

# GAUSS-LAGUERRE AND GAUSS-HERMITE QUADRATURE ON 64, 96 AND 128 NODES

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ABSTRACT. The manuscript provides tables of abscissae and weights for Gauss-Laguerre integration on 64, 96 and 128 nodes, and abscissae and weights for Gauss-Hermite integration on 96 and 128 nodes.

## 1. GAUSS-LAGUERRE

We tabulate abscissae  $x_i$  and weights  $w_i$  for Gauss-Laguerre integration of the form

$$(1.1) \quad \int_0^{\infty} f(x)e^{-x} dx \approx \sum_{i=1}^n w_i f(x_i).$$

The objective is to extend the tables provided for up to  $n = 15$  in the Handbook [1, Tabl 25.9][9], up to 32 points by Krylov [4] and for  $n = 50$  in my thesis [6] to higher numbers of nodes. The abscissae are the zeros of Laguerre Polynomials, which are [1, 22.3.9]

$$(1.2) \quad L_n(x) = L_n^{(0)}(x) = \sum_{i=0}^n \binom{n}{i} \frac{1}{i!} (-x)^i.$$

The weights are related to the derivatives at these nodes [8],

$$(1.3) \quad w_i = \frac{1}{x_i [L_n(x_i)]^2}.$$

Shao, Chen and Frank provide the Newton iteration to converge on the roots  $x_i$  of generalized Laguerre Polynomials  $L_n^{(\alpha)}(x)$  [10, 3]:

$$(1.4) \quad x_i \rightarrow x_i - \frac{f(x_i)}{f'(x_i)} \left[ 1 + \frac{1}{2} \left( 1 - \frac{\alpha + 1}{x_i} \right) \frac{f(x_i)}{f'(x_i)} \right]$$

and the terminating continued fraction

$$(1.5) \quad \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)'}(x)} = \frac{x}{n-} \frac{(n+\alpha)n}{2n+\alpha-1-x-} \frac{(n-1+\alpha)(n-1)}{2n+\alpha-3-x-} \frac{(n-2+\alpha)(n-2)}{2n+\alpha-5-x-} \dots \frac{(1+\alpha)}{1+\alpha-x}.$$

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*Date:* October 2, 2013.

*2010 Mathematics Subject Classification.* Primary 41A55, 65A05; Secondary 65D30.

*Key words and phrases.* Gaussian Integration, Tables, Laguerre Polynomials.

## 2. GAUSS-HERMITE

We also tabulate abscissae  $x_i$  and weights  $w_i$  for Gauss-Hermite integration of the form

$$(2.1) \quad \int_{-\infty}^{\infty} f(x)e^{-x^2} dx \approx \sum_{i=1}^n w_i f(x_i).$$

The objective is to extend the tables provided by Steen et al. up to  $n = 15$  [12], by Shizgal up to  $n = 16$  [11], by Krylov and in the Handbook [1, Table 25.9][4] up to  $n = 20$ , and by Shao, Chen and Frank [10] up to  $n = 64$ . The abscissae are the zeros of Hermite Polynomials, which are [1, 22.3.10]

$$(2.2) \quad H_n(x) = n! \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(-1)^i}{i!(n-2i)!} (2x)^{n-2i}.$$

The weights are related to the derivatives at these nodes [1, 25.4.46],

$$(2.3) \quad w_i = \frac{2^{n-1} n! \sqrt{\pi}}{[nH_{n-1}(x_i)]^2} = \frac{2^{n+1} n! \sqrt{\pi}}{[H'_n(x_i)]^2}.$$

The first order Newton iteration to stabilize the roots  $x_i$  is

$$(2.4) \quad x_i \rightarrow x_i - \frac{f(x_i)}{f'(x_i)},$$

and the supporting terminating continued fraction [13] is

$$(2.5) \quad \frac{H_n(x)}{H'_n(x)} = \frac{1}{2n} \frac{H_n(x)}{H_{n-1}(x)} = \frac{1}{2n} \left[ 2x - \frac{2(n-1)}{2x-} \frac{2(n-2)}{2x-} \dots \frac{2}{2x} \right].$$

