

GAUSS-LAGUERRE AND GAUSS-HERMITE QUADRATURE ON 64, 96 AND 128 NODES

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ABSTRACT. The manuscript provides tables of abscissae and weights for Gauss-Laguerre integration on 64, 96 and 128 nodes, and abscissae and weights for Gauss-Hermite integration on 96 and 128 nodes.

1. GAUSS-LAGUERRE

We tabulate abscissae x_i and weights w_i for Gauss-Laguerre integration of the form

$$(1.1) \quad \int_0^{\infty} f(x)e^{-x} dx \approx \sum_{i=1}^n w_i f(x_i).$$

The objective is to extend the tables provided for up to $n = 15$ in the Handbook [1, Tabl 25.9], up to 32 points by Krylov [2] and for $n = 50$ in my thesis [3] to higher numbers of nodes. The abscissae are the zeros of Laguerre Polynomials, which are [1, 22.3.9]

$$(1.2) \quad L_n(x) = L_n^{(0)}(x) = \sum_{i=0}^n \binom{n}{i} \frac{1}{i!} (-x)^i.$$

The weights are related to the derivatives at these nodes [4],

$$(1.3) \quad w_i = \frac{1}{x_i [L_n(x_i)]' ^2}.$$

Shao, Chen and Frank provide the Newton iteration to converge on the roots x_i of generalized Laguerre Polynomials $L_n^{(\alpha)}(x)$ [5]:

$$(1.4) \quad x_i \rightarrow x_i - \frac{f(x_i)}{f'(x_i)} \left[1 + \frac{1}{2} \left(1 - \frac{\alpha + 1}{x_i} \right) \frac{f(x_i)}{f'(x_i)} \right]$$

and the terminating continued fraction

$$(1.5) \quad \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)'}(x)} = \frac{x}{n-} \frac{(n+\alpha)n}{2n+\alpha-1-x-} \frac{(n-1+\alpha)(n-1)}{2n+\alpha-3-x-} \frac{(n-2+\alpha)(n-2)}{2n+\alpha-5-x-} \dots \frac{(1+\alpha)}{1+\alpha-x}.$$

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2. GAUSS-HERMITE

We also tabulate abscissae x_i and weights w_i for Gauss-Hermite integration of the form

$$(2.1) \quad \int_{-\infty}^{\infty} f(x)e^{-x^2} dx \approx \sum_{i=1}^n w_i f(x_i).$$

The objective is to extend the tables provided for up to $n = 20$ by Krylov and in the Handbook [1, Table 25.9][2], and up to $n = 64$ by Shao, Chen and Frank [5]. The abscissae are the zeros of Hermite Polynomials, which are [1, 22.3.10]

$$(2.2) \quad H_n(x) = n! \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(-1)^i}{i!(n-2i)!} (2x)^{n-2i}.$$

The weights are related to the derivatives at these nodes [1, 25.4.46],

$$(2.3) \quad w_i = \frac{2^{n-1} n! \sqrt{\pi}}{[nH_{n-1}(x_i)]^2} = \frac{2^{n+1} n! \sqrt{\pi}}{[H'_n(x_i)]^2}.$$

The first order Newton iteration to stabilize the roots x_i is

$$(2.4) \quad x_i \rightarrow x_i - \frac{f(x_i)}{f'(x_i)},$$

and the supporting terminating continued fraction [6] is

$$(2.5) \quad \frac{H_n(x)}{H'_n(x)} = \frac{1}{2n} \frac{H_n(x)}{H_{n-1}(x)} = \frac{1}{2n} \left[2x - \frac{2(n-1)}{2x-} \frac{2(n-2)}{2x-} \dots \frac{2}{2x} \right].$$

3. RESULTS

First the x_i and then the associated w_i are shown for the Laguerre polynomials. For Hermite polynomials, the duplicates with the opposite sign are not reproduced.

The originating Maple program is shown in the appendix.

