
6.724673230156266603527437591123e+00 1.221291450409527079860733266896e-02  
1.314219798150161131386382703486e+01 3.215121836142394041855416963962e-05

N=8

1.088780490643117793491480727016e+00 1.454780137933664703045615546923e+00  
4.208368448429612883578963943174e+00 1.144340405198607780507317621372e-01  
8.857597597576316355451915802723e+00 1.579634444640079052649892282238e-03  
1.575066064909643353256005503163e+01 2.513896731059082324490297427435e-06

N=16

9.39776000914173798596605102562e-01 1.366166053291094809358156883950e+00  
3.374746061766199009359792428014e+00 1.945465717780134878765890895980e-01  
6.586602128556441369494039833563e+00 9.849858301748245794555282828847e-03  
1.052906382668069702875518615041e+01 2.314285987325267412846229403666e-04  
1.528108660320025280858377176983e+01 2.405473389364514404243434410742e-06  
  
2.103053302790247540334918518674e+01 9.342497565277832966881730799915e-09  
2.816975000664041667754119669483e+01 9.419665679985758416311408493244e-12  
3.773236037025653414427867668609e+01 9.539885128435901938614018670630e-16

N=32

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 5.327567951107585772456619676312e+00 2.617823740154800118843007161843e-02  
 8.210554693254896360156180602341e+00 1.666741078014349154283200944630e-03  
 1.145690573163925781703529651570e+01 7.246875637119559973697902853995e-05

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N=48

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 4.809169765177271494506232297817e+00 3.861363135978806629620925651077e-02  
 7.316278629204247481906360209014e+00 3.510812268902787001787262056978e-03  
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3.183996852156532247789137329289e+01 1.304918803350326198360458760340e-13  
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N=96

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