## 3. GAUSS-HERMITE OF MOMENTS

The  $w_i$  and  $x_i$  for integrations for the  $m$ -th moment of the form

$$(3.1) \quad \int_{-\infty}^{\infty} f(x)x^m e^{-x^2} dx \approx \sum_{i=1}^n w_i f(x_i)$$

with integer  $m$  are computed with the standard theory from roots of a system of orthogonal polynomials  $p_n$  with norm [2, 5, 14]

$$(3.2) \quad \langle f, g \rangle \equiv \int_{-\infty}^{\infty} f(x)g(x)x^m e^{-x^2} dx$$

and moments  $\mu$  [1, 6.1.12]

$$(3.3) \quad \mu_n \equiv \int_{-\infty}^{\infty} x^{n+m} e^{-x^2} dx = \begin{cases} 0, & n+m \text{ odd;} \\ \Gamma\left(\frac{n+m+1}{2}\right) = \sqrt{\pi} \frac{(n+m-1)!!}{2^{(n+m)/2}}, & n+m \text{ even.} \end{cases}$$

The following subsections summarize the known results [10].  $m$  is always assumed to be a positive even integer—for odd  $m$  the weight function is not always positive over the base interval and the standard theory is not applicable.

**3.1. Generalized Hermite Polynomials.** The set of orthogonal (monic) polynomials  $p_n(x)$  is bootstrapped from

$$(3.4) \quad p_0(x) = 1; \quad p_1(x) = x; \quad p_{n+1}(x) = (x - a)p_n(x) - b_n p_{n-1}(x);$$

$$(3.5) \quad b_n = \begin{cases} n/2; & n \text{ even} \\ (m+n)/2. & n \text{ odd.} \end{cases}$$

The vanishing of the moments  $\mu_n$  at odd indices induces

- that the general term on the right hand side  $x - a$  reduces to  $x$ ,
- a parity  $p_n(x) = (-1)^n p_n(-x)$ .

The first of the orthogonal monic polynomials are for  $m = 2$

$$(3.6) \quad p_2 = x^2 - \frac{3}{2};$$

$$(3.7) \quad p_3 = x^3 - \frac{5}{2}x;$$

$$(3.8) \quad p_4 = x^4 - 5x^2 + \frac{15}{4};$$

$$(3.9) \quad p_5 = x^5 - 7x^3 + \frac{35}{4}x;$$

$$(3.10) \quad p_6 = x^6 - \frac{21}{2}x^4 + \frac{105}{4}x^2 - \frac{105}{8};$$

$$(3.11) \quad p_7 = x^7 - \frac{27}{2}x^5 + \frac{189}{4}x^3 - \frac{315}{8}x;$$

$$(3.12) \quad p_8 = x^8 - 18x^6 + \frac{189}{2}x^4 - \frac{315}{2}x^2 + \frac{945}{16},$$

for  $m = 4$

$$(3.13) \quad p_2 = x^2 - \frac{5}{2};$$

$$(3.14) \quad p_3 = x^3 - \frac{7}{2}x;$$

$$(3.15) \quad p_4 = x^4 - 7x^2 + \frac{35}{4};$$

$$(3.16) \quad p_5 = x^5 - 9x^3 + \frac{63}{4}x;$$

$$(3.17) \quad p_6 = x^6 - \frac{27}{2}x^4 + \frac{189}{4}x^2 - \frac{315}{8};$$

$$(3.18) \quad p_7 = x^7 - \frac{33}{2}x^5 + \frac{297}{4}x^3 - \frac{693}{8}x;$$

$$(3.19) \quad p_8 = x^8 - 22x^6 + \frac{297}{2}x^4 - \frac{693}{2}x^2 + \frac{3456}{16},$$

and for  $m = 6$

$$(3.20) \quad p_2 = x^2 - \frac{7}{2};$$

$$(3.21) \quad p_3 = x^3 - \frac{9}{2}x;$$

$$(3.22) \quad p_4 = x^4 - 9x^2 + \frac{63}{4};$$

$$(3.23) \quad p_5 = x^5 - 11x^3 + \frac{99}{4}x;$$

$$(3.24) \quad p_6 = x^6 - \frac{33}{2}x^4 + \frac{297}{4}x^2 - \frac{693}{8};$$

$$(3.25) \quad p_7 = x^7 - \frac{39}{2}x^5 + \frac{429}{4}x^3 - \frac{1287}{8}x;$$

$$(3.26) \quad p_8 = x^8 - 26x^6 + \frac{429}{2}x^4 - \frac{1287}{2}x^2 + \frac{9009}{16}.$$