Laguerre n=64

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2.241587414670528002281188481904e-02  5.625284233902984574102185450625e-02
1.181225120967704797974664367103e-01  1.190239873124260278149035058888e-01
2.903657440180364839991300663853e-01  1.574964038621445238201964347055e-01
5.392862212279790393181449478122e-01  1.675470504157739478809044116588e-01
8.650370046481139446199550747097e-01  1.533528557792366180854547925640e-01

1.267814040775241398115708877694e+00  1.242210536093297445126137821926e-01
1.747859626059436252829963951288e+00  9.034230098648505773897410920156e-02
2.305463739307508718548070543894e+00  5.947775576835502421224699743972e-02
2.940965156725251840679468152108e+00  3.562751890403607185416573533689e-02
3.654752650207290527035397912093e+00  1.948041043116640604333738027148e-02

4.447266343313094356742550980161e+00  9.743594899382002240107960271379e-03
5.318999254496390343522109859188e+00  4.464310364166275292364824190544e-03
6.270499046923653912911064646334e+00  1.875359581323114826750128222516e-03
7.302370002587395747223498409519e+00  7.226469815750051227191087066195e-04
8.415275239483024194495218591205e+00  2.554875328334967097144484021795e-04

9.609939192796108035762882049548e+00  8.287143534396942179063224039877e-05
1.088715038388637214259455042018e+01  2.465686396788558745973370221716e-05
1.224776450424430161816236929072e+01  6.726713878829668527612545536274e-06

```

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1.522298111152472884800826878343e+01 3.850812981546684414827886110704e-07

1.683966365264873721052883803916e+01 8.068728040990499790415110915505e-08
1.854391817085919052361962597108e+01 1.545723706757688828003703926158e-08
2.033699594873023550114981580640e+01 2.704480147617481409988607446575e-09
2.222024266595087653992215434709e+01 4.316775475427200912314164217177e-10
2.419510487593325398988644388024e+01 6.277752541761452201652955763263e-11

2.626313722711848578512602395478e+01 8.306317376288958063878932234380e-12
2.842601052750102729949977152675e+01 9.984031787220164055897291499895e-13
3.068552076752597177104858239845e+01 1.088353887116662685326123546986e-13
3.304359923643782912552021068053e+01 1.0740174034415901864828048102228e-14
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3.806393216564646826035731791502e+01 7.697028023648586098862970088678e-17
4.073083544445862636573186951324e+01 5.564881137454025366525461526815e-18
4.350563546642152985270318493173e+01 3.609756409010446498298646897469e-19
4.639114297861619207360539994238e+01 2.095095369548946234767748078464e-20
4.939039902562468667923580082269e+01 1.084793301097549361202631562742e-21

5.250669934134630165017927698046e+01 4.994699486363804115792443627063e-23
5.574362241327838046333571129115e+01 2.037836974598822310658063498146e-24
5.910506191901710660884874209180e+01 7.339537564278837039106367650444e-26
6.259526440015139559605501790119e+01 2.323783082198694261304776157500e-27
6.621887325124756438221376267101e+01 6.438234706908762420384341347468e-29

6.998098037714682922853465797223e+01 1.553121095788275270621507732395e-30
7.388718723248296321095740311351e+01 3.244250092019537314465982686763e-32
7.794367743446312031368797587057e+01 5.832386267836201501282207418418e-34
8.215730377831930429519586834224e+01 8.963254833102854061314281996410e-36
8.653569334945651821021627837527e+01 1.168703989550736241198572917330e-37

9.108737561313309014563671534928e+01 1.282055984359980381497645585305e-39
9.582194001552073209476721543650e+01 1.172094937405002291828122995135e-41
1.007502319695139796292592614512e+02 8.835339672328604981296921087574e-44
1.058845994687999493563604278509e+02 5.424955590306186594338147130486e-46
1.112392075244395820634847366376e+02 2.675542666678893828949793699428e-48

1.168304450513064984633866690774e+02 1.042917031411367078110598371701e-50
1.226774602685385765774196905650e+02 3.152902351957772623653127987116e-53
1.288028787692376725127536230543e+02 7.229541910647522339706610677135e-56
1.352337879495258278339804988789e+02 1.224235301230082264459594302229e-58
1.420031214899315190251400382911e+02 1.482168504901910411780618275588e-61

1.491516659000493885872934629324e+02 1.232519348814518808064365504012e-64
1.567310751326711612336169608139e+02 6.691499004571269526814069374687e-68
1.648086026551505229931901090252e+02 2.220465941850448995507392045782e-71
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2.180318519353285163324523844483e+02 2.989484348860634307741312694817e-94
 2.348095791713261647130555297254e+02 2.089063508436952770828154254400e-101

Laguerre n=96

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 1.940394361941514412752508786116e-01 1.158667991339846973607615984049e-01
 3.603184994030137122785822472820e-01 1.338323001322935736406256373056e-01
 5.778305997114498570222491789396e-01 1.364333036164533434786894149718e-01

8.466343334083691664262806795413e-01 1.262862120356477972119000242104e-01
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 1.538417935202955009462473472637e+00 8.532412911159066557538645790078e-02
 1.961582977838289290098638265217e+00 6.314203242520727884377360072333e-02
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 9.060687511578667610371531997438e+00 1.131361618904986583334124750235e-04
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 1.222563911256064169884407585976e+01 5.571480048334918160042235677625e-06
 1.338927742886798323913406764228e+01 1.824020563650868624991610676325e-06
 1.460779595321226557860843179550e+01 5.642346857470526754659778186812e-07
 1.588155542965690531693169690084e+01 1.648822949985393333100870755930e-07

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 3.108056796861862563775135672674e+02 1.333283837705607190258421966741e-134
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 3.399214093107306652234706789151e+02 4.028786213250457687338287704918e-147

3.593576682858397029393826709107e+02 1.957414951969318430250786317709e-155

Laguerre n=128

1.125138826367596296085184031621e-02 2.855184445323972862907317736119e-02
 5.928474126902645428792200896141e-02 6.335021178450511877979781272594e-02
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 1.827997778312350285359845287177e+00 6.032705626566157053893030860025e-02

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 7.527055861229375885855122798418e+00 4.123260885396947309702908308044e-04

8.312801165007770603378843811915e+00 1.976494264425914986205298897833e-04
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 1.529181976688274097174678865517e+01 1.799065980109284720823363388052e-102

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3. Richard J. Mathar, *Untersuchungen zur Dynamik des optischen Stark Effektes in Halbleitern*, Master's thesis, Rheinisch Westf. Techn. Hochschule Aachen, 1988.
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APPENDIX A. MAPLE SOURCE CODE

```

interface(quiet=true):
with(orthopoly) :
Digits := 120 :

# Return the ratio L_n(x)/L_n'(x) of
# the Laguerre Polynomial of index n (log derivative) at abscissa x.
# Parameter x: the abscissa value
# Parameter n: Order of the Laguerre Polynomial.
# Return: L_n(x)/L_n'(x).
ShaoChenFrank := proc(x,n)
    local itr,lnfrac ;
    lnfrac := 0 ;
    for itr from n-1 to 0 by -1 do
        lnfrac := (n-itr)^2/(2*n-2*itr-1-x-lnfrac) ;
    end do:
    lnfrac := x/(n-lnfrac) ;
    return lnfrac ;
end proc:

# Return the ratio H_n(x)/H_n'(x) of
# the Hermite Polynomial of index n (log derivative) at abscissa x.
# Parameter x: the abscissa value
# Parameter n: Order of the Hermite Polynomial.
# Return: H_n(x)/H_n'(x).
Hcfrac := proc(x,n)
    local itr,lnfrac ;
    lnfrac := 1/x ;
    for itr from 2 to n-1 do
        lnfrac := 2*itr/(2*x-lnfrac) ;
    end do:
    lnfrac := (2*x-lnfrac)/(2*n) ;
    return lnfrac ;
end proc:

# Run 3 steps of Newton iterations to refine the estimate of
# a root of L_n(x).
# Parameter x: the estimate of the root.
# Parameter n: Order of the Laguerre Polynomial.
# Return: An improved estimate of the root.
lagroot := proc(x,n)
    local l,xref,ffprime ;
    xref := x ;
    for l from 1 to 3 do
        ffprime := ShaoChenFrank(xref,n) ;
        xref := xref-ffprime*(1+ffprime*(1-1/xref)/2.0) ;
    end do:
    return xref ;
end proc:

```

```

        end do;
        return xref ;
end proc:

# Run 3 steps of Newton iterations to refine the estimate of
# a root of H_n(x).
# Parameter x: the estimate of the root.
# Parameter n: Order of the Hermite Polynomial.
# Return: An improved estimate of the root.
herroot := proc(x,n)
    local l,xref,ffprime ;
    xref := x ;
    for l from 1 to 3 do
        ffprime := Hcfrac(xref,n) ;
        xref := xref-ffprime ;
    end do;
    return xref ;
end proc:

# Print the abscissa and weights for Gauss-Laguerre Quadrature on n nodes.
# Parameter n: The number of nodes. Also the order of the Laguerre Polynomial.
lagGaussian := proc(n)
    local lag,lagprime,ktau,i,xi,wi ;
    # Construct the Laguerre Polynomial L_n(x)
    lag := L(n,x) ;
    lag := evalf(lag) ;
    # Construct the first derivative of L_n(x)
    lagprime := diff(lag,x) ;
    # An upper estimate of the largest root of L_n(x).
    # See Abramowitz and Stegun 22.16.8
    ktau := evalf(n+1/2) ;
    # Find all roots of L_n(x)
    lag := fsolve(lag,x,0..2*ktau+sqrt(4*ktau^2+0.25),fulldigits,complex) ;
    i := 1 ;
    printf("Laguerre n=%d\n",n) ;
    for xi in lag do
        xi := lagroot(Re(xi),n) ;
        wi := 1/xi/(eval(lagprime,x=xi))^2 ;
        printf("%.30e %.30e\n",xi,wi) ;
        if i mod 5 = 0 then
            printf("\n") ;
        end if;
        i := i+1 ;
    end do;
    printf("\n") ;
    return ;
end proc:

# Print the abscissa and weights for Gauss-Hermite Quadrature on n nodes.
# Parameter n: The number of nodes. Also the order of the Hermite Polynomial.
hermGaussian := proc(n)
    local lag,lagprime,ktau,i,xi,wi ;
    # Construct the Hermite Polynomial H_n(x)

```

```

lag := H(n,x) :
lag := evalf(lag) :
# Construct the first derivative of H_n(x)
lagprime := diff(lag,x) ;
# Find all roots of H_n(x)
lag := fsolve(lag,x,0..infinity,fulldigits,complex) ;
i := 1 ;
printf("Hermite n=%d\n",n) ;
for xi in lag do
  if xi >0 then
    xi := herroot(Re(xi),n) ;
  end if;
  wi := 2^(n+1)*n!*sqrt(Pi)/(eval(lagprime,x=xi))^2 ;
  printf("%.30e %.30e\n",xi,wi) ;
  if i mod 5 = 0 then
    printf("\n") ;
  end if;
  i := i+1 ;
end do:
printf("\n") ;
return ;
end proc:

for n in [64, 96, 128] do
  lagGaussian(n) ;
end do:

for n in [96, 128] do
  hermGaussian(n) ;
end do:
URL: http://www.mpia.de/~mathar
HOESCHSTR. 7, 52372 KREUZAU, GERMANY

```