These are conveniently written down separately for even and odd polynomial orders as

$$(3.27) \quad p_{2n} = (-)^n n! L_n^{(m/2-1/2)}(x^2);$$

$$(3.28) \quad p_{2n+1} = (-)^n n! x L_n^{(m/2+1/2)}(x^2).$$

**3.2. Abscissae and Weights.** The standard further steps are

- normalization of the polynomials such that their norm is unity,

$$(3.29) \quad p_n^*(x) \equiv \frac{p_n(x)}{\sqrt{\langle p_n, p_n \rangle}},$$

- computation of all zeros  $x_i$  of  $p_N(x)$  at some degree  $N$ .
- computation of the weights  $w_i$  by

$$(3.30) \quad w_i = - \frac{[x^{N+1}]p_{N+1}^*}{[x^N]p_N^*} \frac{1}{p_{N+1}^*(x_i)p_N^{*'}(x_i)} = - \frac{\langle p_N, p_N \rangle}{p_{N+1}(x_i)p_N'(x_i)},$$

where  $[x^{N+1}]p_{N+1}^*$  and  $[x^N]p_N^*$  are the leading coefficients of the two polynomials after normalization, and where the prime at  $p'$  denotes the derivative with respect to  $x$ .

The final table shows in each line a pair  $(x_i, w_i)$  for  $N$  fixed at 96 or 128. Because the  $p_n$  are even or odd in our case, the  $x_i$  emerge in pairs of the same weight; only the non-negative values need to be shown.

#### 4. RESULTS

First the  $x_i$  and then the associated  $w_i$  are shown for the Laguerre polynomials. For Hermite polynomials, the duplicates with the opposite sign are not reproduced.

The originating Maple program is shown in the appendix.

Laguerre n=64

```
2.241587414670528002281188481904e-02 5.625284233902984574102185450625e-02
1.181225120967704797974664367103e-01 1.190239873124260278149035058888e-01
2.903657440180364839991300663853e-01 1.574964038621445238201964347055e-01
5.392862212279790393181449478122e-01 1.675470504157739478809044116588e-01
8.650370046481139446199550747097e-01 1.533528557792366180854547925640e-01
```

1.267814040775241398115708877694e+00 1.242210536093297445126137821926e-01  
1.747859626059436252829963951288e+00 9.034230098648505773897410920156e-02  
2.305463739307508718548070543894e+00 5.947775576835502421224699743972e-02  
2.940965156725251840679468152108e+00 3.562751890403607185416573533689e-02  
3.654752650207290527035397912093e+00 1.948041043116640604333738027148e-02

4.447266343313094356742550980161e+00 9.743594899382002240107960271379e-03  
5.318999254496390343522109859188e+00 4.464310364166275292364824190544e-03  
6.270499046923653912911064646334e+00 1.875359581323114826750128222516e-03  
7.302370002587395747223498409519e+00 7.226469815750051227191087066195e-04  
8.415275239483024194495218591205e+00 2.554875328334967097144484021795e-04

9.609939192796108035762882049548e+00 8.287143534396942179063224039877e-05  
1.088715038388637214259455042018e+01 2.465686396788558745973370221716e-05  
1.224776450424430161816236929072e+01 6.726713878829668527612545536274e-06  
1.369270784554750515272993257464e+01 1.681785369964088897821201086506e-06  
1.522298111152472884800826878343e+01 3.850812981546684414827886110704e-07

1.683966365264873721052883803916e+01 8.068728040990499790415110915505e-08  
1.854391817085919052361962597108e+01 1.545723706757688828003703926158e-08  
2.033699594873023550114981580640e+01 2.704480147617481409988607446575e-09  
2.222024266595087653992215434709e+01 4.316775475427200912314164217177e-10  
2.419510487593325398988644388024e+01 6.277752541761452201652955763263e-11

2.626313722711848578512602395478e+01 8.306317376288958063878932234380e-12  
2.842601052750102729949977152675e+01 9.984031787220164055897291499895e-13  
3.068552076752597177104858239845e+01 1.088353887116662685326123546986e-13  
3.304359923643782912552021068053e+01 1.074017403441590186482804810228e-14  
3.550232389114120958697877853505e+01 9.575737231574442105585024583281e-16

3.806393216564646826035731791502e+01 7.697028023648586098862970088678e-17  
4.073083544445862636573186951324e+01 5.564881137454025366525461526815e-18  
4.350563546642152985270318493173e+01 3.609756409010446498298646897469e-19  
4.639114297861619207360539994238e+01 2.095095369548946234767748078464e-20  
4.939039902562468667923580082269e+01 1.084793301097549361202631562742e-21

5.250669934134630165017927698046e+01 4.994699486363804115792443627063e-23  
5.574362241327838046333571129115e+01 2.037836974598822310658063498146e-24  
5.910506191901710660884874209180e+01 7.339537564278837039106367650444e-26  
6.259526440015139559605501790119e+01 2.323783082198694261304776157500e-27  
6.621887325124756438221376267101e+01 6.438234706908762420384341347468e-29

6.998098037714682922853465797223e+01 1.553121095788275270621507732395e-30  
7.388718723248296321095740311351e+01 3.244250092019537314465982686763e-32  
7.794367743446312031368797587057e+01 5.832386267836201501282207418418e-34  
8.215730377831930429519586834224e+01 8.963254833102854061314281996410e-36  
8.653569334945651821021627837527e+01 1.168703989550736241198572917330e-37

9.108737561313309014563671534928e+01 1.282055984359980381497645585305e-39  
9.582194001552073209476721543650e+01 1.172094937405002291828122995135e-41  
1.007502319695139796292592614512e+02 8.835339672328604981296921087574e-44  
1.058845994687999493563604278509e+02 5.424955590306186594338147130486e-46  
1.112392075244395820634847366376e+02 2.675542666678893828949793699428e-48

1.168304450513064984633866690774e+02 1.042917031411367078110598371701e-50  
 1.226774602685385765774196905650e+02 3.152902351957772623653127987116e-53  
 1.288028787692376725127536230543e+02 7.229541910647522339706610677135e-56  
 1.352337879495258278339804988789e+02 1.224235301230082264459594302229e-58  
 1.420031214899315190251400382911e+02 1.482168504901910411780618275588e-61

1.491516659000493885872934629324e+02 1.232519348814518808064365504012e-64  
 1.567310751326711612336169608139e+02 6.691499004571269526814069374687e-68  
 1.648086026551505229931901090252e+02 2.220465941850448995507392045782e-71  
 1.734749468364242745221528448669e+02 4.120946094738876249979125791383e-75  
 1.828582046914314636463427945096e+02 3.774399061896489170417616059552e-79

1.931511360370729114793855274172e+02 1.414115052917619417463147014780e-83  
 2.046720284850594559490644333426e+02 1.591833064041367917861031849654e-88  
 2.180318519353285163324523844483e+02 2.989484348860634307741312694817e-94  
 2.348095791713261647130555297254e+02 2.089063508436952770828154254400e-101

Laguerre n=96

1.498247386279735578113381595993e-02 3.787857621904154857294555367357e-02  
 7.894612304880207840185985106866e-02 8.271990609834760360531341593037e-02  
 1.940394361941514412752508786116e-01 1.158667991339846973607615984049e-01  
 3.603184994030137122785822472820e-01 1.338323001322935736406256373056e-01  
 5.778305997114498570222491789396e-01 1.364333036164533434786894149718e-01

8.466343334083691664262806795413e-01 1.262862120356477972119000242104e-01  
 1.166801575278802054604277136035e+00 1.076924888798102793618808477726e-01  
 1.538417935202955009462473472637e+00 8.532412911159066557538645790078e-02  
 1.961582977838289290098638265217e+00 6.314203242520727884377360072333e-02  
 2.436410401399018873095487851099e+00 4.379838230833155826312942207348e-02

2.963028219390472494624798221471e+00 2.854644887954969225023787573827e-02  
 3.541578958083545938690003622685e+00 1.751286619497787142048224525397e-02  
 4.172219874479178368746331029784e+00 1.012564796759256050813924076596e-02  
 4.855123197087293047910857955935e+00 5.522702599754219680646826023698e-03  
 5.5904763910528047486378078442229e+00 2.843408475355105761869086515002e-03

6.378482448916911967984059122961e+00 1.382611675843960566140825599722e-03  
 7.219360208264738897713155255980e+00 6.351647717174251628990736154925e-04  
 8.113344697561966664280773292162e+00 2.757399699878774910439708453509e-04  
 9.060687511578667610371531997438e+00 1.131361618904986583334124750235e-04  
 1.006165721792036504754513846483e+01 4.387518309801511431495637693683e-05

1.111653979632731325771301026208e+01 1.608239776072406157631990197051e-05  
 1.222563911256064169884407585976e+01 5.571480048334918160042235677625e-06  
 1.3389277428867983239134067642228e+01 1.824020563650868624991610676325e-06  
 1.460779595321226557860843179550e+01 5.642346857470526754659778186812e-07  
 1.588155542965690531693169690084e+01 1.648822949985393333100870755930e-07

1.721093677253059231217555823888e+01 4.550601393371598572333222034871e-08  
 1.859634174724747148527200308474e+01 1.185835466607614446944173593309e-08  
 2.003819370093648641201390291985e+01 2.916795610681356328390315858537e-09  
 2.153693834634001257411330749959e+01 6.769611683671967343982507344651e-10

2.309304460278018967012773266987e+01 1.481949983670311204293618226453e-10

2.470700549836557365787677252672e+01 3.058700451737857163777474992448e-11  
2.637933913802545411229107805162e+01 5.949512313209493202229440556218e-12  
2.811058974241966462160890792100e+01 1.090084100615501726449608399095e-12  
2.990132876328396126135791142676e+01 1.880406391828555287344935742008e-13  
3.175215608134195348557972506929e+01 3.052254984068018410341749580814e-14

3.366370129355207656627969531346e+01 4.659257733162024341566967130112e-15  
3.563662509717134091484328114521e+01 6.684573637743514983341244733865e-16  
3.767162077891714874696492629959e+01 9.007624836565320347953647863886e-17  
3.976941581840657130901835782830e+01 1.139275645645738992428945541273e-17  
4.193077361606333874754369394098e+01 1.351496197399314593696786360045e-18

4.415649535682294611578264713579e+01 1.502575840726936195121879000082e-19  
4.64474220225501832608591311646e+01 1.564384020199214922872554537161e-20  
4.880443656518197843096993491383e+01 1.523935207125346406976142000741e-21  
5.122846626253061047821077009198e+01 1.387767030020088746124291318747e-22  
5.372048526403945690663256148001e+01 1.180275279417301142286336764574e-23

5.628151735659690342239985838581e+01 9.365572923149480888948457226280e-25  
5.891263896644579012548334750625e+01 6.926501287689107276208128732857e-26  
6.161498242431210125620526017166e+01 4.769156040289474241447326826417e-27  
6.438973952175925405967762290139e+01 3.053588369363916385087273654705e-28  
6.723816539080926359415386753027e+01 1.815876339567533699955936314641e-29

7.016158274319579814458104875932e+01 1.001623972960673195235157937395e-30  
7.316138651062802220653461483836e+01 5.117644054432073675737621394578e-32  
7.623904893327643203701909184933e+01 2.418537832219781127157564535575e-33  
7.939612515049824600703602616032e+01 1.055571200688440973709765960911e-34  
8.263425935578989393189953246543e+01 4.247854846530332834287086480864e-36

8.595519158731968933103011272068e+01 1.573451260477077523414096110684e-37  
8.936076523643990274261935280560e+01 5.354853087862639709650921850479e-39  
9.285293536965613607385063826609e+01 1.671141386236028863218007942653e-40  
9.643377797507985644032619198955e+01 4.772638592840318669671697980780e-42  
1.001055002629519166797620057781e+02 1.244615423272114008888063805738e-43

1.038704521720840125198861679467e+02 2.956874625061022742177248527240e-45  
1.077311392608829209572922495851e+02 6.383727053789554859703304210991e-47  
1.116902371940922815227430914088e+02 1.249132372568423120891152715930e-48  
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## APPENDIX A. MAPLE SOURCE CODE

```

interface(quiet=true):
with(orthopoly) :
Digits := 200 :

# Return the ratio  $L_n^\alpha(x)/L_n^{\prime\alpha}(x)$  of
# the Laguerre Polynomial of index n (log derivative) at abscissa x.
# Parameter x: the abscissa value
# Parameter n: Order of the Laguerre Polynomial.
# Parameter alpha: Parameter of the Laguerre Polynomial.
# Return:  $L_n^\alpha(x)/L_n^{\prime\alpha}(x)$ .
ShaoChenFrank := proc(x,n,alpha)
    local itr,lnfrac ;
    lnfrac := 0 ;
    for itr from n-1 to 0 by -1 do
        lnfrac := (n-itr+alpha)*(n-itr)/(2*n-2*itr+alpha-1-x-lnfrac) ;
    end do:
    lnfrac := x/(n-lnfrac) ;
    return lnfrac ;
end proc:

# Run 3 steps of Newton iterations to refine the estimate of
# a root of  $L_n^\alpha(x)$ .
# Parameter x: the estimate of the root.
# Parameter n: Order of the generalized Laguerre Polynomial.
# Parameter alpha: Parameter of the generalized Laguerre Polynomial.
# Return: An improved estimate of the root.
lagroot := proc(x,n,alpha)
    local l,xref,ffprime ;
    xref := x ;
    for l from 1 to 3 do
        ffprime := ShaoChenFrank(xref,n,alpha) ;
        xref := xref-ffprime*(1+ffprime*(1-(alpha+1)/xref)/2) ;
    end do:
    return xref ;
end proc:

# Print the abscissa and weights for Gauss-Laguerre Quadrature on n nodes.
# Parameter n: The number of nodes. Also the order of the Laguerre Polynomial.
lagGaussian := proc(n)
    local lag,lagprime,ktau,i,xi,wi ;
    # Construct the Laguerre Polynomial  $L_n(x)$ 
    lag := L(n,x) :
    lag := evalf(lag) :
    # Construct the first derivative of  $L_n(x)$ 
    lagprime := diff(lag,x) ;
    # An upper estimate of the largest root of  $L_n(x)$ .
    # See Abramowitz and Stegun 22.16.8
    ktau := evalf(n+1/2) ;

```

```

# Find all roots of L_n(x)
lag := fsolve(lag,x,0..2*ktau+sqrt(4*ktau^2+0.25),fulldigits,complex) ;
i := 1 ;
printf("Laguerre n=%d\n",n) ;
for xi in lag do
    xi := lagroot(Re(xi),n,0) ;
    wi := 1/xi/(eval(lagprime,x=xi))^2 ;
    printf("%.30e %.30e\n",xi,wi) ;
    if i mod 5 = 0 then
        printf("\n") ;
    end if;
    i := i+1 ;
end do;
printf("\n") ;
return ;
end proc:

# Compute the norm of the polynomial squared.
# Parameter pol: a polynomial of x.
# Parameter m: The exponent in the weight function of the Hermit-Gauss integration.
# return: The norm <pol,pol> with weight x^m*exp(-x^2).
hermNorm := proc(pol,m)
    local no,polsqr,n ;
    # Square the polynomial
    polsqr := expand(pol^2) ;
    no := 0 ;
    # Collect the contribution from each power in the squared polynomial separately.
    for n from 0 to degree(polsqr,x) do
        no := no+ coeff(polsqr,x,n)*GAMMA((n+m+1)/2) ;
    end do:
    no ;
end proc:

# Print the abscissa and weights for Gauss-Hermite Quadrature on n nodes.
# Parameter n: The number of nodes. Also the order of the Hermite Polynomial.
# Parameter m: The (even) power of the moment.
hermGaussian := proc(n,m)
    local lag,lagprime,lagplus,i,xi,wi,nhalf,alpha,roo ;
    nhalf := floor(n/2) ;
    # Construct the generalized Hermite Polynomial p_n(x)
    # and its successor p_{n+1}(x)
    if type(n,'even') then
        alpha := m/2-1/2 ;
        lag := (-1)^nhalf*nhalf!*L(nhalf,alpha,x^2) :
        lagplus := (-1)^nhalf*nhalf!*x*L(nhalf,alpha+1,x^2) :
    else
        alpha := m/2+1/2 ;
        lag := (-1)^nhalf*nhalf!*x*L(nhalf,alpha,x^2) :
        lagplus := -(-1)^nhalf*(1+nhalf)!*L(1+nhalf,alpha-1,x^2) :
    end if;
    lag := evalf(lag) :
    # Construct the first derivative of p_n(x)

```

```

lagprime := diff(lag,x) ;
# Find all roots of p_n(x)
roo := fsolve(lag,x,0..infinity,fulldigits) ;
i := 1 ;
printf("Hermite n=%d m=%d\n",n,m) ;
for xi in roo do
  if xi >0 then
    # the iteration is for functions with argument x, whereas
    # the current representations is L_n^alpha(x^2).
    xi := lagroot(Re(xi)^2,nhalf,alpha) ;
    xi := sqrt(xi) ;
  end if;
  wi := -hermNorm(lag,m)/eval(lagplus*lagprime,x=xi) ;
  printf("%.30e %.30e\n",xi,wi) ;
  if i mod 5 = 0 then
    printf("\n") ;
  end if;
  i := i+1 ;
end do:
printf("\n") ;
return ;
end proc:

for n in [64, 96, 128] do
  lagGaussian(n) ;
end do:

for n in [96, 128] do
  for m in [0,2,4,6,8] do
    hermGaussian(n,m) ;
  end do:
end do:

URL: http://www.mpia.de/~mathar
HOESCHSTR. 7, 52372 KREUZAU, GERMANY

```